

**TREES WITH DISTINGUISHING INDEX EQUAL
DISTINGUISHING NUMBER PLUS ONE**

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Abstract

The distinguishing number (index) $D(G)$ ($D'(G)$) of a graph G is the least integer d such that G has an vertex (edge) labeling with d labels that is preserved only by the trivial automorphism. It is known that for every graph G we have $D'(G) \leq D(G) + 1$. In this note we characterize finite trees for which this inequality is sharp. We also show that if G is a connected unicyclic graph, then $D'(G) = D(G)$.

Keywords: automorphism group, distinguishing index, distinguishing number, tree, unicyclic graph.

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1. INTRODUCTION

Let $G = (V(G), E(G))$ be a graph and let $\text{Aut}(G)$ be its automorphism group. A labeling $\phi : V(G) \rightarrow [r]$ is *distinguishing* if no non-trivial element of $\text{Aut}(G)$ preserves all the labels; such a labeling ϕ is a *distinguishing r -labeling*. More formally, ϕ is a distinguishing labeling if for every $\alpha \in \text{Aut}(G)$, $\alpha \neq \text{id}$, there exists $x \in V(G)$ such that $\phi(x) \neq \phi(\alpha(x))$. The *distinguishing number* $D(G)$ of a graph G is the smallest r such that G admits a distinguishing r -labeling.

The introduction of the distinguishing number in 1996 by Albertson and Collins [1] was a great success, by now about one hundred papers were written motivated by this seminal paper. The core of the research has been done on the invariant D itself, either on finite [6, 11, 15] or infinite graphs [9, 17, 21]; see also the references therein. Extensions to group theory (cf. [14, 23]) and arbitrary relational structures [16] were also investigated, as well as variations of the concept such as the distinguishing chromatic number [5, 8]. Moreover, very recently the game distinguishing number was introduced in [10]. It is hence a bit surprising that the following variation of the distinguishing number—its edge version—was introduced only in 2015 by Kalinowski and Pilśniak [13]. The *distinguishing index* $D'(G)$ of a graph G is the smallest integer d such that G has an edge labeling with d labels that is preserved only by the trivial automorphism.

Generally $D'(G)$ can be arbitrary smaller than $D(G)$, for instance if $p \geq 6$, then $D'(K_p) = 2$ and $D(K_p) = p$. Conversely, there is an upper bound on $D'(G)$ in terms of $D(G)$. In [13, Theorem 11] (see also [18, Theorem 8] for an alternative proof) it is proved that if G is a connected graph of order at least 3, then

$$(1) \quad D'(G) \leq D(G) + 1.$$

In this paper we give a characterisation of the finite trees which achieve equality. We further show that if G is a connected unicyclic graph, then $D'(G) = D(G)$, showing that the inequality is never sharp for unicyclic graphs.

2. PRELIMINARIES

Graphs considered in this note will be simple and connected. With the exception of Section 5, they will also be finite. For a positive integer k we will use the notation $[k] = \{1, \dots, k\}$.

A tree T is *unicentric* if its center (that is, the subgraph induced by the vertices of minimum eccentricity) consists of a single vertex and is *bicentric* otherwise. In the latter case the center is isomorphic to K_2 and will also be identified with its edge.

If T is a bicentric tree with central edge $e = vw$, we denote by T_v and T_w the components of $T - e$, where $v \in T_v$ and $w \in T_w$.

We will treat T_v and T_w as rooted trees with roots v and w respectively. Hence we make the following (obvious) definitions for rooted trees. An *automorphism of a rooted tree* is an automorphism of the underlying unrooted tree which fixes the root. Analogously, an isomorphism of rooted trees is an isomorphism which maps the root of one tree to the root of the other. An edge or vertex labeling of a rooted tree is called *distinguishing*, if the only automorphism (of the rooted tree) which preserves it is the identity.

For rooted trees there is also a natural correspondence between vertex and edge labelings. Let T be a rooted tree with root v . Let $f : V(T) \rightarrow [k]$ be a vertex labeling of T . Define $f'_v : E(T) \rightarrow [k]$ as follows. If $e = xy \in E(T)$, where $d_T(y, v) < d_T(x, v)$, then set $f'_v(e) = f(x)$. Here d_T denotes the standard shortest-path distance function in the tree T . Since each non-root vertex of T has a unique predecessor in T , the labeling f'_v is well-defined. It is also not hard to see, that this procedure is reversible (up to the colour of the root). We will call f'_v the (*edge*) *co-labeling* of f with respect to the root v . The following observation will be useful.

