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DOMINATION GAME CRITICAL GRAPHS

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Abstract

The domination game is played on a graph G by two players who alternately take turns by choosing a vertex such that in each turn at least one previously undominated vertex is dominated. The game is over when each vertex becomes dominated. One of the players, namely Dominator, wants to finish the game as soon as possible, while the other one wants to delay the end. The number of turns when Dominator starts the game on G and both players play optimally is the graph invariant $\gamma_g(G)$, named the game domination number. Here we study the γ_g-critical graphs which are critical with respect to vertex predomination. Besides proving some general properties, we characterize γ_g-critical graphs with $\gamma_g = 2$ and with $\gamma_g = 3$, moreover for each n we identify the (infinite) class of all γ_g-critical ones among the nth powers C_N^n of cycles. Along the way we determine $\gamma_g(C_N^n)$ for all n and N. Results of a computer search for γ_g-critical trees are presented and several problems and research directions are also listed.

Keywords: domination number, domination game, domination game critical graphs, powers of cycles, trees.

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1. Introduction

Critical graphs are indispensable when investigating many central graph invariants. Several different criticality concepts were investigated with respect to the chromatic number and the chromatic index. An important concept in this respect is the one of *color-critical graphs* which are the graphs G such that $\chi(H) < \chi(G)$ holds for any proper subgraph H of G (equivalently, $\chi(G-e) < \chi(G)$ for any edge e of G), see the books [1, Section 14.2] and [24, Section 5.2], and the recent paper [22]. For the parallel concept on the chromatic index we refer to [20] and the references therein. A good source for critical graphs with respect to the independence number is the book [21]. Different criticality concepts were investigated also related to domination. A standard example is formed by γ-critical graphs [5], that is, the graphs for which $\gamma(G-v) < \gamma(G)$ holds for any vertex v of G. For distance domination-critical graphs see [15, 23], while for domination dot-critical graphs see the seminar paper [9] and the recent developments [11, 12].

The domination game [4] is played on an arbitrary graph G by *Dominator* and *Staller*, see [2, 6, 7, 8, 10, 19] for some recent developments on the game and [13, 14] for the total version of the game. The players are taking turns choosing a vertex such that at least one previously undominated vertex becomes dominated. The game ends when no move is possible. Dominator wants to finish the game as soon as possible, while Staller wants to play as long as possible. By *Game* 1 (*Game* 2) we mean a game in which Dominator (respectively, Staller) has the first move. Assuming that both players play optimally, the *game domination number* $\gamma_g(G)$ (the *Staller-start game domination number* $\gamma_g'(G)$) of a graph G, denotes the number of moves in Game 1 (respectively, Game 2). A *partially-dominated graph* is a graph together with a declaration that some vertices are already dominated, that is, they need not be dominated in the rest of the game. Note that a vertex declared to be already dominated can still be played in the course of the game provided it has an undominated neighbor. For a vertex subset S of a graph G, let $G|S$ denote the partially dominated graph in which vertices from S are already dominated. If $S = \{v\}$, then we will simplify the notation $G|\{v\}$ to $G|v$. The following result of Kinnersley, West, and Zamani is a fundamental tool for the domination game and will be used throughout the paper.

Theorem 1 (Lemma 2.1, Continuation Principle [18]). *Let G be a graph and $A, B \subseteq V(G)$. If $B \subseteq A$, then $\gamma_g(G|A) \le \gamma_g(G|B)$ and $\gamma_g'(G|A) \le \gamma_g'(G|B)$.*

In this paper we introduce critical graphs with respect to the domination game as follows. A graph G is *domination game critical* or shortly *γ_g-critical* if $\gamma_g(G) > \gamma_g(G|v)$ holds for every $v \in V(G)$. We also say that G is *k-γ_g-critical* provided that $\gamma_g(G) = k$.

One might think that in view of the standard coloring and domination critical graphs, a more natural option would be to consider vertex removed, or edge removed, or edge added graphs. However, as it turned out, the domination game is not monotone with respect to removing/adding vertices or edges. It was demonstrated in [2] that removing an edge from a graph can change its game domination number by any value from $\{-2,-1,0,1,2\}$. Moreover, removing a vertex from a graph can increase its game domination number by any value, and can also decrease it by 1 or 2. On the other hand, by the Continuation Principle, $\gamma_g(G) \geq \gamma_g(G|v)$ holds for every $v \in V(G)$. Note also that our definition of γ_g-critical graphs is parallel with the chromatic-critical graphs with respect to a vertex removal because removing a vertex just means that it need not be colored.

