Induced cycles in crossing graphs of median graphs

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Abstract

The crossing graph $G^{\#}$ of a partial cube G has the equivalence classes of the Djoković-Winkler relation Θ as vertices, two Θ-classes being adjacent if they appear on some common isometric cycle. The following question from [12, Problem 7.3] is treated: Let G be a median graph and $n \geq 4$. Does an induced cycle C_n in $G^{\#}$ necessarily force an induced cogwheel M_n in G? It is shown that the answer is positive for $n = 4, 5$ and negative for $n \geq 6$. On the other hand it is proved that if $G^{\#}$ contains an induced cycle C_n, $n \geq 4$, then G contains some induced cogwheel M_m, $4 \leq m \leq n$. A refinement of the expansion procedure for partial cubes is obtained along the way.

Key words: partial cube, median graph, crossing graph, cogwheel, (convex) expansion

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1 Problem and results

Let $G = (V, E)$ be a partial cube, that is, an isometric subgraph of some hypercube. Equivalently, partial cubes are precisely isometric subgraphs of the d-dimensional integer lattices $\mathbb{Z}^d$ for some $d \geq 1$, see [9]. The celebrated Djoković-Winkler relation Θ [8, 19] partitions E into the so-called Θ-classes. Then the crossing graph $G^\#$ of G has its Θ-classes as vertices while two Θ-classes are adjacent if they appear on some common isometric cycle.

The concept of the crossing graph was introduced (under the name incompatibility graph) by Bandelt and Dress in [2]. Bandelt and Chepoi proved in [1] that the crossing graph of a median graph G is chordal if and only if G contains no convex cogwheel. Crossings graphs were later implicitly considered in [14] and extensively studied in [12] where it was proved among others that any graph is a crossing graph of some median graph and that cogwheels M_n are the only partial cubes whose crossing graphs are cycles. Additional results on crossing graphs were obtained in [3], for instance, the crossing graph of a median graph G is the join of two graphs A and B if and only if G is a Cartesian product graph. For an extension of the concept of the crossing graph see [6] and for related concepts of the so called τ-graphs and Θ-graphs see [5, 6, 10, 11, 18].

In this paper we consider the following question from [12, Problem 7.3]. Let G be a median graph and $n \geq 4$. Does an induced cycle C_n in $G^\#$ necessarily force an induced cogwheel M_n in G? We prove:

Theorem 1 *Let G be a median graph. If $G^\#$ contains an induced cycle C_n, $n \geq 4$, then G contains some induced cogwheel M_m, $4 \leq m \leq n$.*

Note that Theorem 1 does not extend to partial cubes as can be seen from Fig. 3, see also [12, Theorem 5.4].

It need not be the case that the cogwheel guaranteed by Theorem 1 is M_n. Indeed, let $M_{r,s}$, $3 \leq r \leq s$, be the graph as shown in Fig. 1. Note that $M_{r,s}$ is a median graph and that M_{s+2} is the largest induced cogwheel in $M_{r,s}$. On the other hand, $M_{r,s}^\#$ is obtained from the disjoint union of C_{r+s} and an additional vertex that is adjacent to two vertices of the cycle at distance r. Therefore, $M_{r,s}^\#$ contains an induced C_{r+s}.

This example shows that the answer to [12, Problem 7.3] is in general negative for cycles of length at least 6. It is positive thought for 4-cycles and 5-cycles as our next result asserts.

Theorem 2 *Let G be a median graph and $n \in \{4, 5\}$. If $G^\#$ contains an induced cycle C_n then G contains an induced cogwheel M_n.*

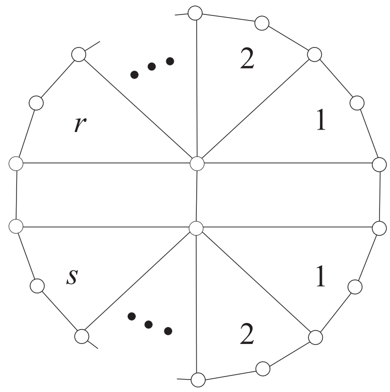

Figure 1: The graph $M_{r,s}$

Note that the converse of Theorem 2 does not hold, that is, an induced cogwheel M_n, $n \in \{4, 5\}$, in a median graph G does not necessarily force an induced cycle C_n in $G^\#$. To see this note that M_n isometrically embeds into Q_n. Then M_n is an induced subgraph of the median graph Q_n, but $Q_n^\# = K_n$ has only triangles as induced cycles. With the same argument we infer that the converse of Theorem 1 also does not hold.

