Constructing uniform central graphs and embedding into them

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Abstract

A graph is called uniform central (UC) if all its central vertices have the same set of eccentric vertices. It is proved that if G is a UC graph with radius at least 3, then substituting a central vertex u of G with an arbitrary graph H and connecting the vertices of H to all neighbors of u (in G), yields a UC graph again. This construction extends several earlier ones and enables a simple argument for the fact that for any $r \geq 2$ and any $r+1 \leq d \leq 2r$, there exists a non-trivial UC graph G with $\mathrm{rad}(G) = r$ and $\mathrm{diam}(G) = d$. Embeddings of graphs into UC graphs are also considered. It is shown that if G is an arbitrary graph with at least one edge then at most three additional vertices suffice to embed G into an r-UC graph with $r \geq 2$. It is also proved that P_3 is the only UC graph among almost self-centered graphs.

Keywords: radius; diameter; uniform central graph; (almost) self-centered graph; central appendage number

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1 Introduction

Graphs considered in this paper are finite, simple and connected. If G is a graph, then the *distance* $d_G(u,v)$ (or $d(u,v)$ for short if the graph is clear from the context) between vertices u and v is the standard shortest-path distance. The *eccentricity* $\mathrm{ecc}_G(u)$ (or $\mathrm{ecc}(u)$ for short) of the vertex u is $\max\{d_G(u,x) : x \in V(G)\}$. The minimum eccentricity and the maximum eccentricity over all vertices of G, respectively, are its *radius* $\mathrm{rad}(G)$ and its *diameter* $\mathrm{diam}(G)$. The *center* $C(G)$ of G is the set of vertices of minimum eccentricity, its elements being called *central vertices*.

Central vertices are important in location theory because when assigning resources to facilities it is often desirable that the resources are located such that eccentricities of related vertices are as small (or, sometimes also as large) as possible. In the extreme case when every vertex is central, that is, when $C(G) = V(G)$ holds, G is called a *self-centered graph* [3, 4, 15, 20]. If $C(G) \neq V(G)$, then G contains at least two non-central vertices; graphs with exactly two non-central vertices were introduced in [17] and named *almost self-centered (ASC) graphs*. Further properties of these graphs were determined in [1, 16]. On the other extreme one finds the so-called *almost-peripheral (AP) graphs* that were introduced in [18] as the graphs in which all but one vertex are peripheral, see also [16]. (A vertex is peripheral if its eccentricity is maximum.) We point out that recently a measure of non-self-centrality was introduced in [22], where ASC graphs and AP graphs (as well as a newly introduced *weakly AP graphs*) play a significant role as extremal graphs. For some recent studies of the (average) eccentricity of graphs we refer to [5, 8, 10, 11, 13, 14] while for an additional role of the eccentricity in network theory see [2, 19, 21].

Sometimes it is desirable that the central vertices of a graph are indistinguishable because in such a network selecting randomly one of the central vertices for a location of a resource makes an optimal choice. A natural way how to give a meaning to "indistinguishable" was proposed in [6] as follows. If G is a graph and $u \in V(G)$, then let $\mathrm{Ecc}(u) = \{x : d(u,x) = \mathrm{ecc}(u)\}$; we say that $\mathrm{Ecc}(u)$ is the *periphery* of u in G. The graph G is a *uniform central graph*, *UC graph* for short, if $u, v \in C(G)$ implies that $\mathrm{Ecc}(u) = \mathrm{Ecc}(v)$. In words, a graph is a UC graph if all its central vertices have the same periphery. If $\mathrm{rad}(G) = r$, then we will say that G is an *r-UC graph*. If $|C(G)| = 1$, that is, when G has a unique central vertex, then G is trivially a UC graph. We will therefore call such graphs *trivial UC graphs*.

Clearly, no self-centered graph on at least two vertices is a UC graph. The variety of UC graphs among the almost-self centered graphs is only a fraction larger as the next result asserts.

Proposition 1.1 *Let G be an ASC graph. Then G is a UC graph if and only if $G = P_3$.*

Proof. By an r-ASC graph we mean an ASC graph of radius r.

Assume first that G is an r-ASC graph, $r \geq 2$. Assume further that G is a UC graph and suppose that $|V(G)| \geq 4$. Let u and v be the two non-central vertices of G. Then $d_G(u, v) = r + 1 \geq 3$. Let P be a shortest u, v-path, and let x and y be the neighbors of u and v on P, respectively. Then $x, y \in C(G)$ and because $d_G(u, v) \geq 3$, we infer that $x \neq y$. But then clearly x and y do not have the same periphery. So $|V(G)| \leq 3$ must hold and hence we are left with $G = P_3$ which is a 1-ASC graph, a contradiction with the assumption $r \geq 2$.