We proceed as follows. In the next section we prove some general results that we need in later sections. One of them establishes the relation $\gamma_g(G|v) \geq \gamma_g(G) - 2$, valid for every graph G and every $v \in V(G)$. In Section 3 we concentrate on γ_g-critical graphs. Especially, we show that if G is γ_g-critical then $\gamma_g'(G)$ equals either $\gamma_g(G)$ or $\gamma_g(G)-1$; moreover, we characterize 2-γ_g-critical and 3-γ_g-critical graphs. In Section 4, the game domination number $\gamma_g(C_N^n)$ and the Staller-start game domination number $\gamma_g'(C_N^n)$ of the powers of cycles are determined for each $n \geq 1$ and $N \geq 2n+1$. As a consequence, for every $k \geq 2$ we identify the infinite class of k-γ_g-critical graphs which are powers of cycles. In Section 5, we present all γ_g-critical trees up to order 17 (obtained by computer search) and discuss possible infinite families of γ_g-critical trees. In the last section we describe some further typical examples of γ_g-critical graphs and raise some open problems.

2. Preliminaries

A graph G *realizes* the pair (k,ℓ) if $\gamma_g(G) = k$ and $\gamma_g'(G) = \ell$. It was proved in [4, 18] that $|\gamma_g(G)-\gamma_g'(G)| \leq 1$ holds for any graph G. Hence any graph realizes either $(k,k+1)$, (k,k), or $(k,k-1)$ (for some integer k) and is consequently called a plus graph, an equal graph, or a minus graph, respectively. Moreover, we say that G is a *no-minus graph* [10], if for any $S \subseteq V(G)$, $\gamma_g(G|S) \leq \gamma_g'(G|S)$ holds (in other words, $G|S$ is not a minus).

A variation of the domination game when Dominator (respectively, Staller) is allowed, but not obligated, to skip exactly one move in the course of the game, is called the *Dominator-pass game* (respectively, *Staller-pass game*). The number of moves in such a game, where both players are playing optimally, is denoted by $\gamma_g^{dp}(G)$ (respectively, $\gamma_g^{sp}(G)$) when Dominator starts the game (unless he decides to pass already the first move) and by $\gamma_g'^{dp}(G)$ (respectively, $\gamma_g'^{sp}(G)$) when Staller starts the game. These variants of the domination game turned out to be very useful, see [3, 4, 10, 18]. For our purposes we recall the following result.

Proposition 2 (Lemma 2.2, Proposition 2.3 [10]). *If S is a subset of vertices of a graph G, then $\gamma_g^{sp}(G|S) \leq \gamma_g(G|S)+1$. Moreover, if G is a no-minus graph, then $\gamma_g^{sp}(G|S) = \gamma_g^{dp}(G|S) = \gamma_g(G|S)$.*

Now we can prove:

Theorem 3. *If u is a vertex of a graph G, then $\gamma_g(G|u) \geq \gamma_g(G) - 2$ holds. Moreover, if G is a no-minus graph, then $\gamma_g(G|u) \geq \gamma_g(G) - 1$ holds.*

Proof. We assume that Dominator plays two games at the same time. The real game is played on G, while Dominator also imagines another game being played on $G|u$. The strategy of Dominator is to consider the imagined game as a Staller-pass game and to play optimally in it. Throughout the game Dominator will ensure that every vertex that is dominated in the real game is also dominated in the imagined game. Clearly this is true at the beginning. Dominator optimally plays in the imagined game and copies his moves to the real game. Since the described property is preserved, all of his moves are legal in the real game. Every move of Staller in the real game is copied to the imagined game. If her move is not legal in the imagined game, then the only new dominated vertex in the real one must be u. In this case Dominator just skips her move in the imagined game (which is fine because Dominator is playing a Staller-pass game). From this move on, the sets of dominated vertices are the same in both games, hence all the moves until the end will be legal, and the number of moves still needed to finish the game is equal in both games.

Let p and q be the number of moves played in the real game and in the imagined game, respectively. Then, since Staller plays optimally in the real game (but Dominator might not), we have $\gamma_g(G) \leq p$. Since in the imagined game it is possible that one move from the real game was skipped, we have $p \leq q + 1$. Moreover, since Dominator is playing optimally on $G|u$ (but Staller might not), we also infer that $q \leq \gamma_g^{sp}(G|u)$. Putting these inequalities together we get

$$\gamma_g(G) \leq p \leq q + 1 \leq \gamma_g^{sp}(G|u) + 1\,,$$

that is, $\gamma_g(G) \leq \gamma_g^{sp}(G|u) + 1$. Therefore, by the first assertion of Proposition 2, $\gamma_g(G) \leq \gamma_g(G|u)+2$. For the case when G is a no-minus graph we can analogously apply the second statement of Proposition 2 to show that $\gamma_g(G) \leq \gamma_g(G|u) + 1$ holds. ∎

Vertices u and v of a graph G are *twins* in G if their closed neighborhoods are the same, $N[u] = N[v]$. In particular, twins are adjacent vertices. A graph is called *twin-free* if it contains no twins.