For a graph X we denote with $\mathcal{E}(X)$ the class of all partial cubes for which their crossing graph is isomorphic to X:

$$\mathcal{E}(X) = \{G \mid G \text{ is partial cube with } G^\# = X\}\,.$$

While proving Theorem 1 we also obtain the following result that could be of independent interest.

Theorem 3 *Let G be a partial cube and X an induced subgraph of $G^\#$. Then G can be obtained by an expansion procedure from some member of $\mathcal{E}(X)$.*

This theorem can be considered as a refinement of Chepoi's expansion theorem which claims that any partial cube can be obtained by an expansion procedure from the one vertex graph K_1 [7].

In the next section we define the concepts used in this paper and recall some known results. In the last section we give proofs of Theorems 1-3.

2 Definitions and preliminaries

The wheel W_n, $n \geq 3$, consists of the n-cycle C_n together with an extra vertex joined to all the vertices of the cycle. The cogwheel M_n is obtained from the wheel W_n by subdividing all the edges of the outer cycle. See Fig. 2 for W_5 and M_5. Cogwheels are also known as bipartite wheels. The central vertex of M_n is the *center* of the wheel and the edges incident with the center are the *spokes* of the wheel.

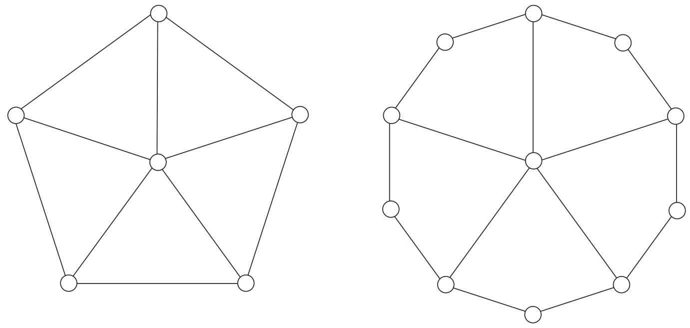

Figure 2: W_5 and M_5

Let u and v be vertices of a connected graph G. Then $d_G(u,v)$, or $d(u,v)$ for short, denotes the length of a shortest u,v-path in G. The *interval* $I(u,v)$ is the set of all vertices on shortest u,v-paths. A subgraph H of G is *isometric* if $d_H(u,v) = d_G(u,v)$ for all $u,v \in V(H)$ and *convex* if $I(u,v) \subseteq V(H)$ for any $u,v \in V(H)$.

A graph G is a *partial cube* if for some n, G is an isometric subgraph of the n-cube Q_n. A connected graph is a *median graph* if for every triple u,v,w of its vertices $|I(u,v) \cap I(u,w) \cap I(v,w)| = 1$. Median graphs are partial cubes [16]. For more information on median graphs see recent papers [4, 17] and references therein.

The Djoković-Winkler relation Θ [8, 19] is defined on the edge set of a graph G in the following way. Edges $e = xy$ and $f = uv$ of G are in relation Θ if

$$d(x,u) + d(y,v) \neq d(x,v) + d(y,u)\,.$$

Let G be a partial cube and $ab \in E(G)$. Winkler [19] proved that among bipartite graphs Θ is transitive precisely for partial cubes. Hence in this case Θ is an equivalence relation. For an edge ab of G set $W_{ab} = \{w \in V \mid d(a,w) < d(b,w)\}$. Note that if G is bipartite then $V = W_{ab} \cup W_{ba}$. If G is a partial cube then the Θ-class containing ab consists of all edges that are incident with a vertex from W_{ab} and a vertex from W_{ba}.