Let now G be a 1-ASC graph. According to [17], G must be isomorphic to $K_n - e$, that is, to the complete graph K_n with one edge removed. Now, $K_3 - e = P_3$, while $K_n - e$, $n \geq 4$, is not a UC graph. □

The rest of the paper is organized as follows. In the next section we give a construction of UC central graphs and apply it to show that for any $r \geq 2$ and any $r + 1 \leq d \leq 2r$, there exists a non-trivial r-UC graph G with $\mathrm{diam}(G) = d$. This result has been first proved in [7], the advantage of our construction is its simplicity. Then, in Section 3, we show that the construction from the previous section covers several earlier ones, for instance a couple of results from [12] on the so-called central appendage number of a graph. We also consider minimal embeddings of graphs into UC graphs and prove that any graph can be embedded into an r-UC graph, $r \geq 2$, with the addition of at most three vertices.

2 Constructing UC graphs

Let G and H be graphs and let $u \in V(G)$. Then the graph $G \otimes_u H$ is obtained from G by replacing the vertex u with the graph H and joining all the vertices of H with all the neighbors of u (in G). In other words, $G \otimes_u H$ is obtained from the disjoint union of $G - \{u\}$ and H by making a join between H and the open neighborhood $N(u)$ of u in G. Using this operation, we can construct many UC graphs as follows.

Theorem 2.1 *Let G be an r-UC graph, $r \geq 3$, and let H be an arbitrary graph. If $u \in C(G)$ and $X = G \otimes_u H$, then X is an r-UC graph with $C(X) = (C(G) - \{u\}) \cup V(H)$. Moreover,* $\mathrm{rad}(X) = \mathrm{rad}(G)$ *and* $\mathrm{diam}(X) = \mathrm{diam}(G)$.

Proof. We start with the following:

Claim A: The vertices in H are not capable of shortening the shortest paths in G.
Let P be a shortest path in X with $|P| \geq 3$. Note first that P cannot lie completely in H. Indeed, for otherwise P is not a shortest path because $|P| \geq 3$ and the first and the last vertex of P are adjacent to (at least one) common vertex in $N(u)$. Hence we may assume without loss of generality that the first vertex from $V(P) \cap V(H)$ has a neighbor x' in $N(u)$. If P contains another vertex of H, then, x' being adjacent to another vertex of $V(P) \cap V(H)$, we see again that P is not a shortest path. We have thus proved that if $|P| \geq 3$, then $|V(P) \cap V(H)| \leq 1$. So let P be a shortest path in X and $x \in V(P) \cap V(H)$. By the above, x is on P adjacent to $x', x'' \in N(u)$. But then the $x' - x - x''$ subpath of P can be replaced with $x' - u - x''$ to obtain a path in G of the same length. Claim A is proved.

Consider now the eccentricity of a typical vertex x of X. Suppose first that $x \in C(G)$. (Clearly, as x is a vertex of X, we have $x \neq u$.) Let $y \in V(G)$. If $d_X(x, y) < d_G(x, y)$ would hold, then by Claim A a shortest x, y-path P would necessarily contain exactly one vertex from H, but then an x, y-path in G would exist of the same length as P. Indeed, just replace the vertex of P from H

with u. It follows that $d_X(x,y) = d_G(x,y)$ holds for any vertex $y \in V(G)$. In addition, $d_X(x,y) = d_G(x,u)$ holds for any vertex $y \in V(H)$. It follows that $\mathrm{ecc}_X(x) = \mathrm{ecc}_G(x) = r$.

Consider next the situation when $x \in V(H)$. Note that then $d_X(x,y) \leq 2$ holds for any vertex $y \in V(H)$, $y \neq x$. Furthermore, if $y \in V(G)$, then $d_X(x,y) = d_G(u,y) \leq r$, where the last inequality holds since $u \in C(G)$. Moreover, since $\mathrm{rad}(G) = r$, we find that also in this case $\mathrm{ecc}_X(x) = r$.

Assume finally that $x \in V(G) - C(G)$. Then there exists a vertex $y \in V(G)$ such that $d_G(x,y) > r$. But then, using Claim A again, also $d_X(x,y) > r$ and so $x \notin C(X)$. We conclude that $C(X) = (C(G) - \{u\}) \cup V(H)$.

Note that above arguments in particular imply that $\mathrm{rad}(X) = \mathrm{rad}(G)$ and $\mathrm{diam}(X) = \mathrm{diam}(G)$.