Lemma 4. *If u and v are twins in G, then $\gamma_g(G) = \gamma_g(G|u) = \gamma_g(G|v)$.*

Proof. Suppose that the same game is played on G and on $G|u$, that is, the same vertices are selected in both games. Then we claim that a move is legal in the game played on G if and only if the move is legal in the game played on $G|u$. Clearly, a legal move on $G|u$ is also legal on G. On the other hand, if at some point a move is legal on G but not on $G|u$, then u would be the only newly dominated vertex in the game played on G. This would mean that the vertex v has already been dominated, but this is not possible as u and v are twins. This proves the claim, from which the lemma follows immediately. ∎

3. Properties of γ_g-Critical Graphs

The concept of γ_g-critical graphs is interesting only for Game 1. Indeed, suppose that γ_g'-critical graphs are defined analogously, that is, as graphs G for which $\gamma_g'(G) > \gamma_g'(G|v)$ holds for every $v \in V(G)$. Let G be an arbitrary graph and let u be an optimal start vertex for Staller in Game 2 on G. This implies that $\gamma_g'(G) = 1 + \gamma_g(G|N[u])$. Assuming that u has at least one neighbor in G, it follows that Staller can play u on her first move in Game 2 on the graph $G|u$. While this may not be an optimal first move for Staller on $G|u$, we still get $\gamma_g'(G|u) \geq 1 + \gamma_g(G|N[u]) = \gamma_g'(G)$. It follows that there are no non-trivial γ_g'-critical graphs.

Consider next the graph G from Figure 1. It has 13 vertices and is 7-critical. What appears to be quite surprising is that $\gamma_g(G|u) = \gamma_g(G|w) = 5$ holds. Hence in the definition of the γ_g-critical graphs, the condition $\gamma_g(G) > \gamma_g(G|v)$ cannot be replaced with the condition $\gamma_g(G) = \gamma_g(G|v) + 1$. On the other hand it follows from Theorem 3 that the decrease $\gamma_g(G) - \gamma_g(G|v) = 2$ is largest possible.

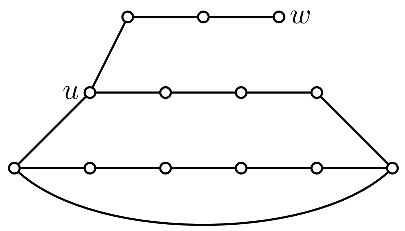

Figure 1. A critical graph on 13 vertices.

A general property of γ_g-critical graphs is that they cannot be PLUS.

Proposition 5. *If G is a γ_g-critical graph, then G is either a* MINUS *graph or an* EQUAL *graph.*

Proof. Suppose s is an optimal start vertex in Game 2 for Staller on G. Then, using the Continuation Principle,

$$\gamma_g'(G) = 1 + \gamma_g(G|N[s]) \leq 1 + \gamma_g(G|s)\,.$$

On the other hand, since G is critical, $\gamma_g(G|s) \leq \gamma_g(G) - 1$. Hence

$$\gamma_g'(G) - 1 \leq \gamma_g(G|s) \leq \gamma_g(G) - 1\,.$$

In conclusion, $\gamma_g'(G) \leq \gamma_g(G)$, that is, G is not a PLUS graph. ∎

The only 1-γ_g-critical graph is K_1. As the next result asserts, 2-γ_g-critical graphs are precisely the cocktail party graphs. Recall that the *cocktail party* graph $K_{k\times 2}$, $k \geq 1$, is the graph obtained from K_{2k} by deleting a perfect matching. In particular, $K_{1\times 2}$ is the disjoint union of two vertices and $K_{2\times 2} = C_4$.

Proposition 6. *The following conditions are equivalent for a graph G.*

(i) *G is 2-γ_g-critical,*

(ii) *$G = K_{k\times 2}$, for some $k \geq 1$,*

(iii) *$\gamma(G) = 2$ and every pair of vertices of G forms a dominating set.*

Proof. Let $|V(G)| = n$.

(i) $\Rightarrow$ (ii) Assuming that G is a 2-γ_g-critical graph, for every vertex u of G there exists a vertex $u' \neq u$ which is an optimal first choice of Dominator on $G|u$. Since $\gamma_g(G) = 2$, $\deg(u') \leq n - 2$. On the other hand, because $\gamma_g(G|u) = 1$, u' is a dominating vertex in $G - u$. Therefore, $N[u'] = V(G) \setminus \{u\}$. Since this is true for each vertex of G, we conclude that G is isomorphic to the graph obtained from K_{2k} by deleting a perfect matching for some $k \geq 1$, that is, $G = K_{k\times 2}$.

(ii) $\Rightarrow$ (iii) This implication is obvious.

(iii) $\Rightarrow$ (i) Let x be an arbitrary vertex of G. Since $\gamma(G) = 2$, $\deg(x) \leq n-2$. On the other hand, if x is adjacent to neither x' nor to x'', then $\{x', x''\}$ is not a dominating set. Therefore, $\deg(x) = n - 2$. Suppose now that G is not 2-γ_g-critical, so that there exists a vertex u such that $\gamma_g(G|u) = 2$. Then any vertex $x \neq u$ must be non-adjacent with exactly one vertex $x' \neq u$, which in turn implies that x is adjacent to u (since $\deg(x) = n-2$). Since x was arbitrary, we conclude that $\deg(u) = n - 1$ and so $\gamma(G) = 1$, the final contradiction. ∎

The equivalence between (ii) and (iii) was earlier proved in [16].