Let $[G]_\Theta$ stands for the set of all Θ-classes of a partial cube G. Let $E, F \in [G]_\Theta$ and $ee' \in E$, $ff' \in F$. Then E and F *cross*, $E \#_G F$, if

$$W_{ee'} \cap W_{ff'} \neq \emptyset, W_{ee'} \cap W_{f'f} \neq \emptyset, W_{e'e} \cap W_{ff'} \neq \emptyset, \text{ and } W_{e'e} \cap W_{f'f} \neq \emptyset\,.$$

In [1, 2] the so-called splits $\{W_{ee'}, W_{e'e}\}$ and $\{W_{ff'}, W_{f'f}\}$ satisfying the above condition are called *incompatible*. As noted above there is a bijective correspondence between Θ-classes and splits of a partial cube.

The crossing relation can be described in several equivalent ways. We say that Θ-classes E and F *alternate on a cycle* C if they both occur in C and we encounter them alternately while walking along C.

Lemma 4 [12] *Let G be a partial cube G and $E, F \in [G]_\Theta$. Then the following statements are equivalent:*

(i) $E \#_G F$.

(ii) *E and F alternate on an isometric cycle of G.*

(iii) *E and F occur on an isometric cycle of G.*

(iv) *Each of the Θ-classes E and F appear exactly twice on a cycle of G and they alternate.*

For a partial cube G, the *crossing graph* $G^\#$ of G has elements of $[G]_\Theta$ as vertices, two vertices E and F being adjacent if $E \#_G F$. See Fig. 3 for a partial cube G and its crossing graph. The marked Θ-classes of G induce the outer 6-cycle of $G^\#$ and the remaining three classes the inner triangle.

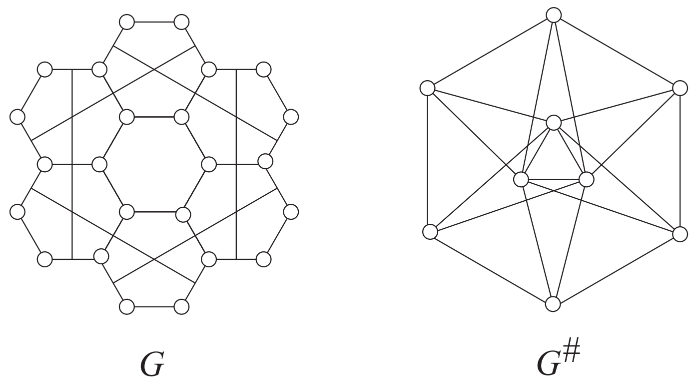

Figure 3: A partial cube and its crossing graph

An *isometric cover* G_1, G_2 of a connected graph G consists of two isometric subgraphs G_1 and G_2 of G such that $G = G_1 \cup G_2$ and $G_1 \cap G_2 \neq \emptyset$. Let $\widetilde{G}_1$ and $\widetilde{G}_2$ be isomorphic copies of G_1 and G_2, respectively. For any vertex $u \in G_i$, $1 \leq i \leq 2$, let $\widetilde{u}_i$ be the corresponding vertex in $\widetilde{G}_i$. The *expansion of G with respect to G_1, G_2* is the graph $\widetilde{G}$ obtained from the disjoint union of $\widetilde{G}_1$ and $\widetilde{G}_2$, where for any $u \in G_1 \cap G_2$ the vertices $\widetilde{u}_1$ and $\widetilde{u}_2$ are joined by an edge. A *contraction* is the reverse operation to the expansion.

The expansion is called *convex* if vertices of $G_1 \cap G_2$ induce a convex subgraph of G. Mulder [15, 16] proved that a graph is a median graph if and only if it can be obtained from K_1 by a sequence of convex expansions.

3 Proofs

Let $\widetilde{G}$ be the expansion of a partial cube G with respect to an isometric cover G_1, G_2 and let G_0 be the intersection of the cover. For an edge e of $G \setminus E(G_0)$ let $\widetilde{e}$ be the corresponding edge of $\widetilde{G}$, while for an edge e of G_0 we will denote the two corresponding edges of $\widetilde{G}$ with $\widetilde{e}_1$ and $\widetilde{e}_2$, where $\widetilde{e}_1 \in \widetilde{G}_1$ and $\widetilde{e}_2 \in \widetilde{G}_2$.

