



Monophonic position sets of Cartesian and lexicographic products of graphs

Ullas Chandran S. V.^{1,7} · Sandi Klavžar^{2,3,4} · P. K. Neethu¹ · James Tuite^{5,6}

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Abstract

The general position problem in graph theory asks for the number of vertices in a largest set S of vertices of a graph G such that no shortest path of G contains more than two vertices of S . The analogous monophonic position problem is obtained from the general position problem by replacing “shortest path” by “induced path.” In this paper the monophonic position number is studied on Cartesian and lexicographic products of graphs. It is proved that in Cartesian products, a monophonic position set can only be in one of three canonical forms, named layered, varied, and cliquey. The monophonic position number of an arbitrary Cartesian product is bounded from below and above. The two bounds coincide if neither of the factors has simplicial vertices. A formula for the monophonic position number of a lexicographic product is given which only contains the clique number and the structure of monophonic sets of the second factor.

Keywords General position set · General position number · Monophonic position set · Monophonic position number · Cartesian product · Lexicographic product

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✉ Sandi Klavžar
sandi.klavzar@fmf.uni-lj.si

Ullas Chandran S. V.
svuc.math@gmail.com

P. K. Neethu
p.kneethu.pk@gmail.com

James Tuite
james.t.tuite@open.ac.uk

- ¹ Department of Mathematics, Mahatma Gandhi College, University of Kerala, Thiruvananthapuram, India
- ² Faculty of Mathematics and Physics, University of Ljubljana, Ljubljana, Slovenia
- ³ Institute of Mathematics, Physics and Mechanics, Ljubljana, Slovenia
- ⁴ Faculty of Natural Sciences and Mathematics, University of Maribor, Maribor, Slovenia
- ⁵ School of Mathematics and Statistics, Open University, Milton Keynes, UK
- ⁶ Department of Informatics and Statistics, Klaipėda University, Klaipėda, Lithuania
- ⁷ Faculty of Mathematics and Informatics, Vilnius University, Vilnius, Lithuania

1 Introduction

The general position problem for graphs originates in Dudeney's 'Puzzle with Pawns' in his book 'Amusements in Mathematics' (Dudeney 1917) from 1917. This problem asked the reader to find the largest number of pawns that can be placed on a chessboard without any three pawns lying on a common straight line in the plane. This problem was generalised to the setting of graph theory independently in Chandran S. V. and Parthasarathy (2016); Manuel and Klavžar (2018) as follows: a set of vertices S in a graph G is in *general position* if no shortest path of G contains more than two vertices of S . The problem then consists of finding the largest set of vertices in general position for a given graph G . The structure of general position sets in graphs was described in Anand et al. (2019).

The general position problem has been the subject of intensive research, with some 48 papers appearing on the subject since 2018. For some recent developments see Araujo et al. (2025); Chandran et al. (2024); Di Stefano et al. (2025); Iršič et al. (2024); Klavžar et al. (2023); Roy et al. (2025); Chandran et al. (2026); Thomas et al. (2024a); Tian and Xu (2024); Tian et al. (2023); Yao et al. (2022). A survey of the problem is given in Chandran et al. (2025). One case of particular interest is the general position number, that is, the number of vertices in a largest general position set, of product graphs. The general position problem for Cartesian products has been treated in many papers, including (Klavžar et al. 2021a; Klavžar and Rus 2021; Korže and Vesel 2023; Krufft Welton et al. 2025; Tian and Xu 2021; Tian et al. 2021). General position numbers of strong and lexicographic products were investigated in Dokyeesun et al. (2026).

Several variations on the general position problem have also been considered. For example, we can replace 'shortest path' in the definition of the problem by some other family of paths. In Klavžar et al. (2021b) the authors restrict attention to shortest paths of bounded length, whilst (Haponenko and Kozerenko 2024) considers the problem for the widest possible family, i.e. all paths. Another important family is the *induced* or *monophonic* paths. Partially inspired by the extensive literature on monophonic convexity (a recent example is Neethu et al. (2024)), the *monophonic position problem* was introduced in Thomas et al. (2024b). Some extremal problems for monophonic position sets are discussed in Tuite et al. (2025), smallest maximal monophonic position sets are treated in Di Stefano et al. (2025) and graph colourings in which every colour class is in monophonic position are explored in Chandran et al. (2026). In the present article we explore the monophonic position problem for various graph products.

The plan of this paper is as follows. The next section contains the definitions we need, and we also call up some of the basic results needed later on. In Sect. 3 we focus on the monophonic position number of Cartesian products. We prove that a monophonic position set of a Cartesian product must take one of three canonical forms, which we call layered, varied, and cliquey. We bound the monophonic position number of an arbitrary Cartesian product from below and above and prove that if neither of the factors has simplicial vertices, then the bounds coincide. We also give a different upper bound for Cartesian products of triangle-free graphs. In Sect. 4 we turn our attention to lexicographic products and give a formula for the monophonic position number which involves the clique number and the structure of monophonic sets of the second factor.

2 Preliminaries

We now define the terminology that will be used in this paper. By a graph $G = (V(G), E(G))$ we mean a finite, undirected, simple graph. We will write $u \sim v$ if vertices u and v are adjacent. The *open neighbourhood* $N(u)$ of $u \in V(G)$ is $\{v \in V(G) : u \sim v\}$, whilst the *closed neighbourhood* $N[u]$ is defined by $N[u] = N(u) \cup \{u\}$. The *degree* of a vertex is the number of vertices in its open neighbourhood; in particular, a vertex of degree one is a *leaf*. The *distance* $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest u, v -path in G , and any such shortest u, v -path is a *geodesic*. A path P in G is *induced* or *monophonic* if G contains no chords between non-consecutive vertices of P . For two distinct vertices u, v of a graph G , the *monophonic interval* $J_G[u, v]$ is the set of all vertices lying on at least one monophonic u, v -path.