Clearly, $\Delta(G) \leq |V(G)| - 3$ holds in a 3-γ_g-critical graph G since otherwise $\gamma_g(G) \leq 2$ would hold. Now we can characterize 3-γ_g-critical graphs as follows.

Theorem 7. *Let $G = (V, E)$ be a graph of order n and with $\Delta(G) \leq n-3$. Then G is 3-γ_g-critical if and only if G is twin-free, and for any $v \in V$ there exists a vertex $u \in V$ such that $uv \notin E$ and $\deg(u) = n - 3$.*

Proof. Assume first that G is twin-free, and that for any $v \in V$ there exists a vertex $u \in V$ such that $uv \notin E$ and $\deg(u) = n - 3$. Since $\Delta(G) \leq n - 3$, after the first move of Dominator at least two vertices remain undominated. Let x and y be such vertices. Because G is twin-free, Staller can force at least two more moves to be made to finish the game. On the other hand, if Dominator selects a vertex of degree $n-3$ (which exists by our second assumption), the game will finish in the next two moves. We conclude that $\gamma_g(G) = 3$. Consider now any $v \in V$ and the game on $G|v$. Because of the second assumption, Dominator can play a vertex u such that only one vertex remains undominated. Thus, any legal move of Staller finishes the game in the second turn. Therefore, G is 3-γ_g-critical.

To prove the other direction, assume that G is 3-γ_g-critical. By Lemma 4, G must be twin-free. Take any vertex $v \in V$ and consider the domination game on $G|v$. Let an optimal first choice of Dominator be u and assume that $\deg(u) < n-3$ in G. Hence, in $G|v$ we have two different undominated vertices, say x and y, after the choice of u. Since Staller finishes the game after her first move, there exists no vertex that dominates only one of x and y, hence $N[x] = N[y]$. This contradicts Lemma 4. So after Dominator's first move in $G|v$, there exists only one undominated vertex w. Since $\gamma_g(G) > 2$ we get that v is not adjacent to u. From $\gamma_g(G|v) = 2$ we conclude that $\deg(u) = n-3$. Because u is not adjacent to v we are done. ■

The two conditions of Theorem 7 are independent. For instance, P_5 is twin-free and does not fulfill the second condition (for the middle vertex). Of course, it is then not 3-γ_g-critical by the theorem. Similarly, the graph from Figure 2 is not twin-free (u and v are twins), fulfills the second condition, and is not 3-γ_g-critical. Indeed, if Dominator plays w as the first move, then the only undominated vertices left are the twins u and v, hence Staller is forced to finish the game in the next move.

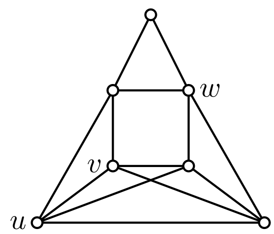

Figure 2. A graph with twins u and v.

It follows from Theorem 7 that the class of 3-γ_g-critical graphs is quite rich (in contrast with the class of 2-γ_g-critical graphs). For instance, the class of 3-γ_g-critical graphs includes complements of cycles $\overline{C}_n$, $n \geq 5$. In addition, we also get 3-γ_g-critical graphs by removing the edges of two disjoint cycles C_p and C_q from K_k, where $k \geq 6$, $p, q \geq 3$ and $p + q = k$. Moreover,

Corollary 8. *The join of any two 3-γ_g-critical graphs is a 3-γ_g-critical graph.*

Proposition 6 can be reformulated in the way parallel to Theorem 7. However, as we will see in the next section (cf. Corollary 10), Theorem 7 does not extend to k-critical graphs with $k \geq 4$. Note also that Corollary 8 does not hold for k-γ_g-critical graphs with $k \geq 4$ because if G is a join of two graphs, then $\gamma_g(G) \leq 3$.

4. Powers of Cycles

For a positive integer n, the nth power G^n of a graph G is the graph with $V(G^n) = V(G)$ and two vertices are adjacent in G^n if and only if their distance in G is at most n. In this section we consider the powers of cycles and determine their game domination number, Staller-start game domination number, and classify which are γ_g-critical. In this way we extend the result on cycles from [17] and also obtain infinite families of k-γ_g-critical graphs for any $k \geq 2$.

Theorem 9. *For every $n \geq 1$ and $N \geq 3$,*

$$\gamma_g(C_N^n) = \begin{cases} \left\lceil \frac{N}{n+1} \right\rceil; & N \bmod (2n+2) \in \{0, 1, \ldots, n+1\}, \\ \left\lceil \frac{N}{n+1} \right\rceil - 1; & N \bmod (2n+2) \in \{n+2, \ldots, 2n+1\}. \end{cases}$$

Moreover, for every $n \geq 1$ and $N \geq 2n+1$,

$$\gamma_g'(C_N^n) = \begin{cases} \left\lceil \frac{N}{n+1} \right\rceil; & N \bmod (2n+2) \in \{0\}, \\ \left\lceil \frac{N}{n+1} \right\rceil - 1; & N \bmod (2n+2) \in \{1, \ldots, n+1, 2n+1\}, \\ \left\lceil \frac{N}{n+1} \right\rceil - 2; & N \bmod (2n+2) \in \{n+2, \ldots, 2n\}. \end{cases}$$