Let $E \in [G]_\Theta$. Then E expands to the Θ-class $\widetilde{E}$ of $\widetilde{G}$ with $E \subseteq \widetilde{E}$. Note that if $E \cap G_0 = \emptyset$ then $E = \widetilde{E}$. Otherwise, $|\widetilde{E}| = |E| + 2|E \cap G_0|$. More precisely, the following part of the folklore result holds.

Lemma 5 *Let $\widetilde{G}$ be the expansion of a partial cube G with respect to the cover G_1, G_2 and let $G_0 = G_1 \cap G_2$. Then*

$$\widetilde{E} = E \cup_{e \in E \cap G_0} \{\widetilde{e}_1, \widetilde{e}_2\}.$$

For our purposes we specialize [13, Lemma 6.4] to partial cubes as follows.

Lemma 6 *Let G be a partial cube, $F \in [G]_\Theta$, and C an isometric cycle of G. Then one of the following holds:*

(1) *$C \cap F = \emptyset$; or*

(2) *C meets F in two opposite edges.*

Lemma 7 *Let $\widetilde{G}$ be an expansion of a partial cube G and let $E, F \in [G]_\Theta$. Then $E \#_G F$ if and only if $\widetilde{E} \#_{\widetilde{G}} \widetilde{F}$.*

Proof. Let $\widetilde{G}$ be the expansion of G with respect to the cover G_1, G_2 and let $G_0 = G_1 \cap G_2$.

Suppose first $E \#_G F$. By Lemma 4 we may assume that E and F appear exactly twice on a cycle C of G and they alternate.

Assume C is completely contained in one of the parts of the cover, say G_1. Then C expands to the cycle $\widetilde{C}$ that is isomorphic to C and lies completely in $\widetilde{G}_1$. Then $\widetilde{E} \#_{\widetilde{G}} \widetilde{F}$ by Lemma 5.

Assume $C \cap (G_1 \setminus G_2) \neq \emptyset$ and $C \cap (G_2 \setminus G_1) \neq \emptyset$. We construct a cycle $\widetilde{C}$ of $\widetilde{G}$ as follows. Start at any vertex x of $G_1 \setminus G_2$ and follow C until a vertex y is reached such that the next vertex of C, call it z, belongs to $G_2 \setminus G_1$. Then the first part of $\widetilde{C}$ will consist of the vertices from $\widetilde{x}_1$ to $\widetilde{y}_1$ followed with $\widetilde{y}_2$ and $\widetilde{z}_2$. Now follow C from z to w, where w is the last vertex after z that belongs to G_2. Append to the before constructed beginning of $\widetilde{C}$ the vertices between $\widetilde{z}_2$ and $\widetilde{w}_2$ and add also the vertex $\widetilde{w}_1$. Proceed in the same manner until we return to x in C. In this way $\widetilde{C}$ consists of copies of all the edges from C together with some extra edges that all belong to the new Θ-class of $\widetilde{G}$. By Lemma 5, $\widetilde{C}$ contains precisely two edges from $\widetilde{E}$ and two edges from $\widetilde{F}$ and they alternate on $\widetilde{C}$. Therefore also in this case $\widetilde{E} \#_{\widetilde{G}} \widetilde{F}$.

Conversely, assume $\widetilde{E} \#_{\widetilde{G}} \widetilde{F}$. Then $\widetilde{E}$ and $\widetilde{F}$ appear on some isometric cycle C of $\widetilde{G}$ and they alternate. By Lemma 6 cycle C meets the new Θ-class of $\widetilde{G}$ in two opposite edges. When we contract $\widetilde{G}$ to G, E and F appear on cycle C', where $|C'| = |\widetilde{C}| - 2$ and they still alternate on $\widetilde{C}$. Therefore $E \#_G F$. □

Corollary 8 *Let $\widetilde{G}$ be an expansion of a partial cube G. Then $\widetilde{G}^{\#}$ is obtained from the disjoint union of $G^{\#}$ and a vertex x, where x is adjacent to some vertices of $G^{\#}$.*

We are now in a position to prove Theorem 3. Let $F_1, \ldots, F_k \in [G]_\Theta$ be the classes that correspond to the vertices of X. Contract all the Θ-classes from $[G]_\Theta \setminus \{F_1, \ldots, F_k\}$ in some order. Let H be the obtained graph. Then Corollary 8 implies that $H^{\#} = X$, that is, $H \in \mathcal{E}(X)$. Therefore, G can be obtained from $H \in \mathcal{E}(X)$ by an expansion procedure and Theorem 3 is proved.