Observation 2.1. *A labeling f of a rooted tree is distinguishing if and only if the co-labeling f'_v is distinguishing. In particular (since we can reverse the construction and the colour of the root plays no role) the distinguishing index and the distinguishing number of rooted trees are always equal.*

To conclude the section we describe the class of trees $\mathcal{B}(h, d)$ from [13] which forms a key motivation for this note. Its precise definition is lengthy, hence we define it here a bit briefly; for additional details see [13].

Let $T_{h,d}$ be a unicentric tree with the central vertex v_0 in which all leaves are at distance h from v_0 and all non-leaves are of degree d . Similarly, let $T'_{h,d}$ be a unicentric tree with the central vertex v_0 which is of degree $d - 1$, all leaves are at distance h from v_0 , and all the other vertices have degree d . Let now $h \geq 1$ and $d \geq 2$. Then in $T'_{h,d}$ select ℓ of its levels h_1, \dots, h_ℓ , where $0 \leq h_i \leq h - 2$, $i \in [\ell]$. Select ℓ trees $T'_{k_1,d}, \dots, T'_{k_\ell,d}$, where $1 \leq k_i \leq h - h_i - 1$, and construct the tree $T'_{h,d} \left([T'_{k_1,d}]_{h_1}, \dots, [T'_{k_\ell,d}]_{h_\ell} \right)$ by attaching $T'_{k_i,d}$, $i \in [\ell]$, to every vertex of the h_i -th level of $T'_{h,d}$. Iteratively, we can repeat this operation for any tree attached in the previous stages. Let T_0 be the constructed tree. The construction is completed by taking two copies of T_0 and joining their central vertices by an edge. In this way a tree from $\mathcal{B}(h, d)$ is obtained.

3. EXTREMAL TREES

A characterisation of finite trees T for which $D'(G) = D(G) + 1$ was suggested in [13, Theorem 9]. While equality holds for every tree in the class $\mathcal{B}(h, d)$,

there are further trees for which the inequality is sharp as the following example demonstrates. Let T be the tree which consists of a central edge with four paths of length 2 attached to each endpoint. The tree T together with a distinguishing 2-labeling demonstrating that $D(T) = 2$ is shown in Figure 1. On the other hand, it is easy to verify that $D'(T) = 3$. But T does not belong to the set $\mathcal{B}(h, d)$ which was claimed to contain all trees with $D' = D + 1$.

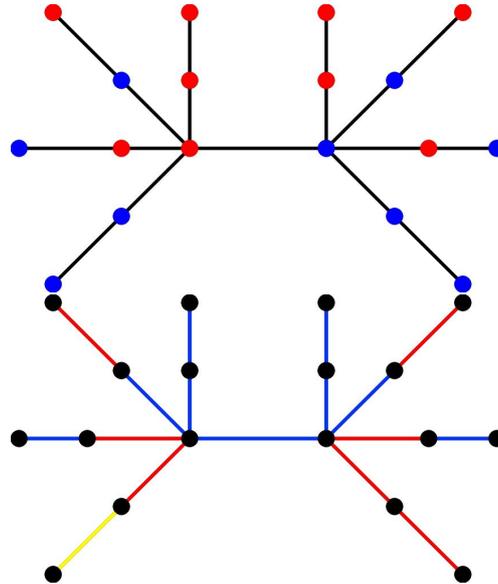


Figure 1. A tree T with $D(T) = 2$, $D'(T) = 3$ and $T \notin \mathcal{B}(h, d)$.

In this section, we give a complete characterisation of finite trees with $D' = D + 1$, thus correcting the flaw in [13, Theorem 9]. Define a family \mathcal{T} as follows. It consists of those trees T of order at least 3, for which the following conditions are fulfilled.

1. T is a bicentric tree with the central edge $e = vw$.
2. There is an isomorphism between the rooted trees T_v and T_w .
3. There is a unique distinguishing edge-labeling of the rooted tree T_v using $D(T)$ labels.