Proof. If $N \leq 2n+1$, then C_N^n is a complete graph and hence $\gamma_g(C_N^n) = 1$ and $\gamma_g'(C_{2n+1}^n) = 1$, thus the statements clearly hold. Hence we assume $N \geq 2n+2$. After the first turn, throughout the game we always have $2n+1$ consecutive vertices of the cycle C_N, each one being dominated. Hence, Staller may choose a vertex such that only one new vertex becomes dominated. On the other hand, Dominator cannot dominate more than $2n+1$ vertices in a turn. Thus, Staller has a strategy which ensures that in any two consecutive turns at most $2(n+1)$ vertices become newly dominated.

It follows for Game 1 that if Staller finishes this game, the number of turns is at least $2\left\lceil \frac{N}{2(n+1)} \right\rceil$, while if the last choice is made by Dominator, it is at least

$2\left\lceil \frac{N-2n-1}{2(n+1)} \right\rceil + 1$. Hence, we have

$$\gamma_g(C_N^n) \geq \min\left\{2\left\lceil \frac{N}{2(n+1)} \right\rceil, 2\left\lceil \frac{N-2n-1}{2(n+1)} \right\rceil + 1\right\}.$$

From these inequalities, checking the cases due to the residues modulo $2n+2$, we obtain that the formulae given in the theorem are lower bounds on $\gamma_g(C_N^n)$.

In Game 2, the inequality

$$\gamma_g'(C_N^n) \geq \min\left\{2\left\lceil \frac{N-2n-1}{2(n+1)} \right\rceil + 1,\ 2\left\lceil \frac{N-4n-2}{2(n+1)} \right\rceil + 2\right\}$$

must hold. Checking the cases due to the residues, we obtain that the formulae given for $\gamma_g'(C_N^n)$ in the theorem are lower bounds.

For the upper bounds consider the following strategy of Dominator, where a *run* means a non-extendable set of consecutive dominated vertices which induce a path on C_N.

- In the first turn (if it is his turn) he is free to choose any vertex.
- In his later turns, he prefers to extend a run by exactly $2n+1$ vertices; if it is not possible, he selects a vertex which dominates all the vertices between the ends of two runs.

Extending an idea from [17] used there to determine the game domination number of cycles, we consider function

$$P(m) = u + (n+1)m + 2nr,$$

where u is the number of undominated vertices and r is the number of runs after the mth turn of the game.

Now, we consider the Dominator-start game and prove the following relations by induction on m:

$$P(m) \leq \begin{cases} N+n; & m \text{ is odd,} \\ N+2n; & m \text{ is even.} \end{cases}$$

After the first turn we have $P(1) = N - (2n+1) + (n+1) + 2n = N + n$.

- If in the mth turn, for an $m \geq 3$ odd, Dominator extends a run with $2n+1$ vertices, then

  $$P(m) \leq P(m-1) - (2n+1) + (n+1) = P(m-1) - n \leq N + n$$

  holds. In the other case, when Dominator decreases the number of runs, the induction hypothesis implies again

  $$P(m) \leq P(m-1) - 1 + (n+1) - 2n = P(m-1) - n \leq N + n.$$

- If m is even, then in the mth turn Staller either does not increase the number of runs and we have

$$P(m) \leq P(m-1) - 1 + (n+1) = P(m-1) + n \leq N + 2n,$$

or a new run arises and then exactly $2n+1$ vertices become dominated. In the latter case,

$$P(m) \leq P(m-1) - (2n+1) + (n+1) + 2n = P(m-1) + n \leq N + 2n.$$

Further, we note that if Staller finishes the game in the mth turn, then the number of runs necessarily decreases by at least 1, hence

$$P(m) \leq P(m-1) - 1 + (n+1) - 2n = P(m-1) - n \leq N$$

must be true for the number m of turns if m is even.

Consequently, if the game finishes in the mth turn, we have

$$P(m) = m(n+1) \leq \begin{cases} N+n; & m \text{ is odd,} \\ N; & m \text{ is even.} \end{cases}$$

Let us write N as $N = 2s(n+1)+x$, where $s = \left\lfloor \frac{N}{2(n+1)} \right\rfloor$ and $0 \leq x \leq 2n+1$ is the residue of N modulo $(2n+2)$. Checking the conditions proved for the value $P(m)$, we obtain the following inequalities.

- If $x = 0$, the condition $(2s+1)(n+1) \leq N+n$ is not true, hence $\gamma_g(C_N^n) \leq 2s = \left\lceil \frac{N}{n+1} \right\rceil$.
- If $x \geq 1$ then $(2s+2)(n+1) \leq N$ cannot hold, and $\gamma_g(C_N^n) \leq 2s+1$ is concluded. Then, for $1 \leq x \leq n+1$ we have $\gamma_g(C_N^n) \leq \left\lceil \frac{N}{n+1} \right\rceil$, while for $n+2 \leq x \leq 2n+1$, $\gamma_g(C_N^n) \leq \left\lceil \frac{N}{n+1} \right\rceil - 1$ is obtained.