Recall that a set S of vertices in a graph G is a *general position set* if no shortest path in G contains more than two vertices of S . The number of vertices in a largest general position set of G is called the *general position number* of G and is denoted by $\text{gp}(G)$. A set $M \subseteq V(G)$ is a *monophonic position set* of G if no three vertices of M lie on a common monophonic path in G . The *monophonic position number* or *mp-number* $\text{mp}(G)$ of G is the number of vertices in a largest monophonic position set of G . Observe that every monophonic position set of a graph G is also in general position, and hence $\text{mp}(G) \leq \text{gp}(G)$. Any pair of vertices is in monophonic position, so for graphs with order $n(G) \geq 2$ we have $2 \leq \text{mp}(G) \leq n(G)$. Trivially $\text{mp}(G) = n(G)$ holds for a connected graph G if and only if G is complete. It was shown in Thomas et al. (2024b); Tuite et al. (2025) that for any $2 \leq a \leq b$ there exists a graph G with $\text{mp}(G) = a$ and $\text{gp}(G) = b$. Interestingly, the largest possible number of edges in a graph with order n and monophonic position number a is quadratic in n (with constructions close in size to the a -partite Turán graph with order n), whereas for fixed general position number the largest size is linear in n , see Tuite et al. (2025). Thus the general and monophonic position problems are intrinsically different.

We will also make use of the following terminology to simplify our arguments: a monophonic path containing three or more vertices of a set S is *S-bad* (or if S is clear from the context, we will simply say *bad*). Hence, if S is assumed to be in monophonic position, then the appearance of a bad path constitutes a contradiction.

We will denote the subgraph of G induced by a subset $S \subseteq V(G)$ by $G[S]$. A vertex is *simplicial* if its neighbourhood induces a clique. We define $\sigma(G) = 1$ if G contains at least one simplicial vertex and $\sigma(G) = 0$ otherwise.

The *clique number* $\omega(G)$ of G is the number of vertices in a maximum clique in G and the *independence number* $\alpha(G)$ is the number of vertices of a maximum independent set. The path of order ℓ will be written as P_ℓ and the cycle of length ℓ as C_ℓ . For any positive integer k , we fix $[k] = \{1, \dots, k\}$.

We will need the following result on monophonic position sets.

Lemma 2.1 (Thomas et al. 2024b, Lemma 3.1). *Let G be a connected graph and $M \subseteq V(G)$ be a monophonic position set. Then $G[M]$ is a disjoint union of k cliques and $G[M] = \bigcup_{i=1}^k W_i$, where each W_i is a clique. If $k \geq 2$, then for each $i \in [k]$ any two vertices of W_i have a common neighbour in $G - M$.*

Let G and H be graphs. In this paper we discuss the monophonic position number of the *Cartesian product* $G \square H$ and the *lexicographic product* $G \circ H$. Both products have vertex set $V(G) \times V(H)$. Let $(g_1, h_1), (g_2, h_2) \in V(G) \times V(H)$. In the Cartesian product $G \square H$ the vertices (g_1, h_1) and (g_2, h_2) are adjacent if (i) $g_1 \sim g_2$ in G and $h_1 = h_2$ or (ii) $g_1 = g_2$

and $h_1 \sim h_2$ in H . In the lexicographic product $G \circ H$ these vertices are adjacent if (i) $g_1 \sim g_2$ or (ii) $g_1 = g_2$ and $h_1 \sim h_2$.

If $*$ $\in \{\square, \circ\}$, then the *projection mappings* $\pi_G : G * H \rightarrow G$ and $\pi_H : G * H \rightarrow H$ are given by $\pi_G(u, v) = u$ and $\pi_H(u, v) = v$, respectively. If $S \subseteq V(G \square H)$, then the set $\{g \in V(G) : (g, h) \in S \text{ for some } h \in V(H)\}$ is the *projection* $\pi_G(S)$ of S on G . The projection $\pi_H(S)$ of S on H is defined analogously. We adopt the following conventions. If $v_0, v_1, \dots, v_{\ell-1}, v_\ell$ is a path Q in H , then for any $u \in V(G)$, by ${}_u Q$ we denote the path

$$(u, v_0), (u, v_1), \dots, (u, v_{\ell-1}), (u, v_\ell)$$

in $G * H$. Similarly, if $u_0, u_1, \dots, u_{\ell-1}, u_\ell$ is a path P in G , then in $G * H$, the path

$$(u_0, v), (u_1, v), \dots, (u_{\ell-1}, v), (u_\ell, v)$$

is denoted by P_v . Furthermore, a tilde over a path will denote that it is traversed in the opposite direction; for example, if P is the path $u_0, u_1, \dots, u_\ell$, then $\tilde{P}$ is the path $u_\ell, \dots, u_1, u_0$. Finally, for $u \in V(G)$ and $v \in V(H)$ we define ${}^u H$ to be the subgraph of $G * H$ induced by $\{u\} \times V(H)$, which we call a *H-layer*, whilst the *G-layer* G^v is the subgraph induced by $V(G) \times \{v\}$.

3 Cartesian products

In this section we investigate the monophonic position number of the Cartesian product of connected graphs. In the main result (Theorem 3.16) we give sharp upper and lower bounds for $\text{mp}(G \square H)$ in general, and the exact formula for the case when the factors do not contain simplicial vertices. In Theorem 3.18 we also give a different upper bound in the case that both factors are triangle-free.

3.1 Structure of mp-sets

Note that if $S \subseteq V(G \square H)$, then due to the commutativity of the Cartesian product, any result that holds for $\pi_G(S)$ is also valid for $\pi_H(S)$ and vice versa. We begin with the following straightforward lower bound for $\text{mp}(G \square H)$.

Observation 3.1 For any graphs G and H , $\text{mp}(G \square H) \geq \max\{\omega(G), \omega(H)\}$.

Proof The observation follows from the fact that for any graph X we have $\text{mp}(X) \geq \omega(X)$, and the clique number of the Cartesian product is given by $\omega(G \square H) = \max\{\omega(G), \omega(H)\}$. $\square$

In order to prove the main result of this section, Theorem 3.16, we need a sequence of lemmas.

Lemma 3.2 Let G and H be connected graphs and S be a monophonic position set of $G \square H$. If $(u, v) \in S$, then $V({}^u H) \cap S = \{(u, v)\}$ or $V(G^v) \cap S = \{(u, v)\}$.