The following theorem now states that the family \mathcal{T} contains all finite trees with $D'(T) = D(T) + 1$.

Theorem 3.1. *Let T be a finite tree of order at least 3. Then*

$$D'(T) = \begin{cases} D(T) + 1 & \text{if } T \in \mathcal{T}, \\ D(T) & \text{otherwise.} \end{cases}$$

Before we prove this theorem, we state and prove a couple of auxiliary results. Both of them are essentially contained in the proof of [13, Theorem 9].

Lemma 3.2. *If T is a unicyclic tree, then $D'(T) = D(T)$.*

Proof. This follows from Observation 2.1 by noting that every automorphism of a unicyclic tree must fix the central vertex (and thus can be seen as an automorphism of a rooted tree). ■

Lemma 3.3. *If $T \neq K_2$ is a bicentric tree, then $D(T) \leq D'(T) \leq D(T) + 1$.*

Proof. Throughout the proof let $e = vw$ be the central edge of T . Note that every automorphism of T either fixes both v and w , or swaps them.

For the first inequality let f' be a distinguishing edge labeling and pick a vertex labeling f such that f' is the co-labeling of f on T_v and T_w . Observation 2.1 makes sure that no automorphism which fixes both v and w preserves the labeling f . If there is an automorphism which swaps v and w and preserves the labeling f , then this automorphism would also preserve the labeling f' .

For the second inequality start with a distinguishing vertex labeling f and label the edges of T_u and T_v by the corresponding co-labelings respectively. Label the central edge e arbitrarily. This ensures by Observation 2.1 that no automorphism which fixes both v and w preserves the resulting edge-labeling. To ensure that the same is true for every automorphism which swaps v and w , relabel one of the edges in T_v using an additional label $D(T) + 1$. ■

As we already said in the introduction, the right-hand side inequality in Lemma 3.3 actually holds for all connected graphs.

Proof of Theorem 3.1. Observe first that the result is clearly true if T is an asymmetric tree, because in this case $D(T) = D'(T) = 1$ and $T \notin \mathcal{T}$. If T is a unicyclic tree, then $D'(T) = D(T)$ holds by Lemma 3.2. Hence assume in the rest of the proof that T is a bicentric tree with the central edge $e = vw$ and $D(T) \geq 2$. By Lemma 3.3 we get $D(T) \leq D'(T) \leq D(T) + 1$. Let T_v and T_w be the components of $T - vw$ with $v \in T_v$ and $w \in T_w$.

Suppose first that T_v and T_w are not isomorphic. Then for any $\alpha \in \text{Aut}(T)$ we have $\alpha(v) = v$ and $\alpha(w) = w$. Let $f : V(T) \rightarrow [D(T)]$ be a distinguishing vertex labeling. Let $f' : E(T) \rightarrow [D(T)]$ be defined as follows. Set $f'(vw) = 1$, and on T_v and T_w let f' coincide with the co-labelings of $f|_{T_v}$ and $f|_{T_w}$, respectively. Then f' is a distinguishing edge labeling and hence $D'(T) \leq D(T)$ and equality holds by Lemma 3.3.

Suppose next that T_v and T_w are isomorphic and that T_v admits two non-isomorphic distinguishing edge labelings (as a rooted tree) with $D(T)$ labels, say g' and g'' . Let now $f' : E(T) \rightarrow [D(T)]$ be defined as follows. Set $f'(vw) = 1$,

and on T_v and T_w let f' coincide with g' and g'' , respectively. Then f' is a distinguishing edge labeling and hence again $D'(T) = D(T)$.

Until now we have proved that $D'(T) = D(T)$ unless $T \in \mathcal{T}$. To complete the proof we need to show that if $T \in \mathcal{T}$, then $D'(T) = D(T) + 1$. Suppose on the contrary that this is not the case. So let $T \in \mathcal{T}$ be such that $D'(T) = D(T)$ and let $f' : E(T) \rightarrow [D(T)]$ be a distinguishing edge labeling of T . Thus the restrictions of f' to T_v and T_w , respectively, are distinguishing edge labelings of $D(T)$ labels. Since T_v and consequently its isomorphic copy T_w admit unique v -distinguishing edge $D(T)$ -labelings, there exists $\alpha \in \text{Aut}(T)$ that exchanges T_v with T_w and preserves f , a contradiction. We conclude that $D'(T) = D(T) + 1$. ■

The example from Figure 1 can be generalized as follows. Let $k \geq 2$ and let T_k be a bicentric tree in which at each of the endvertices of the central edge precisely t^k paths of length t are attached. (Note that T_2 is the tree from Figure 1.) Then it is straightforward to verify that $T_k \in \mathcal{T}$.