These bounds together prove our formulae stated for $\gamma_g(C_N^n)$.

Similarly, in any Staller-start game $P(1) = N + n$ holds, and if Dominator follows the strategy described above then

$$P(m) \leq \begin{cases} N+n; & m \text{ is odd,} \\ N; & m \text{ is even} \end{cases}$$

is valid for every m. In addition, if Staller finishes the game with the mth turn, $P(m) \leq N - n$ must be fulfilled. Again, we may consider the form $N = 2s(n+1) + x$ and check the different cases. This yields that the formulae given for $\gamma_g'(C_N^n)$ in the theorem are upper bounds.

These facts together prove our statements. ∎

The game domination number $\gamma_g(C_N^n|v)$, where v is any vertex of C_N^n, can be determined in a similar way. We note that in this case

$$P(m) = m(n+1) \leq \begin{cases} N+n-1; & m \text{ is odd,} \\ N-1; & m \text{ is even} \end{cases}$$

must be true, where $P(m) = u + (n+1)m + 2nr$ and m is the length of the game on $C_N^n|v$. Then, we obtain

$$\gamma_g(C_N^n|v) = \begin{cases} \left\lceil \frac{N-1}{n+1} \right\rceil; & N \bmod (2n+2) \in \{1, \ldots, n+2\}, \\ \\ \left\lceil \frac{N-1}{n+1} \right\rceil - 1; & N \bmod (2n+2) \in \{0, n+3, \ldots, 2n+1\}. \end{cases}$$

Comparing it with the result of Theorem 9, we immediately get that $\gamma_g(C_N^n|v) < \gamma_g(C_N^n)$ holds if and only if $N \equiv 0$ or $1 \bmod (2n+2)$.

Corollary 10. *For every $n \geq 1$ and $k \geq 1$, the graph $C^n_{2(n+1)k}$ is $(2k)$-γ_g-critical, and the graph $C^n_{2(n+1)k+1}$ is $(2k+1)$-γ_g-critical. Further, if $2 \leq x \leq 2k+1$ then $C^n_{2(n+1)k+x}$ is not γ_g-critical.*

Note that Corollary 10 in particular asserts that $C^n_{2n+2} = K_{(n+1)\times 2}$ are 2-γ_g-critical graphs and that $C^n_{2n+3} = \overline{C}_{2n+3}$ are 3-γ_g-critical. Moreover, by Theorem 9 the graphs $C^n_{2(n+1)k}$ are EQUAL graphs (and $(2k)$-γ_g-critical), while the graphs $C^n_{2(n+1)k+1}$ are MINUS (and $(2k+1)$-γ_g-critical).

We also emphasize the following consequence:

Corollary 11. *For every ℓ and $k \geq 2$, there exist infinitely many k-γ_g-critical graphs of order greater than ℓ.*

5. On γ_g-Critical Trees

In this section we present γ_g-critical trees that were found by computer. The computational results indicate that the appearance of such trees is somehow random. In Figure 3 all γ_g-critical trees up to 17 vertices are shown. There are no γ_g-critical trees on up to 12 vertices, two γ_g-critical trees on 13 vertices, no such trees on 14 and 15 vertices, another (and only) one on 16 vertices, and ten on 17 vertices.

Let $T_{p,q,r}$ be the graph obtained from disjoint paths P_{4p+1}, P_{4q+1}, and P_{4r+1} by identifying three end-vertices, one from each of the paths. Hence $|V(T_{p,q,r})| = 4(p+q+r)+1$. Note that $T_{1,1,1}$ is one of the two γ_g-critical trees on 13 vertices and that $T_{1,1,2}$ also appears in Figure 3. Moreover, we have verified by computer that