Proof Suppose that the result is not true, i.e. that there exist $u' \neq u$ in G and $v' \neq v$ in H such that (u, v) , (u', v) and (u, v') all belong to S . Let P be a monophonic u', u -path in G and Q a monophonic v, v' -path in H . Then the concatenation of P_v and ${}_u Q$ would be a monophonic (u', v) , (u, v') -path in $G \square H$ passing through (u, v) , a contradiction. $\square$

Table 1 Construction of S -bad paths

Case	Section 1	Section 2	Section 3
$u_1 = u_2 \neq u_3$	$u_1 P$	R_{v_3}	
$u_1 = u_3 \neq u_2$	Q_{v_1}	$u_2 P$	$\tilde{Q}_{v_3}$
$u_1 = u_2 = u_3$	$u_1 P$		

Lemma 3.3 *If S is a monophonic position set of $G \square H$, then $\pi_H(S)$ is a monophonic position set of H .*

Proof Suppose for a contradiction that $\pi_H(S)$ is not in monophonic position of H . Then there exists a set $S' = \{v_1, v_2, v_3\} \subseteq \pi_H(S)$ such that there is an induced v_1, v_3 -path P in H that passes through v_2 . Since $\{v_1, v_2, v_3\} \subseteq \pi_H(S)$, there exist (not necessarily distinct) vertices u_1, u_2, u_3 of G such that (u_1, v_1) , (u_2, v_2) and (u_3, v_3) belong to S ; we will derive a contradiction by constructing a monophonic path in $G \square H$ from (u_1, v_1) to (u_3, v_3) that passes through (u_2, v_2) . Let Q and R be monophonic paths in G from u_1 to u_2 and from u_2 to u_3 respectively. Also, recall that $\tilde{Q}$ is the reverse path of Q , i.e. the path formed by traversing Q in the opposite direction from u_2 to u_1 . In view of Lemma 3.3, without loss of generality, there are three possibilities to consider: (i) $u_1 = u_2 \neq u_3$, (ii) $u_1 = u_3 \neq u_2$, and (iii) $u_1 = u_2 = u_3$. Note that the path $u_2 P$ passes through (u_2, v_2) .

The desired monophonic paths in $G \square H$ are constructed by concatenating the paths in Table 1 in order. Each section of the paths is monophonic by construction and it is easily verified that there are no chords between different sections of the displayed paths. Observe that $d_H(v_1, v_3) \geq 2$, so that there is no edge in $G \square H$ between a vertex (u, v_1) and (u, v_3) . $\square$

Note that the converse of Lemma 3.3 is not true. For a simple counterexample, consider $K_2 \square K_2 \cong C_4$. Of course $V(C_4)$ is not in monophonic position, but both projections onto the first and second factors have order two, and so trivially are in monophonic position in the factors. The same observation holds using any pair of edges in the Cartesian product of any pair of non-empty graphs.

Lemma 3.4 *If $S = \{(u_i, v_i) : i \in [r]\}$ is a monophonic position set of $G \square H$ for some $r \geq 2$, then one of the following holds:*

- S lies in a single G -layer, or S lies in a single H -layer,
- $u_1, \dots, u_r$ are distinct vertices of G and $v_1, \dots, v_r$ are distinct vertices of H , and neither of these sets induce a clique, or
- $\pi_G(S)$ is a clique of G with order at least 2 and $v_1, \dots, v_r$ are distinct vertices of H , or $\pi_H(S)$ is a clique of H with order at least 2 and $u_1, \dots, u_r$ are distinct vertices of G .

Proof Suppose that S satisfies neither of the first two statements; without loss of generality, we can assume that $u_1 = u_2$ and there is some $u_i \in \pi_G(S)$ with $u_i \neq u_1$. We will show that this implies that $\pi_G(S)$ induces a clique in G and that the vertices $v_1, \dots, v_r$ of H are all distinct.

Let P be a monophonic u_1, u_i -path in G and Q and R be monophonic v_1, v_2 - and v_2, v_i -paths in H . As the concatenated path $u_1 Q, P_{v_2, u_i} R$ passes through (u_1, v_1) , (u_2, v_2) and (u_i, v_i) , there must be a chord between the paths $u_1 Q$ and $u_i R$, which is the case if and only

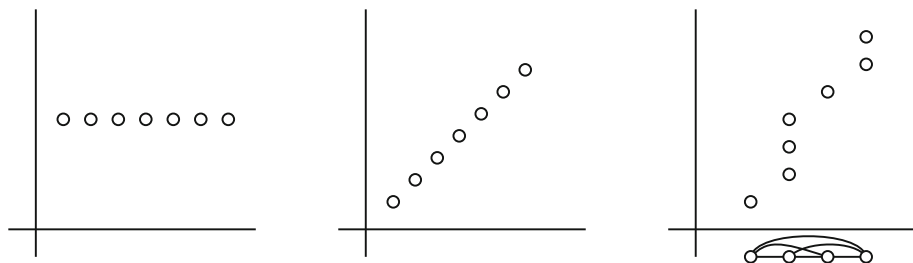


Fig. 1 Layered (left), varied (middle), and cliquey (right) monophonic position sets

if $u_1 \sim u_i$ in G and the paths Q and R intersect in H . Therefore, $u_1 \sim u_i$ in G for every i such that $u_1 \neq u_i$, so that $\pi_G(S)$ induces a connected subgraph of G . It now follows from Lemmas 2.1 and 3.3 that $\pi_G(S)$ induces a clique in G .

Suppose now that not all vertices $v_1, \dots, v_r$ are distinct, say $v_i = v_j$ (and hence $u_i \neq u_j$). As we are assuming that S does not lie in a single layer, there must be a $v_k \in \pi_H(S)$ with $v_k \neq v_i$. Let P' be a monophonic v_i, v_k -path in H . As $\pi_G(S)$ is a clique, the path formed by following the edge $(u_i, v_i) \sim (u_j, v_i)$, the path $u_j P'$ and then the edge $(u_j, v_k) \sim (u_k, v_k)$ (if $u_k = u_j$ we omit the final edge) will be monophonic unless $u_i = u_k$ and there is an edge $v_i \sim v_k$ in H ; however, in this case we can interchange u_i and u_j in the argument to produce a contradiction. $\square$

Definition 3.5 We will call a monophonic position set of Type (a) *layered*, of Type (b) *varied*, and of Type (c) *cliquey*. These three types of monophonic position sets are shown schematically in Fig. 1.