4. UNICYCLIC GRAPHS

In this section we prove that among the unicyclic graphs the upper bound (1) is never attained.

Theorem 4.1. *If G is a connected unicyclic graph, then $D'(G) = D(G)$.*

Proof. Let $C = v_1v_2 \cdots v_tv_1$ be the cycle of G , where $3 \leq t \leq n$. Let T_i , $i \in [t]$, be the maximal subgraph of G that contains v_i and no other vertex of C . Then T_i is a tree, consider it as a rooted tree with the root v_i . It is possible that T_i is a single vertex graph. If $G = C_t$, then the result holds because $D' = D$ holds for all cycles, see [13, Proposition 5]. The result also clearly holds if $\text{Aut}(G)$ is trivial, hence assume in the rest of the proof that $D(G) \geq 2$ and $D'(G) \geq 2$.

We first show that $D(G) \leq D'(G)$. For this purpose, let f' be a distinguishing edge labeling of G and define a vertex labeling f as follows. On $V(T_i) \setminus \{v_i\}$, let f be such that f' is the co-labeling of f restricted to T_i . Then, by Observation 2.1, f is a distinguishing labeling of $V(T_i) \setminus \{v_i\}$ provided that v_i is fixed. If $t \geq 6$, then let $f|C$ be a distinguishing 2-labeling of C . If $3 \leq t \leq 5$ and $f'|C$ uses at least three labels, then let $f|C$ be a distinguishing 3-labeling of C . In the last case we have $3 \leq t \leq 5$ and $f'|C$ uses two labels. Then label the vertices of C with two colors as shown in Figure 2 for all possible edge labelings of C with two colors. In all the cases one can verify that if an automorphism α of C preserves $f|C$, then α also preserves $f'|C$. Since f' is distinguishing we conclude that f is also distinguishing, and consequently $D(G) \leq D'(G)$.

To show that also $D'(G) \leq D(G)$ holds, we proceed similarly as above. Let f be a distinguishing vertex labeling of G and define an edge labeling f' as follows.

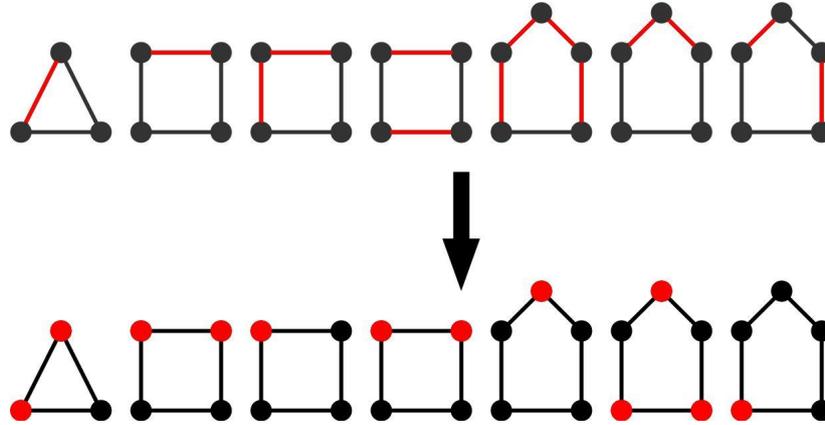


Figure 2. All non-equivalent edge 2-labelings of C and their respective transformations to vertex 2-labelings of C .

On each T_i let f' be the co-labeling of $f|T_i$ (with respect to the root v_i). If $t \geq 6$, then set $f'|C$ to be a distinguishing edge 2-labeling of C . If $3 \leq t \leq 5$ and $f|C$ uses at least three labels, then let $f'|C$ be a distinguishing edge 3-labeling of C . Finally, if $3 \leq t \leq 5$ and $f|C$ uses two labels, then let $f'|C$ be as shown in Figure 3 for all possible vertex labelings of C , respectively.