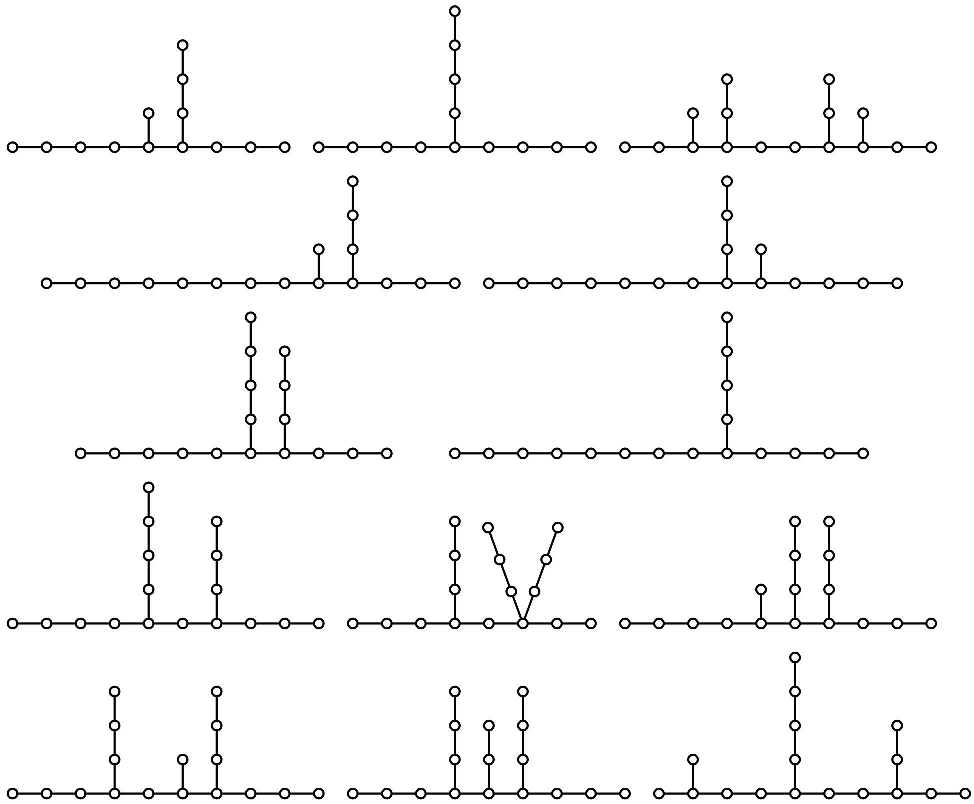

Figure 3. Critical trees on 13, 16, and 17 vertices.

$T_{p,q,r}$ is $2((p+q+r)+1)$-critical for $1 \leq p, q, r \leq 3$. These computations naturally lead to a conjecture that $T_{p,q,r}$ is a γ_g-critical tree for any $p, q, r \geq 1$. Using the existing tools, a possible proof of this conjecture would be a technical, lengthy case analysis, hence new techniques to prove γ_g-criticality would be welcome.

One of the things we would need to determine in order to prove that the trees $T_{p,q,r}$ are γ_g-critical, is $\gamma_g(T_{p,q,r})$. Note that $T_{p,q,r}$, $p, q \geq 1$, $r \geq 2$, is obtained by attaching P_4 to a leaf of $T_{p,q,r-1}$ corresponding to the parameter $r-1$. Because of that it would be nice if it would hold in general that if T' is obtained from a tree T by attaching P_4 to one of its leaves, then $\gamma_g(T') = \gamma_g(T) + 2$. However, this is not true in general. There is no such example on at most 8 vertices, but there is a unique such tree T on 9 vertices shown in Figure 4. First, $\gamma_g(T) = 5$. On the other hand, if T' is the tree obtained from T by attaching a P_4 at x, then $\gamma_g(T') = 6$. Moreover, if T'' is a tree obtained from T by attaching a P_4 at y, then $\gamma_g(T'') = 7$. The same holds also when a P_4 is attached at z.

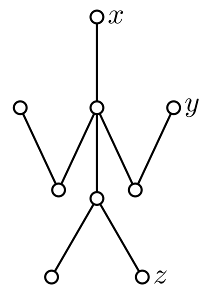

Figure 4. Tree T.

6. Concluding Remarks and Open Problems

Using computer we have found more sporadic examples of γ_g-critical graphs. The broken ladder BL_k, $k \geq 0$, is the graph obtained from the Cartesian product $P_2 \,\square\, P_4$ by amalgamating an edge of the cycle C_{4k+2} with an edge of $P_2 \,\square\, P_4$ whose end-points are of degree 2, see [19]. In particular, $BL_0 = P_2 \,\square\, P_4$. It was verified by computer that BL_k is $(2k + 4)$-critical for $0 \leq k \leq 5$. Similarly, let H_k, $k \geq 0$, be the graph obtained from $P_2 \,\square\, P_4$ by amalgamating an edge of the cycle C_{4k+2} with a middle P_4-layer edge of $P_2 \,\square\, P_4$. It was verified for $k \leq 2$ that H_k is γ_g-critical.

If G is a vertex-transitive graph, then we only need to compare $\gamma_g(G)$ with $\gamma_g(G|v)$ for some vertex v of G. That is, a vertex-transitive graph G is either γ_g-critical or $\gamma_g(G) = \gamma(G|v)$ holds for any vertex v of G.

Every graph contains a color-critical subgraph with the same chromatic number. Naturally extending the concept of γ_g-criticality to all partially dominated graphs, the same conclusion holds also for this concept. Indeed, let G be an arbitrary graph. If it is not γ_g-critical, then it contains a vertex u such that $\gamma_g(G|u) = \gamma_g(G)$. Now, if $G|u$ is γ_g-critical (in the sense of the extended definition), we are done, otherwise we repeat the procedure until finally a γ_g-critical partially dominated graph $G|S$ with $\gamma_g(G|S) = \gamma_g(G)$ is found, where $S \subseteq V(G)$. Hence we pose:

Problem 12. Investigate γ_g-criticality of partially dominated graphs. In particular, compare it with the concept studied in this paper.