Corollary 3.6 If G and H are connected graphs, then

$$\text{mp}(G \square H) \leq \max\{\text{mp}(G), \text{mp}(H)\}.$$

Proof By Lemma 3.4, $|\pi_G(S)| = |S|$ or $|\pi_H(S)| = |S|$ (or both), and so the conclusion follows from Lemma 3.3. $\square$

Corollary 3.6 implies that the difference $\text{gp}(G \square H) - \text{mp}(G \square H)$ can be arbitrarily large. The gear H_r is the graph of order $2r + 1$ obtained by inserting an extra vertex between each pair of adjacent vertices of the outer cycle of a wheel graph W_r , cf. Peabody et al. (2013). It is shown in Thomas et al. (2024b) that for $r \geq 4$ the gear satisfies $\text{mp}(H_r) = 2$ and $\text{gp}(H_r) = r$. By Corollary 3.6 it follows that for $r \geq 4$ we have $\text{mp}(H_r \square H_r) = 2$, but $\text{gp}(H_r \square H_r) \geq \Delta(H_r \square H_r) = 2r$.

As the bounds in Observation 3.1 and Corollary 3.6 coincide for products of paths and cycles, the monophonic position numbers of grid, cylinder and torus graphs follow immediately.

Corollary 3.7 For paths P_m, P_n of order at least two and cycles C_r, C_s of length at least four,

$$\text{mp}(P_m \square P_n) = \text{mp}(P_n \square C_r) = \text{mp}(C_r \square C_s) = 2.$$

Our bounds also make it easy to evaluate $\text{mp}(K_n \square H)$ for graphs H with small mp -number.

Corollary 3.8 If H is a connected graph and $n \geq \text{mp}(H)$, then $\text{mp}(K_n \square H) = n$.

Corollary 3.8 yields $\text{mp}(K_n \square P_m) = n$ for $n \geq 2, m \geq 4$; $\text{mp}(K_n \square C_m) = n$ for $n \geq 3$; and $\text{mp}(K_n \square K_m) = \max\{n, m\}$.

3.2 Varied sets

In this subsection we show that varied monophonic position sets can only contain at most two vertices. This also incidentally allows us to determine the smallest maximal monophonic position sets in any Cartesian product.

Lemma 3.9 *If $u, u' \in V(G)$ and $v, v' \in V(H)$ are such that $u' \notin N_G[u]$ and $v' \notin N_H[v]$, then the set $\{(u, v), (u', v')\}$ is a maximal monophonic position set of $G \square H$.*

Proof Suppose that (u, v) and (u', v') satisfy the stated conditions, but that there exists a third vertex (x, y) of $G \square H$ such that $\{(u, v), (u', v'), (x, y)\}$ is a monophonic position set. It follows from Lemmas 3.3 and 3.4 that $\{u, u', x\}$ is a set of distinct vertices of G in monophonic position, and likewise $\{v, v', y\}$ is a monophonic position set of H of order three.

Let P and P' be monophonic u, x - and x, u' -paths in G respectively and Q and Q' be monophonic v, y - and y, v' -paths in H respectively. Consider the path formed by the concatenation of $P_v, {}_xQ, P'_y$ and ${}_{u'}Q'$. Trivially there are no chords between consecutive sections of the concatenated path, e.g. between P_v and ${}_xQ$. There is no chord between P_v and ${}_{u'}Q'$; otherwise there is an edge $(w, v) \sim (u', z)$ in $G \square H$, where w lies on P and z lies on Q' and, since either $w = u'$ or $z = v$ we would have a violation of Lemma 3.3.

Finally, if we assume that $x \approx u'$ and $v \approx y$, then there are no chords between P_v and P'_y or between ${}_xQ$ or ${}_{u'}Q'$. This contradicts our assumption that $\{(u, v), (u', v'), (x, y)\}$ is a monophonic position set, so we conclude that either $x \sim u'$ or $y \sim v$. By symmetry, we also have $x \sim u$ or $y \sim v'$. By Lemma 3.3 we cannot have both $x \sim u$ and $x \sim u'$, or both $y \sim v$ and $y \sim v'$, so without loss of generality $x \sim u$ and $y \sim v$. Let P'' be a monophonic u, u' -path in G and Q'' be a monophonic v, v' -path in H . Then as x does not lie on P'' and y does not lie on Q'' , the path formed by the edges $(x, y) \sim (x, v)$ and $(x, v) \sim (u, v)$ followed by the path ${}_uQ''$ and $P''_{v'}$ would be a monophonic $(x, y), (u', v')$ -path passing through (u, v) , a contradiction. We conclude that $\{(u, v), (u', v')\}$ is a maximal monophonic position set. $\square$

Lemma 3.10 *If uu' is an edge of G and vv' an edge of H , then $\{(u, v), (u', v')\}$ is a maximal monophonic position set of $G \square H$.*

Proof Let uu' and vv' be edges in G and H respectively. We claim that $M = \{(u, v), (u', v')\}$ is a maximal monophonic position set. Trivially (u, v') and (u', v) cannot be added to M . Suppose that for some $y \notin \{v, v'\}$ a vertex (u, y) can be added to M to form a monophonic position set M' . Then by Lemma 3.3 $\{v, v', y\}$ is a monophonic position set of H . Hence there is an induced y, v -path Q that does not pass through v' . The path ${}_uQ$ followed by $(u, v) \sim (u', v) \sim (u', v')$ would then be an induced path through all three vertices of M' . The case of adding a vertex from the u' -, v - or v' -layers is analogous.

Finally, consider a set $M' = M \cup \{(x, y)\}$, where $x \notin \{u, u'\}$ and $y \notin \{v, v'\}$. If M' is in monophonic position, then by Lemma 3.3 the sets $\{u, u', x\}$ and $\{v, v', y\}$ are in monophonic position in G and H respectively. Let P be an induced x, u' -path in G that does not contain u and Q be an induced y, v' -path in H that does not contain v . Then the concatenation of ${}_xQ$ and $P_{v'}$ followed by the path $(u', v') \sim (u', v) \sim (u, v)$ would be an induced path through all three vertices of M' . $\square$

The lower monophonic position number $\text{mp}^-(G)$ of a graph G is the cardinality of a smallest maximal monophonic position set of G (Di Stefano et al. 2025). Lemma 3.10 allows us to immediately deduce the lower monophonic position number of any Cartesian product.