Again we can verify that if an automorphism α of C preserves $f'|C$, then α also preserves $f|C$. So f' is distinguishing and we conclude that $D'(G) \leq D(G)$. ■

5. CONCLUDING REMARKS

In our main result, Theorem 3.1, we have characterized the finite trees T for which $D'(T) = D(T) + 1$ holds. The three conditions that define the corresponding class \mathcal{T} are conceptually simple. Nevertheless, it would be interesting to have a structural description of the class \mathcal{T} . Hence we pose the following question.

Problem 5.1. Find a constructive characterization of the class of trees \mathcal{T} .

The second question implicit in this paper is, whether Theorem 4.1 can be extended to graphs with multiple cycles. We note that a proof similar to ours also works for graphs with exactly two cycles: they have to be edge disjoint, whence every automorphism either fixes both of them or swaps them and hence there is a vertex or an edge that must be fixed by every automorphism. If there are exactly three cycles, then either they are edge disjoint or there are two cycles sharing an edge and the third is the symmetric difference of the two, and again there is an edge or a vertex fixed by every automorphism.

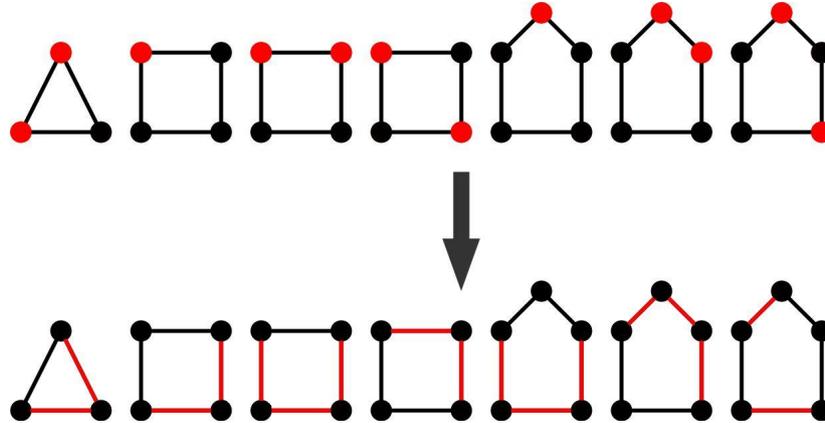


Figure 3. All non-equivalent vertex 2-labelings of C and their respective transformations to edge 2-labelings of C .

Unfortunately, these explicit structural descriptions soon become too complicated to write out. Recently, Lehner and Smith [19] announced a proof of the fact that any graph containing a cycle satisfies $D'(G) \leq D(G)$, but with a far more involved proof. The fact that our proof still works for small numbers of cycles suggests that there may be a simpler proof for an arbitrary number of cycles as well. We hence pose the following problem.

Problem 5.2. Is there an elementary proof for the fact that $D'(G) \leq D(G)$ for any graph with at least one cycle?

Finally we briefly discuss the case of infinite trees. Imrich *et al.* [12] extended Theorem [13, Theorem 11] to infinite graphs, and proved that if G is a connected infinite graph, then $D'(G) \leq D(G) + 1$.

Similar to the finite case, we can construct examples where equality is achieved. For this purpose, call a tree *rayless*, if it does not contain a ray, i.e., a one-sided infinite path. By a result of Schmidt [20], every rayless tree contains a finite subtree which is fixed by every automorphism. Hence we can call a rayless tree unicycric or bicycric, depending on whether this finite subtree has one or two central vertices. Hence the definition of the class \mathcal{T} also makes sense for infinite, rayless trees, and the same arguments as in the proof of Theorem 3.1 show that these are the only rayless trees with $D(T) = D'(T) + 1$. Finally, let us note that by [19], if a tree contains a ray, then $D(T) = D'(T)$.