The graphs that are in a way complementary to the γ_g-critical graphs are the graphs G such that $\gamma_g(G) = \gamma_g(G|v)$ holds for every $v \in V(G)$. For instance, among the powers of cycles, such graphs are precisely those that are not critical. Thus:

Problem 13. Study the graphs G for which $\gamma_g(G) = \gamma_g(G|v)$ holds for every $v \in V(G)$. In particular, establish their connections with the γ_g-critical graphs.

Finally, in view of [10], we also pose:

Problem 14. Consider the behavior of the γ_g-criticality on the disjoint union of graphs. In particular, if G is a γ_g-critical graph, when is $G \cup K_1$ γ_g-critical?

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References

[1] J.A. Bondy and U.S.R. Murty, Graph Theory (Springer, New York, 2008). doi:10.1007/978-1-84628-970-5

[2] B. Brešar, P. Dorbec, S. Klavžar and G. Košmrlj, *Domination game: Effect of edge- and vertex-removal*, Discrete Math. **330** (2014) 1–10. doi:10.1016/j.disc.2014.04.015

[3] B. Brešar, S. Klavžar, G. Košmrlj and D.F. Rall, *Domination game: extremal families of graphs for the* 3/5*-conjectures*, Discrete Appl. Math. **16** (2013) 1308–1316. doi:10.1016/j.dam.2013.01.025

[4] B. Brešar, S. Klavžar and D.F. Rall, *Domination game and an imagination strategy*, SIAM J. Discrete Math. **24** (2010) 979–991. doi:10.1137/100786800

[5] R.C. Brigham, P.Z. Chinn and R.D. Dutton, *Vertex domination-critical graphs*, Networks **1** (1988) 173–179. doi:10.1002/net.3230180304

[6] Cs. Bujtás, *Domination game on trees without leaves at distance four*, in: Proceedings of the 8th Japanese-Hungarian Symposium on Discrete Mathematics and Its Applications, A. Frank, A. Recski, G. Wiener, Eds., June 4–7 (2013) Veszprém, Hungary, 73–78.

[7] Cs. Bujtás, *Domination game on forests*, Discrete Math. **338** (2015) 2220–2228. doi:10.1016/j.disc.2015.05.022

[8] Cs. Bujtás, *On the game domination number of graphs with given minimum degree*, Electron. J. Combin. **22** (2015) #P3.29.

[9] T. Burton and D.P. Summer, *Domination dot-critical graphs*, Discrete Math. **306** (2006) 11–18. doi:10.1016/j.disc.2005.06.029

[10] P. Dorbec, G. Košmrlj and G. Renault, *The domination game played on unions of graphs*, Discrete Math. **338** (2015) 71–79.
doi:10.1016/j.disc.2014.08.024

[11] M. Furuya, *Upper bounds on the diameter of domination dot-critical graphs with given connectivity*, Discrete Appl. Math. **161** (2013) 2420–2426.
doi:10.1016/j.dam.2013.05.011

[12] M. Furuya, *The connectivity of domination dot-critical graphs with no critical vertices*, Discuss. Math. Graph Theory **34** (2014) 683–690.
doi:10.7151/dmgt.1752

[13] M.A. Henning, S. Klavžar and D.F. Rall, *Total version of the domination game*, Graphs Combin. **31** (2015) 1453–1462.
doi:10.1007/s00373-014-1470-9

[14] M.A. Henning, S. Klavžar and D.F. Rall, *The* 4/5 *upper bound on the game total domination number*, Combinatorica, to appear.

[15] M.A. Henning, O.R. Oellermann and H.C. Swart, *Distance domination critical graphs*, J. Combin. Math. Combin. Comput. **44** (2003) 33–45.

[16] S.R. Jayaram, *Minimal dominating sets of cardinality two in a graph*, Indian J. Pure Appl. Math. **28** (1997) 43–46.

[17] W.B. Kinnersley, D.B. West and R. Zamani, *Game domination for grid-like graphs*, manuscript, 2012.

[18] W.B. Kinnersley, D.B. West and R. Zamani, *Extremal problems for game domination number*, SIAM J. Discrete Math. **27** (2013) 2090–2107.
doi:10.1137/120884742

[19] G. Košmrlj, *Realizations of the game domination number*, J. Comb. Optim. **28** (2014) 447–461.
doi:10.1007/s10878-012-9572-x

[20] X. Li and B. Wei, *Lower bounds on the number of edges in edge-chromatic-critical graphs with fixed maximum degrees*, Discrete Math. **334** (2014) 1–12.
doi:10.1016/j.disc.2014.06.017

[21] L. Lovász and M.D. Plummer, Matching Theory (AMS Chelsea Publishing, Providence, RI, 2009).

[22] W. Pegden, *Critical graphs without triangles: an optimum density construction*, Combinatorica **33** (2013) 495–513.
doi:10.1007/s00493-013-2440-1

[23] F. Tian and FJ.-M. Xu, *Distance domination-critical graphs*, Appl. Math. Lett. **21** (2008) 416–420.
doi:10.1016/j.aml.2007.05.013

[24] D.B. West, Introduction to Graph Theory (Prentice Hall, Inc., Upper Saddle River, NJ, 1996).

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