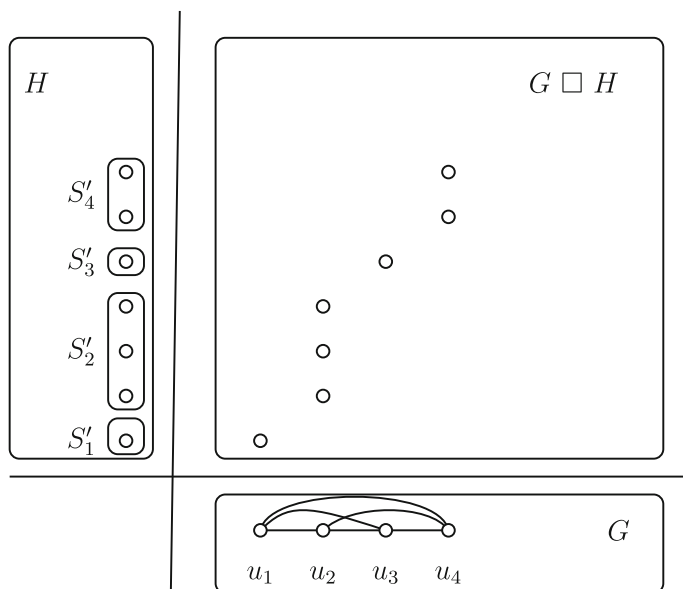


Fig. 2 Sets S'_i

Corollary 3.11 *If G, H are graphs with order at least two, then $\text{mp}^-(G \square H) = 2$.*

We now show that Lemmas 3.9 and 3.10 yield an upper bound on the number of vertices in a varied monophonic position set of $G \square H$.

Corollary 3.12 *If S is a varied monophonic position set of $G \square H$, then $|S| \leq 2$.*

Proof Assume that S is a varied monophonic position set of $G \square H$ and suppose for a contradiction that $|S| \geq 3$. Suppose that $\pi_G(S)$ is an independent set; then by Lemma 3.4 the set $\pi_H(S)$ would induce a clique, a contradiction. Thus, since $\pi_G(S)$ induces neither a clique nor an independent set in G , we can choose three vertices (u_1, v_1) , (u_2, v_2) and (u_3, v_3) in S such that $\{u_1, u_2, u_3\}$ induces neither a triangle nor an independent set. Hence by Lemma 3.3 we can assume that $u_1 \sim u_2$, $u_1 \approx u_3$ and $u_2 \approx u_3$. Hence by Lemmas 3.9 and 3.10 we must have $v_1 \sim v_3$, $v_2 \sim v_3$ and $v_1 \approx v_2$; however, this implies that v_1, v_3, v_2 is an induced path, contradicting Lemma 3.3. $\square$

3.3 Cliques and layered sets

We now restrict our attention to *layered* and *cliquey* monophonic position sets. Without loss of generality, we will assume that layered sets lie in a H -layer induced by $\{u\} \times V(H)$ and that if S is cliquey, then $\pi_G(S)$ is a clique of order at least two. We will use the following labelling convention for layered and cliquey monophonic position sets S of $G \square H$: with each vertex $u_i \in \pi_G(S)$ we associate the set $S'_i = \pi_H(S \cap (\{u_i\} \times V(H)))$ (hence if S is layered there is just one such set). Recall that by Lemma 3.4 the sets S'_i are pairwise disjoint, so that these sets partition $\pi_H(S)$. The sets S'_i are schematically represented in Fig. 2.

Since we can find monophonic position sets in $G \square H$ of order $\max\{\omega(G), \omega(H)\}$, we also tacitly assume that the monophonic sets under discussion have order strictly greater than the lower bound in Observation 3.1.

Lemma 3.13 *Let S be either a cliquey or a layered monophonic position set of $G \square H$ with $|S| > \max\{\omega(G), \omega(H)\}$, and let $u \in V(G)$, $v \in V(H)$. Then:*

- (a) *If $\{u\} \times S' \subseteq S$, where $S' \subseteq V(H)$, then S' induces an independent set in H .*
- (b) *If $S'' \times \{v\} \subseteq S$, where $S'' \subseteq V(G)$, then S'' induces an independent set in G .*

Proof By the commutativity of $G \square H$, it suffices to prove the first part. Suppose that $\{u\} \times S' \subseteq S$. We show firstly that S' is either a clique or an independent set. If $|S'| \leq 2$, then there is nothing to prove, so we assume that $|S'| \geq 3$. By Lemmas 2.1 and 3.3, S' is an independent union of cliques, so if S' is not a clique or an independent set, then there are vertices $v_1, v_2, v_3 \in S'$ such that $v_1 \sim v_2$, $v_1 \approx v_3$ and $v_2 \approx v_3$ in H . Let u' be any neighbour of u in G and Q be a monophonic v_2, v_3 -path in H . Since $\pi_H(S)$ is a monophonic position set of H , it is clear that the vertex v_1 does not belong to the path Q . Then the path formed by concatenating the path (u, v_1) , (u, v_2) , (u', v_2) with the path $u'Q$ and then the edge $(u', v_3) \sim (u, v_3)$ in that order would be $\{u\} \times S'$ -bad. Thus S' is either a clique or an independent set.

If S is layered, and S' is a clique, we would immediately have $|S| \leq \omega(H)$, so we assume that S is cliquey. Suppose that there are $v_1, v'_1 \in S'$ with $v_1 \sim v'_1$ in H and let $u_2 \in \pi_G(S) \setminus \{u\}$, where $(u_2, v_2) \in S$. Let Q be a monophonic v'_1, v_2 -path. Then the path (u, v_1) , (u, v'_1) , (u_2, v'_1) followed by u_2Q is a bad path, so there is no such edge in S' . $\square$

Lemma 3.14 *Let S be a cliquey monophonic position set of $G \square H$ with $|S| > \max\{\omega(G), \omega(H)\}$, where $\pi_G(S)$ is a clique. Then $\pi_H(S)$ is an independent set.*

Proof Lemma 3.13 shows that each set S'_i is an independent set in H . It further follows from Lemma 3.10 that there cannot be an edge in H between sets S'_i and S'_j for $i \neq j$. $\square$

Lemma 3.15 *If S is a layered or cliquey monophonic position set of $G \square H$ (where $\pi_G(S)$ is a clique) and $|S| > \max\{\omega(G), \omega(H)\}$, then every vertex from $\pi_G(S)$ is simplicial.*

Proof Suppose that $u_i \in \pi_G(S)$ is not simplicial, with $w_1, w_2 \in N_G(u_i)$ and $w_1 \approx w_2$ in G . Recall that $S'_i = \pi_H(S \cap (\{u_i\} \times V(H)))$. Firstly assume that $|S'_i| \geq 3$, with $\{v_1, v_2, v_3\} \subseteq S'_i$. Let Q and Q' be monophonic v_1, v_2 - and v_2, v_3 -paths in H , respectively. By Lemmas 3.3 and 3.13, the path formed by concatenating the edge $(u_i, v_1) \sim (w_1, v_1)$, the path w_1Q , the path (w_1, v_2) , (u_i, v_2) , (w_2, v_2) , the path w_2Q' and finally the edge $(w_2, v_3) \sim (u_i, v_3)$ in that order would be bad.