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REFERENCES

- [1] M.O. Albertson and K.L. Collins, *Symmetry breaking in graphs*, Electron. J. Combin. **3** (1996) #R18.
- [2] S. Alikhani and S. Soltani, *Distinguishing number and distinguishing index of certain graphs*, Filomat **31** (2017) 4393–4404.
doi:10.2298/FIL1714393A
- [3] V. Arvind, C. Cheng and N. Devanur, *On computing the distinguishing numbers of planar graphs and beyond: a counting approach*, SIAM J. Discrete Math. **22** (2008) 1297–1324.
doi:10.1137/07068686X
- [4] V. Arvind and N. Devanur, *Symmetry breaking in trees and planar graphs by vertex coloring*, in: Proceedings of the 8th Nordic Combinatorial Conference (Aalborg University, Aalborg, Denmark, 2004).
- [5] M. Cavers and K. Seyffarth, *Graphs with large distinguishing chromatic number*, Electron. J. Combin. **20** (2013) #P19.
- [6] M. Chan, *The distinguishing number of the augmented cube and hypercube powers*, Discrete Math. **308** (2008) 2330–2336.
doi:10.1016/j.disc.2006.09.056
- [7] C. Cheng, *On computing the distinguishing numbers of trees and forests*, Electron. J. Combin. **13** (2006) #R11.
- [8] K.L. Collins and A.N. Trenk, *The distinguishing chromatic number*, Electron. J. Combin. **13** (2016) #R16.
- [9] E. Estaji, W. Imrich, R. Kalinowski, M. Pilśniak and T. Tucker, *Distinguishing Cartesian products of countable graphs*, Discuss. Math. Graph Theory **37** (2017) 155–164.
doi:10.7151/dmgt.1902
- [10] S. Gravier, K. Meslem, S. Schmidt and S. Slimani, *A new game invariant of graphs: the game distinguishing number*, Discrete Math. Theor. Comput. Sci. **19** (1) (2017) Paper No. 2.
doi:10.23638/DMTCS-19-1-2
- [11] P. Immel and P.S. Wenger, *The list distinguishing number equals the distinguishing number for interval graphs*, Discuss. Math. Graph Theory **37** (2017) 165–174.
doi:10.7151/dmgt.1927
- [12] W. Imrich, R. Kalinowski, M. Pilśniak and M.H. Shekarriz, *Bounds for distinguishing invariants of infinite graphs*, Electron. J. Combin. **24** (2017) #P3.6.
- [13] R. Kalinowski and M. Pilśniak, *Distinguishing graphs by edge colourings*, European J. Combin. **45** (2015) 124–131.
doi:10.1016/j.ejc.2014.11.003
- [14] S. Klavžar, T.-L. Wong and X. Zhu, *Distinguishing labelings of group action on vector spaces and graphs*, J. Algebra **303** (2006) 626–641.
doi:10.1016/j.jalgebra.2006.01.045

- [15] D. Kim, Y.S. Kwon and J. Lee, *The distinguishing numbers of Merged Johnson graphs*, Bull. Korean Math. Soc. **52** (2015) 395–408.
doi:10.4134/BKMS.2015.52.2.395
- [16] C. Laflamme, L. Nguyen Van Thé and N. Sauer, *Distinguishing number of countable homogeneous relational structures*, Electron. J. Combin. **17** (2010) #R20.
- [17] F. Lehner, *Distinguishing graphs with intermediate growth*, Combinatorica **36** (2016) 333–347.
doi:10.1007/s00493-015-3071-5
- [18] F. Lehner, *Breaking graph symmetries by edge colourings*, J. Combin. Theory Ser. B **127** (2017) 205–214.
doi:10.1016/j.jctb.2017.06.001
- [19] F. Lehner and S.M. Smith, *On symmetries of edge and vertex colourings of graphs*, preprint.
- [20] R. Schmidt, *Ein Ordnungsbegriff für Graphen ohne unendliche Wege mit einer Anwendung auf n -fach zusammenhängende Graphen*, Arch. Math. **40** (1983) 283–288.
doi:10.1007/BF01192782
- [21] S.M. Smith and M.E. Watkins, *Bounding the distinguishing number of infinite graphs and permutation groups*, Electron. J. Combin. **21** (2014) #P3.40.
- [22] M. Watkins and X. Zhou, *Distinguishability of locally finite trees*, Electron. J. Combin. **14** (2007) #R29.
- [23] T.-L. Wong and X. Zhu, *Distinguishing labeling of group actions*, Discrete Math. **309** (2009) 1760–1765.
doi:10.1016/j.disc.2008.02.022

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