We next prove Theorem 1 and for this sake recall the following result.

Theorem 9 [12, Theorem 5.3] *For any $n \geq 4$, $\mathcal{E}(C_n) = \{M_n\}$.*

Let $F_1, \ldots, F_n \in [G]_\Theta$ be the vertices of an induced cycle of $G^{\#}$ on at least four vertices. Contract all the other Θ-classes of G. Then Corollary 8 and Theorem 9 imply that the obtained graph is a cogwheel with at least four spokes. Consider now the sequence of expansions that reverse the performed sequence of contractions and let H be a graph obtained during this sequence. We claim that $\widetilde{H}$ contains some cogwheel provided that H does it.

Let M be a cogwheel of H. Note that M exists at the beginning of the expansion procedure by the above argument. If M is completely contained in one part of the isometric cover of H, then a copy of M is clearly contained in $\widetilde{H}$.

Assume in the rest that M is contained in $H_1 \setminus H_2$ and in $H_1 \setminus H_2$, where H_1, H_2 is the isometric cover of H. Hence M is also contained in $H_0 = H_1 \cap H_2$. Suppose the center x of M lies in $H_1 \setminus H_2$. Let u be a vertex of M from $H_2 \setminus H_1$ with $d(x, u) = 2$. Clearly such a vertex exists since otherwise M is completely contained in one part of the isometric cover of H. Let $w \in M$ be a common neighbor of x and u. Then $w \in H_0$. The other common neighbor w' of u and x must also lie in

H_0. Since $d(w, w') = 2$ this implies that H_0 is not convex which is not possible. Therefore, $x \in H_0$.

We next claim that H_0 contains at least two non-neighboring spokes of M.

Let u be a vertex of M with $d(x, u) = 2$ and $u \in H_0$. Let w and w' be the neighbors of u in M. Then it is not possible that $w \in H_1 \setminus H_2$ and $w' \in H_2 \setminus H_1$ (or vice verse), for otherwise H_0 is not convex.

Let xu_1 and xu_2 be spokes of M, where $u_1 \in H_1 \setminus H_2$ and $u_2 \in H_2 \setminus H_1$. Note that such spokes exist by the convexity of H_0 and the assumption that M intersects $H_1 \setminus H_2$ as well as $H_2 \setminus H_1$. The spokes xu_1 and xu_2 are not neighboring, for otherwise the common neighbor of u_1 and u_2 would lie in H_0 which would again violate the convexity of H_0. Following the spokes from xu_1 to xu_2 along M in one direction gives a spoke xw_1 with $w_1 \in H_0$, while following the spokes from xu_1 to xu_2 along M in the other direction gives another spoke xw_2 with $w_2 \in H_0$. Then xw_1 and xw_2 are the claimed non-neighboring spokes of H_0.

Hence H_0 contains the center x and at least two non-neighboring spokes of M. Therefore, $\widetilde{H_1}$ contains a cogwheel M_r whose spokes are the expanded spokes of M from H_1 together with a new spoke $\widetilde{x}_1\widetilde{x}_2$. Similarly, the spokes of M from H_2 together with the spoke $\widetilde{x}_1\widetilde{x}_2$ yield another cogwheel M_s in $\widetilde{H}$, more precisely in $\widetilde{H_2}$. Finally, as the spokes xu_1 and xu_2 of M are not adjacent, $4 \leq r, s \leq n$, hence Theorem 1 is proved.

It remains to verify Theorem 2. Note that by the above considerations, an M_4 from H either remains unchanged while expanding to $\widetilde{H}$ or expands to two copies of M_4 in $\widetilde{H}$. Similarly, an M_5 from H either remains unchanged while expanding to $\widetilde{H}$ or expands to one M_4 and one M_5 in $\widetilde{H}$. In any case, a cogwheel M_n, $n \in \{4, 5\}$, from H leads to at least one M_n in $\widetilde{H}$.

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