Now suppose that $|S'_i| = 2$. If S is layered, then we could not have $|S| > \max\{\omega(G), \omega(H)\}$, so S must be cliquey and there is a $u_j \in \pi_G(S) \setminus \{u_i\}$. Let $v_1, v_2 \in S'_i$, $v_3 \in S'_j$, Q and Q' be monophonic v_1, v_2 - and v_2, v_3 -paths respectively and P be a shortest path from $\{w_1, w_2\}$ to u_j , which we assume without loss of generality to be a w_2, u_j -path (notice that Q has length at most two and that $w_2 = u_j$ is possible). Then the following concatenated path is bad: the edge $(u_i, v_1) \sim (w_1, v_1)$, the path w_1Q , the path (w_1, v_2) , (u_i, v_2) , (w_2, v_2) , the path w_2Q' , and finally the path P_{v_3} . Thus $|S'_i| = 1$.

Finally suppose that $|S'_i| = 1$. As $|S| > \max\{\omega(G), \omega(H)\}$, S is cliquey and there is a $u_j \in \pi_G(S) \setminus \{u_i\}$ with $|S'_j| \geq 2$, say $S'_i = \{v_1\}$ and $\{v_2, v_3\} \subseteq S'_j$. Since $|S'_j| \geq 2$, the arguments in the previous case show that u_j is simplicial. Consequently, u_j is not adjacent to at least one of w_1 and w_2 , say w_1 . Let Q and Q' be monophonic v_1, v_2 - and v_2, v_3 -paths respectively and P be a monophonic w_1, u_j -path. The path formed as the concatenation of the edge $(u_i, v_1) \sim (w_1, v_1)$ and the paths w_1Q , P_{v_2} , and u_jQ' in that order would be bad. $\square$

A vertex subset S of a graph that is simultaneously an independent set and in monophonic position is called an *independent monophonic position set*. The largest order of an independent monophonic position set is the *independent monophonic position number* of G , denoted by $\text{mp}_i(G)$. With this notation in hand, we prove the following tight bounds. Recall that for any graph $\sigma(G) = 1$ if G contains a simplicial vertex and $\sigma(G) = 0$ otherwise.

Theorem 3.16 *If G and H are connected graphs, then*

$$\max\{\omega(G), \omega(H)\} \leq \text{mp}(G \square H) \leq \max\{\omega(G), \omega(H), \sigma(G) \text{mp}_i(H), \sigma(H) \text{mp}_i(G)\}.$$

Furthermore, if neither G nor H has simplicial vertices, then

$$\text{mp}(G \square H) = \max\{\omega(G), \omega(H)\}.$$

Proof Suppose that $G \square H$ contains a monophonic position set S with $|S| > \max\{\omega(G), \omega(H)\}$. As G and H are connected, $\max\{\omega(G), \omega(H)\} \geq 2$ and so $|S| \geq 3$. Hence it follows from Corollary 3.12 that S is either cliquey or layered. As before, we assume that $\pi_G(S)$ is a clique. Then by Lemma 3.15 it is necessary that every vertex of $\pi_G(S)$ is simplicial, so that $\sigma(G) = 1$. Lemmas 3.3 and 3.14 show that $\pi_H(S)$ is an independent monophonic position set of H and hence $\text{mp}(G \square H) = |S| = |\pi_H(S)| \leq \text{mp}_i(H)$. The case that $\pi_H(S)$ is a clique is similar and would yield an upper bound of $\sigma(H) \text{mp}_i(G)$. $\square$

We conclude this subsection by a lower bound for graphs which contain leaves. To do so, for a vertex of any graph G we will write $\ell(u)$ for the number of neighbours of u that are leaves, and set $\Delta_1(G) = \max\{\ell(u) : u \in V(G)\}$.

Proposition 3.17 *If G and H are graphs with order at least three that both contain leaves, then*

$$\text{mp}(G \square H) \geq \max\{\Delta_1(G), \Delta_1(H)\}.$$

Proof Let u be a leaf of G and L a set of leaves adjacent to some vertex v of H . We show that the set $\{u\} \times L$ is in monophonic position in $G \square H$. If $|L| \leq 2$, then this is trivial. Otherwise, suppose that P is a monophonic $(u, x_1), (u, x_3)$ -path in $G \square H$ passing through (u, x_2) for some $x_1, x_2, x_3 \in L$. The vertex (u, x_2) has degree two in $G \square H$ and so one of its neighbours is (u, v) , which is also adjacent to (u, x_1) and (u, x_2) , a contradiction. $\square$

As an example, consider $K_{1,n} \square K_{1,m}$, $n \geq m \geq 2$, $n \geq 4$. Let $V(K_{1,k}) = \{0, 1, \dots, k\}$, where 0 is the central vertex. Then the set $T = [n] \times \{1\}$ is a maximum monophonic position set of $K_{1,n} \square K_{1,m}$ and hence $\text{mp}(K_{1,n} \square K_{1,m}) = n = \text{mp}_i(K_{1,n})$. This demonstrates that the upper bound in Theorem 3.16 and the lower bound in Proposition 3.17 are sharp. In addition, it follows from (Tian et al., 2023, Theorem 2.1) that $\text{gp}(K_{1,n} \square K_{1,m}) = m + n$, hence this is another example that shows that the mp-number and the gp-number on product graphs can be arbitrarily far apart.