It remains to prove that X is a UC graph. If $x \in C(G)$, then u is not in the periphery of x (cf. [6, Theorem 3]), and consequently no vertex of H is in the periphery of x. It follows that x has the same periphery in X as in G. Let now $x \in V(H)$. Then no vertex $y \in V(H)$ is in periphery of x since $d_X(x,y) \leq 2$ and $r \geq 3$. But then the periphery of x in X is the same as the periphery of u in G. We conclude that all the vertices from $C(X) = (C(G) - \{u\}) \cup V(H)$ have the same periphery. □

Note that Theorem 2.1 enables us to transform a trivial r-UC graph into a non-trivial one by choosing an H with $|V(H)| > 1$.

The reason that we have defined $G \otimes_u H$ such that the vertex u is removed is that in the case when $|C(G)| = 1$, the constructed graph $X = G \otimes_u H$ has the property $C(X) = V(H)$. This will be useful to us in the sequel. On the other hand, the construction can be made a bit simpler by adding H to G without removing a vertex of G. Also in this case the conclusion holds, more precisely, we have the following:

Corollary 2.2 *Let G be an r-UC graph, $r \geq 3$, let H be an arbitrary graph, and let $u \in C(G)$. Let X be the graph obtained from the disjoint union of G and H by making a join between H and the open neighborhood of u in G. Then X is an r-UC graph with $C(X) = C(G) \cup V(H)$.*

Proof. Let $H' = H \cup K_1$. Then X is isomorphic to $G \otimes_u H'$ because the vertex of K_1 plays the role of u. By Theorem 2.1 the conclusion follows. $\square$

As observed in [6], all 1-UC graphs are trivial UC-graphs. But as soon $r \geq 2$, there exist non-trivial r-UC graphs for any possible diameter, that is, for any diameter between $r+1$ and $2r$. This result was first proved by Choi and Guan in [7] in order to demonstrate that a conjecture from [6] asserting that an r-UC graph has diameter at least $r + \lfloor (r+1)/2 \rfloor$ does not hold.

Let us briefly describe the construction of Choi and Guan. For given radius r and diameter d, where $r < d \leq 2r$, set $m = d-r$. Start with the cycle C_{4m}. Then take $4m$ disjoint paths of length $r-1$ and respectively identify an end-vertex of each with a vertex of C_{4m}. Let X be the set of $4m$ vertices of degree 1 in the so far constructed graph. Finally add two more vertices a and b, and add the edge between them as well as all edges between a and b and the vertices from X.

Using Theorem 2.1 we now reprove the above result since our construction is simpler and the result is stronger.

Theorem 2.3 *For any $r \geq 2$ and any $r+1 \leq d \leq 2r$ there exists a non-trivial r-UC graph G with $\mathrm{diam}(G) = d$. Moreover, if $r \geq 3$, and H is an arbitrary graph, then G can be selected such that H is an induced subgraph of G.*

Proof. Let Q_3^- be the graph obtained from the 3-cube Q_3 by removing one of its vertices. Then it is straightforward to verify that $Q_3^- \otimes_u K_2$ is a non-trivial 2-UC graph of diameter 3, where u is the center of Q_3^-, and that $P_5 \otimes_u K_2$ is a non-trivial 2-UC graph of diameter 4, where u is the center of P_5. In the rest of the proof we may hence assume that $r \geq 3$.

Let $G_{r,r+\ell}$, $r \geq 3$, $1 \leq \ell \leq r$, be the graph constructed as follows. Take $2\ell+2$ disjoint paths of length r, select one end-vertex in each of them, and identify these vertices. In other words, the graph so far is obtained from $K_{1,2\ell+2}$ by subdividing $r-1$ times each of its edges. Finally, connect the pendant vertices of this temporary graph such that they induce a path. The graph $G_{r,r+\ell}$ is schematically presented in Fig. 1, where the notation for its vertices to be used in the sequel is also introduced.