3.4 Triangle-free graphs

In this subsection we employ the structural properties established in the earlier lemmas about the monophonic position sets in the Cartesian product of graphs to give a tighter bound on the monophonic position number of Cartesian products of triangle-free graphs. Observe that any simplicial vertex of a triangle-free graph must be a leaf.

Theorem 3.18 *If G and H are connected triangle-free graphs of order at least 3, then*

$$\text{mp}(G \square H) \leq \max\{2, \sigma(G)\Delta(H), \sigma(H)\Delta(G)\}.$$

Proof Let S be a largest monophonic position set of $G \square H$. If $|S| \geq 3$, then S is either cliquy or layered by Corollary 3.12. Suppose that S is cliquy and $\pi_G(S) = \{u_1, u_2\}$ induces a clique, that is, $u_1 \sim u_2$ in G . However, as the order of G is at least three, the vertices u_1 and u_2 cannot both be leaves, contradicting Lemma 3.15.

Hence S must be layered. Again by Lemma 3.15, we may assume that $\pi_G(S) = \{u\}$, where u is a leaf of G . Let u' be any vertex of G with $d_G(u, u') = 2$ and P be any shortest u, u' -path. As in the proof of Theorem 3.16, we conclude that $\pi_H(S)$ is an independent monophonic position set of H .

We first show that all vertices of $\pi_H(S)$ are at distance two from each other. Let $k = \max\{d_H(v, v') : v, v' \in \pi_H(S)\}$. Assume for a contradiction that $k \geq 3$ and let v_1, v_2 be a pair of vertices of $\pi_H(S)$ with $d_H(v_1, v_2) = k$. Let v_3 be any third vertex of $\pi_H(S)$ (v_3 exists, since $|\pi_H(S)| = |S| \geq 3$). Amongst all shortest v_1, v_2 -paths in H , let Q be a path that minimises $d_H(Q, v_3)$, where $d_H(Q, v_3) = \min\{d_H(x, v_3) : x \in V(Q)\}$. Let Q be the path $v_1 = x_0, x_1, \dots, x_{k-1}, x_k = v_2$.

Let x_i be the vertex with the smallest value of i on Q such that $d_H(x_i, v_3) = d_H(Q, v_3)$. Let Q^+ be the monophonic v_3, v_1 -path in H formed from the concatenation of a shortest v_3, x_i -path Q^* and the x_i, v_1 -section of Q . If $i = k$, then Q^+ would be a monophonic path containing all three of v_1, v_2, v_3 , so we have $0 \leq i \leq k - 1$.

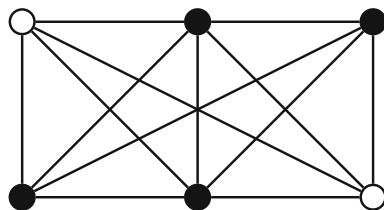
Suppose that $0 \leq i \leq k - 2$. Consider the concatenation of the four paths ${}_u Q^+, P_{v_1}, {}_{u'} Q$ and $\tilde{P}_{v_2}$ in that order. As v_2 does not lie on Q^+ , $d_G(u, u') = 2$ and $v_1 \sim v_2$ in H , this concatenated path would be a monophonic path containing (u, v_1) , (u, v_2) and (u, v_3) unless v_2 has a neighbour y on Q^* . By our assumption that $i \leq k - 2$, $y \neq x_i$. In fact, y must be the neighbour of x_i on Q^* , for otherwise Q^* would not be a shortest path from v_3 to Q . As Q is a shortest v_1, v_2 -path, it follows that $i = k - 2$ and the path formed by following the v_1, v_{k-2} -section of Q followed by the path v_{k-2}, y, v_2 would be a shortest v_1, v_2 -path closer to v_3 than Q , contradicting the definition of Q . Hence $i = k - 1$.

Now the vertex v_2 has no neighbours on Q^* other than x_{k-1} , for otherwise H would either contain a triangle or else Q^* would not be a shortest path from v_3 to Q . Therefore the v_3, v_2 -path Q^{++} formed by appending the edge $x_{k-1}v_2$ to Q^* is monophonic. By definition of Q^* and the condition that $d_H(v_1, v_2) = k \geq 3$ there is no edge from v_1 to Q^{++} and it follows that the concatenation of the four paths ${}_u Q^{++}, P_{v_2}, {}_{u'} \tilde{Q}$ and $\tilde{P}_{v_1}$ is a monophonic path containing each of (u, v_1) , (u, v_2) and (u, v_3) . Therefore our assumption that $k \geq 3$ is false and all pairs of vertices from $\pi_H(S)$ are at distance two.

Suppose now that $\text{mp}(G \square H) > \Delta(H)$. Fix two vertices $v_1, v_2 \in \pi_H(S)$ and let Q be a shortest path v_1, x, v_2 between them. As $|\pi_H(S)| > \Delta(H)$, there must be a vertex v_3 of $\pi_H(S)$ that is not adjacent to x . Let Q' be a shortest path v_2, y, v_3 . As there is no edge from v_3 to $\{v_1, x, v_2\}$, it follows that the concatenation of ${}_u Q, P_{v_2}, {}_{u'} Q'$ and $\tilde{P}_{v_3}$ in that order would be a monophonic path containing (u, v_1) , (u, v_2) and (u, v_3) , a contradiction. Therefore $\text{mp}(G \square H) \leq \Delta(H)$. The claimed inequality follows when we allow the roles of G and H to be reversed. $\square$

Again the example of the Cartesian product of two stars at the end of the previous subsection shows that the bound of Theorem 3.18 is tight.

Fig. 3 A monophonic position set of $G \circ H$ need not project to a monophonic position set in H



4 Lexicographic products

The variety of general position sets was introduced in Tian and Klavžar (2025) and has already been investigated on lexicographic products in Dokyeesun et al. (2026). In this section, we determine the monophonic position number of arbitrary lexicographic products. Before stating the main result of this section, we recall the distance function of lexicographic products (see Hammack et al. 2011, Proposition 5.12) and state two lemmas, where by $\deg_G(g)$ we denote the degree of $g \in V(G)$.