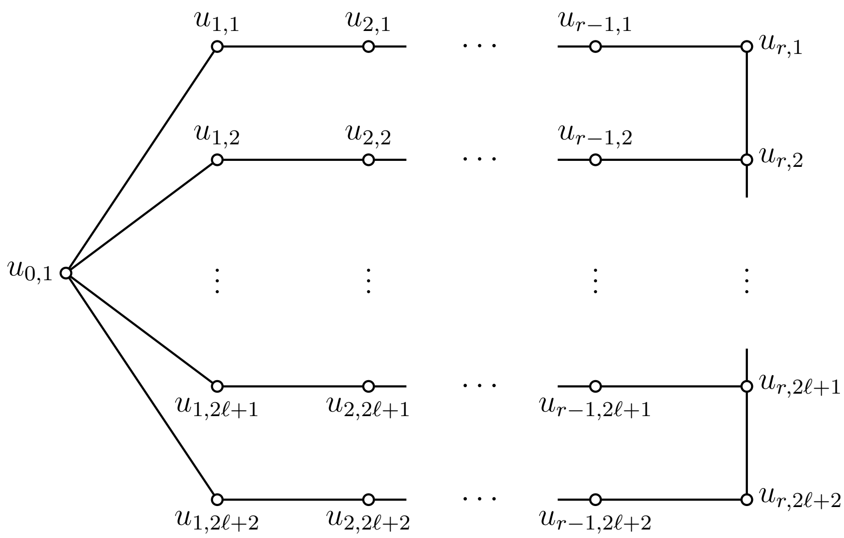

Figure 1: The graph $G_{r,r+\ell}$

To shorten the notation set $G = G_{r,r+\ell}$ for the rest of the proof. We claim that $\mathrm{diam}(G) = r+\ell$. For $1 \le i < j \le 2\ell+2$, let C_{ij} be the following cycle of G:

$$C_{i,j}: \; u_{0,i}, u_{1,i}, \ldots, u_{r,i}, u_{r,i+1}, \ldots, u_{r,j}, u_{r-1,j}, \ldots, u_{1,j}, u_{0,1}\,.$$

The longest of these cycles is $C_{1,2\ell+2}$, its length is $2(k+\ell)+1$. Since each pair of vertices x, y of G lies on some cycle $C_{i,j}$, we infer that $d_G(x,y) \le d_{C_{i,j}} x, y \le k+\ell$. Moreover,

$$\begin{aligned} d_G(u_{r-(r-\ell),1}, u_{r,2\ell+1}) &= d_G(u_{r-(r-\ell),1}, u_{0,1}) + d_G(u_{0,1}, u_{r,2\ell+1}) \\ &= (r-(r-\ell)) + r = r+\ell\,, \end{aligned}$$

which proves that $\mathrm{diam}(G) = r+\ell$.

It is straightforward to see that the vertex vertex $u_{0,1}$ is the unique central vertex of $G = G_{r,r+\ell}$. Consequently, $G_{r,r+\ell}$ is a trivial r-UC graph. Using Theorem 2.1 we can construct a non-trivial r-UC graph $G_{r,r+\ell} \otimes_u H$ which contains H as an induced subgraph. Finally, $\mathrm{diam}(G_{r,r+\ell} \otimes_u H) = \mathrm{diam}(G_{r,r+\ell}) = r+\ell$. $\square$

3 Earlier constructions and embeddings into UC graphs

Note that Theorem 2.1 does not hold for $r = 2$. Consider for instance P_5 which is a trivial 2-UC graph, and denote with u its central vertex. Then the graph $P_5 \otimes_u \overline{K}_2$ is not a 2-UC graph, the reason being that the vertices of $\overline{K}_2$ (two independent vertices) are pairwise peripheral in $P_5 \otimes_u \overline{K}_2$. On the other hand, the construction works also for $r = 2$ if the graph H is complete. Then $P_5 \otimes_u K_n$ is a 2-UC graph. We also note that the the 2-UC graph $Q_3^- \otimes_u K_2$ from the proof of Theorem 2.3 is the graph from [6, Fig. 1].

In [12] the *central appendage number* $A_{\rm ucg}(G)$ of a graph G was introduced as the smallest number of vertices to be added to G such that the resulting graph H is a UC graph with $C(H) = V(G)$. It was proved that for any connected graph G of order at least two, $4 \le A_{\rm ucg}(G) \le 6$. Moreover, $A_{\rm ucg}(G) = 4$ if and only if G is a complete graph, while for any non-complete graph G, $A_{\rm ucg}(G) = 6$ holds. To prove that if G is an arbitrary graph on at least two vertices, then $A_{\rm ucg}(G) \le 6$ holds, Gu actually considered

$$X = P_7 \otimes_u G\,,$$

where u is the center of P_7. As Theorem 2.1 implies, X is then a 3-UC graph with $C(X) = C(G)$ as required. Moreover, Gu also considered

$$Y = P_5 \otimes_u K_n\,,$$

where u is the center of P_5. Then by the discussion in the first paragraph of this section, Y is a 2-UC graph with K_n as its center.