Proposition 4.1 *If (g, h) and (g', h') are distinct vertices of $G \circ H$, then*

$$d_{G \circ H}((g, h), (g', h')) = \begin{cases} d_G(g, g'); & g \neq g', \\ d_H(h, h'); & g = g', \deg_G(g) = 0, \\ \min\{d_H(h, h'), 2\}; & g = g', \deg_G(g) \neq 0. \end{cases}$$

Lemma 4.2 *If S is a monophonic position set of $G \circ H$, then $\pi_G(S)$ is a monophonic position set of G .*

Proof Suppose for a contradiction that $\pi_G(S)$ is not a monophonic position set in G . Then there exist vertices $u_1, u_2, u_3 \in \pi_G(S)$ and a monophonic u_1, u_3 -path P in G which passes through u_2 . As $u_1, u_2, u_3 \in \pi_G(S)$, there exist vertices $v_1, v_2, v_3 \in V(H)$ such that $(u_1, v_1), (u_2, v_2), (u_3, v_3) \in S$. But now it is straightforward to lift the path P to a monophonic $(u_1, v_1), (u_3, v_3)$ -path in $G \circ H$ which passes through (u_2, v_2) , a contradiction. $\square$

Let $u \in V(G)$ and let P be a monophonic path in H . Then the isomorphic copy of P in the layer ${}^u H$ of $G \circ H$ is a monophonic path of $G \circ H$. This fact implies the second lemma of this section.

Lemma 4.3 *If S is a monophonic position set of $G \circ H$, then for any $u \in \pi_G(S)$, $\pi_H({}^u H \cap S)$ is a monophonic position set of H .*

Note that $\pi_H(S)$ need not be a monophonic position set of H . As an example consider $P_2 \circ P_3$ (See Fig. 3). Let $V(P_2) = \{u_1, u_2\}$, $V(P_3) = \{v_1, v_2, v_3\}$ and let $S = \{(u_1, v_1), (u_1, v_2), (u_2, v_2), (u_2, v_3)\}$. As S is a clique, it is a monophonic position set of $P_2 \circ P_3$. But $\pi_{P_3}(S) = \{v_1, v_2, v_3\}$ is not a monophonic position set in P_3 .

In view of Lemma 2.1, in the rest of this section, for a monophonic position set M of G we denote the components of $G[M]$ by: $A_1, A_2, \dots, A_k, B_1, \dots, B_r$, where $|A_i| \geq 2$ for each $i \in [k]$ and $|B_j| = 1$ for each $j \in [r]$. Also we fix $n_M = \sum_{i=1}^k |A_i|$ and write $r_M = r$ to emphasise that r is a function of M . Then for any monophonic position set M of G we have $|M| = n_M + r_M$. Now we are ready for our main result of this section.

Theorem 4.4 *Let G be a connected graph of order at least 2 and let $\mathcal{M}$ be the collection of all monophonic position sets of G . Then*

$$\text{mp}(G \circ H) = \max_{M \in \mathcal{M}} \{n_M \cdot \omega(H) + r_M \cdot \text{mp}(H)\}.$$

Proof Let S be a monophonic position set of $G \circ H$. Set $M = \pi_G(S)$, which is a monophonic position set of G by Lemma 4.2. To show that $|S| \leq \max_{M \in \mathcal{M}} \{n_M \cdot \omega(H) + r_M \cdot \text{mp}(H)\}$ it suffices to show that if $u \in M$ lies in a component A_i of $G[M]$ of order greater than one, then $\pi_H({}^u H \cap S)$ is a clique. Suppose for a contradiction that there are $v_1, v_2 \in {}^u H \cap S$ such that $v_1 \approx v_2$ in H . Let $u' \in A_i \setminus \{u\}$ and v' be a vertex of H such that $(u', v') \in S$. Then $(u, v_1), (u', v'), (u, v_2)$ would be a bad monophonic path and the claimed bound holds.

For the converse, we show that any subset S of $V(G \circ H)$ with the properties that

- $M = \pi_G(S)$ is in monophonic position in G ,
- $\pi_H({}^u H \cap S)$ is a clique of H if u belongs to a component of $G[M]$ of order greater than one, and
- $\pi_H({}^u H \cap S)$ is a monophonic position set of H if u is an isolated vertex of $G[M]$,

then S is in monophonic position in $G \circ H$. Let S be such a subset, set $M = \pi_G(S)$ and define the sets A_i and B_j as above. Assume for a contradiction that there are vertices $(u_1, v_1), (u_2, v_2)$ and (u_3, v_3) of S and a monophonic $(u_1, v_1), (u_3, v_3)$ -path P of $G \circ H$ passing through (u_2, v_2) .

If u_1, u_2, u_3 are distinct vertices of G , then $\pi_G(P)$ would be a monophonic path of G containing u_1, u_2, u_3 , contradicting our assumption that M is in monophonic position in G . Suppose that $u_1 = u_3$. Then we must have $v_1 \approx v_3$ in H , for otherwise P would have length just one; therefore $u_1 = u_3 \in B_j$ for some $j \in [r]$. We have $d_{G \circ H}((u_1, v_1), (u_3, v_3)) = 2$. Then either $u_2 = u_1$ and v_1, v_2, v_3 is a monophonic path in H , which contradicts Lemma 4.3, or else $u_2 \sim u_1$, which is impossible since $u_1 \in B_j$.

Hence we can assume that $u_1 = u_2 \neq u_3$. Then if (u', v') is the first vertex on the $(u_2, v_2), (u_3, v_3)$ -subpath of P with $u' \neq u_2$ there is a chord $(u_1, v_1) \sim (u', v')$, so that again P is not monophonic. Thus S is in monophonic position and the monophonic position number of $G \circ H$ is the maximum value of $n_M \omega(H) + r_M \text{mp}(H)$ over all such sets. $\square$

It follows from Lemma 2.1 that any monophonic position set of a triangle-free graph G is either an independent set or consists of a pair of adjacent vertices, so if the order of the graph is $n \geq 3$ we have $\text{mp}(G) = \text{mp}_t(G)$. As $\omega(H) \leq \text{mp}(H)$ this provides an exact value for $\text{mp}(G \circ H)$ when G is triangle-free.

Corollary 4.5 *If G is a connected triangle-free graph with order $n \geq 3$ and H is a connected graph, then $\text{mp}(G \circ H) = \text{mp}(G) \cdot \text{mp}(H)$.*

Corollary 4.6 *If H is a connected graph and $n \geq 2$, then*

$$\text{mp}(K_n \circ H) = \max\{n \cdot \omega(H), \text{mp}(H)\}.$$

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Declarations

Conflict of interest The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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