The requirement that $C(H) = C(G)$ can be relaxed by changing the condition that $V(G)$ is the center of the UC graph that contains G to the condition that G is an induced subgraph of it. Hence we set $\widehat{A}_{\rm ucg}(G)$ to be the smallest number of vertices to be added to G such that the resulting graph H is an r-UC graph for some $r \ge 2$, that is,

$$\widehat{A}_{\rm ucg}(G) = \min\{|V(H)| - |V(G)| :\ G \text{ induced in } H, H\ r\text{-UC graph}, r \ge 2\}\,.$$

We impose the latter technical condition (that is, $r \geq 2$) to avoid trivialities. For instance, if G contains a vertex x adjacent to all other vertices, then attaching to x a pendant vertex yields a 1-UC graph. For this new graph invariant we have:

Theorem 3.1 *If G is an arbitrary graph with at least one edge, then $\widehat{A}_{\rm ucg}(G) \leq 3$. Moreover, the bound is sharp.*

Proof. Let uv be an edge of G. Let H be the graph obtained from G by adding the vertices x, y, z, the edges ux, xy, yz, and the edges $\{xw : w \in V(G), d_G(w,u) \geq 2\}$. It is now straightforward to check that ${\rm ecc}_H(x) = 2$ while ${\rm ecc}_H(w) \geq 3$ holds for any vertex $w \neq x$. This means that H is a 2-UC graph and so $\widehat{A}_{\rm ucg}(G) \leq 3$.

To demonstrate that the bound is sharp we are going to prove that $\widehat{A}_{\rm ucg}(K_n) = 3$ holds for any $n \geq 2$. Since K_n is self-centered, $\widehat{A}_{\rm ucg}(K_n) \geq 1$. If we add a new vertex and connect it to at least one vertex of K_n, we obtain a graph of radius 1. Therefore, $\widehat{A}_{\rm ucg}(K_n) \geq 2$. Assume now that $\widehat{A}_{\rm ucg}(K_n) = 2$ holds and let H be an r-UC graph, $r \geq 2$, with the vertex set $V(H) = V(G) \cup \{x, y\}$. Suppose first that x and y both have a neighbor in K_n, say x' and y', respectively. Note first that $x' \neq y'$, for otherwise ${\rm rad}(H) = 1$. From the same reason, $x'y \notin E(H)$ and $xy' \notin E(H)$. But now $d_H(x, y) = 3$, $x \in {\rm Ecc}(y')$, and $x \notin {\rm Ecc}(x')$, which is not possible since $x', y' \in C(H)$. It remains to consider the case that x has a neighbor in K_n, say x', while y does not have a neighbor in K_n. Then clearly $xy \in E(H)$. But now x' and x must lie in $C(H)$ and since y is in the periphery of x' but not in the periphery of x, we conclude that this case is also impossible. It follows that $\widehat{A}_{\rm ucg}(K_n) \geq 3$ and hence by the first paragraph $\widehat{A}_{\rm ucg}(K_n) = 3$. The bound is thus sharp. $\square$

Note that the graph H from the proof of Theorem 3.1 which demonstrates that $\widehat{A}_{\rm ucg}(G) \leq 3$ holds for any graph G has a unique central vertex x. Hence H is a trivial UC graph. One might prefer that the super graph of G would be a non-trivial UC graph. In the case that H is an r-UC graph with $r \geq 3$, then in view of Theorem 2.1, a non-trivial UC-graph can be constructed with the addition of a single additional vertex. In general, however, we leave the investigation of such embeddings for future research.

4 Concluding remarks

In Section 3 we have considered embeddings of graphs into UC graphs in view of the least number of vertices required to be added to a given graph in order to turn into an UC graph. This definition is in lines with several earlier related investigations; the central appendage number [12] was our primary motivation, but see also [9, 17, 18] where the same approach has been followed. On the other hand, adding few additional vertices might require that a lot of edges must be added. Hence we pose the following:

Problem 4.1 *Study the problem of determining the least number of edges which need to be added to a given graph in order the obtained graph is an UP graph. In particular, compare the efficiency of such embeddings with the ones from Section 3.*

Note that the embedding concept from the above problem is well-defined since in the worst case we end up with a complete graph on the same vertex set.

The graph H from the proof of Theorem 3.1 which demonstrates that $\widehat{A}_{\rm ucg}(G) \leq 3$ holds for any graph G has a unique central vertex x. Hence H is a trivial UC graph. One might prefer that the super graph of G would be a non-trivial UC graph. Hence we pose:

Problem 4.2 *Prove a general upper bound on the number of vertices needed to be added to an arbitrary graph such that the obtained graph is a* **non-trivial** *UC graph.*

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