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Absolute retracts of split graphs

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Abstract

It is proved that a split graph is an absolute retract of split graphs if and only if a partition of its vertex set into a stable set and a complete set is unique or it is a complete split graph. Three equivalent conditions for a split graph to be an absolute retract of the class of all graphs are given. It is finally shown that a reflexive split graph G is an absolute retract of reflexive split graphs if and only if G has no retract isomorphic to some J_n, $n \geqslant 3$. Here J_n is the reflexive graph with vertex set $\{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n\}$ in which the vertices $x_1, x_2, \ldots, x_n$ are mutually adjacent and the vertex y_i is adjacent to $x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$.

1. Introduction

The main motivation for this paper is the investigation [1] by Bandelt et al. in which the internal structure of the absolute retracts of bipartite graphs is clarified. They proved several nice characterizations of bipartite absolute retracts and applied these graphs to the competitive location theory. However, the investigation of the absolute retracts of bipartite graphs started as early as in 1972 with the Ph.D. Thesis of Hell [8], who gave, among others, two characterizations of the absolute retracts of bipartite graphs. Split graphs are 'half-way' between bipartite graphs and their complements. Since there are many structural characterizations of the absolute retracts of bipartite graphs, one may hope that something similar holds for the absolute retracts of split graphs as well.

All graphs considered in this paper are finite, undirected, connected, and simple. Sometimes we add a loop at each vertex and say we have a *reflexive graph*. The *degree* $d_G(x)$ of a vertex $x \in V(G)$ is the number of edges incident with x, not counting loops. The *neighbourhood* $N_G(x)$ of a vertex $x \in V(G)$ is the set of vertices adjacent to x excluding x if there is a loop at x. Hence we think of the degree and the neighbourhood of a vertex of a reflexive graph G just as if G were a simple graph.

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A subset $K \subseteq V(G)$ is a *complete set* if K induces a complete subgraph of G. A subset $S \subseteq V(G)$ is a *stable set* if no pair of distinct vertices of S is adjacent in G. The number of vertices in a maximum complete set of G is denoted by $\omega(G)$ and the number of vertices in a stable set of a maximum cardinality is denoted by $\alpha(G)$.

For graphs G and H an *edge-preserving map* of G to H is a map $f: V(G) \to V(H)$ such that $(f(x), f(y)) \in E(H)$ whenever $(x, y) \in E(G)$. Note that in reflexive graphs an edge-preserving map can identify adjacent vertices. A subgraph H of a (reflexive) graph G is a *retract* of G if there is an edge-preserving map $r: V(G) \to V(H)$ with $r(x) = x$, for all $x \in V(H)$. The map r is called a *retraction*. If H is a retract of a (reflexive) graph G then H is an *isometric* subgraph of G, that is $d_G(x, y) = d_H(x, y)$ for all $x, y \in V(H)$, where $d_G(x, y)$ denotes the length of a shortest path in G between x and y. A simple graph G is *n-chromatic* if there is an edge-preserving map from G onto the complete graph K_n, and n is the smallest such natural number. This n is called the *chromatic number* of G and is denoted by $\chi(G)$. The corresponding edge-preserving map is called an *n-colouring* of G. If H is a retract of a simple graph G then $\chi(H) = \chi(G)$, i.e., H is an *isochromatic* subgraph of G.

Let $\mathscr{C}$ be a class of graphs. A graph $G \in \mathscr{C}$ is an *absolute retract of graphs of* $\mathscr{C}$ if G is a retract of any graph $H \in \mathscr{C}$ containing G as an isometric and isochromatic subgraph. If $\mathscr{C} = \mathscr{G}$, i.e. $\mathscr{C}$ is the class of all graphs then we call an absolute retract of all graphs simply an *absolute retract*. If $\mathscr{C}$ is a class of reflexive graphs then call a graph $G \in \mathscr{C}$ an *absolute retract of reflexive graphs of* $\mathscr{C}$ if G is a retract of any graph $H \in \mathscr{C}$ containing G as an isometric subgraph. If $\mathscr{C}$ is the class of all reflexive graphs then we call such a graph an *absolute reflexive retract*.

In the literature one can find also some other definitions of absolute retracts. In fact, any set of necessary conditions for a retraction would yield a definition. For example, in [10] the only condition for an absolute retract is to be an induced subgraph while in [11] a graph H is called an absolute reflexive retract if it is a retract of any graph G provided that every hole of H is separated in G.

In the next section we recall some basic facts about split graphs and introduce complete split graphs and graphs J_n. In Section 3 we characterize absolute retracts of split graphs, and prove that an n-chromatic split graph G is an absolute retract if and only if there is a unique n-colouring of G. The graphs J_n, which are defined in Section 2, were considered earlier; see for instance [4, 12–14]. They came up in a game-theoretic problem solved by Nowakowski and Winkler [14]. The graphs J_n were shown to be irreducible by Nowakowski and Rival in [13], and these graphs are also important in a classification of reflexive graphs, see [4, 12]. In the last section we add a new aspect of the graphs J_n: They are precisely the forbidden retracts of the absolute retracts of reflexive split graphs.

2. Split graphs

A (reflexive) graph G is called *split* if there is a partition $V(G) = K + S$ of its vertex set into a complete set K and a stable set S. Such a partition may not be unique, as

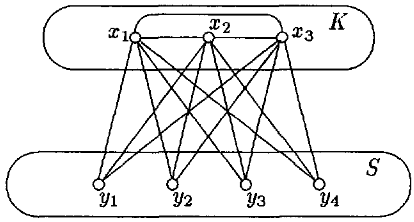

Fig. 1. A complete split graph.

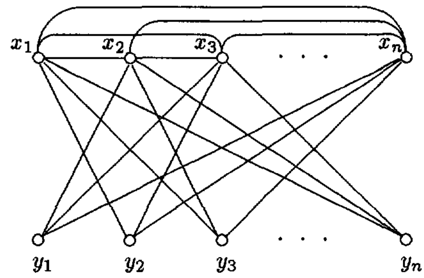

Fig. 2. The graph J_n.

can be seen on Fig. 1. One of its partitions is indicated, while the others are $(K \cup \{y_i\})+(S-\{y_i\})$ for $i=1,2,3,4$.

Let H be an induced subgraph of a split graph G, $V(G)=K+S$. Clearly, H is a split graph and the partition $V(H)=K'+S'$ of its vertex set which satisfies $K' \subseteq K$ and $S' \subseteq S$ will be called an *induced partition*. Földes and Hammer [5] proved that the graphs $2K_2$, C_4 and C_5 are the minimal forbidden induced subgraphs of split graphs. It follows that every split graph with at least one edge consists of the nontrivial connected component and several isolated vertices. Hence our assumption that all the graphs are connected is not a restriction.

A graph G is a *complete split graph* if there is a partition $V(G)=K+S$ of its vertex set into a complete set K and a stable set S such that every vertex from S is adjacent to every vertex from K. Note that in such a partition any one vertex from S together with the set K forms a complete subgraph, hence $\omega(G)=|K|+1$. In a trivial case, a complete split graph is isomorphic to a complete graph K_n. We refer again to Fig. 1, which shows a complete split graph with a corresponding partition.

The (reflexive) graph J_n, $n \geqslant 2$, has the vertex set $X_n \cup Y_n$ where $X_n=\{x_1, x_2, \ldots, x_n\}$ and $Y_n=\{y_1, y_2, \ldots, y_n\}$. The subset X_n forms a complete set, the subset Y forms a stable set, and otherwise, two vertices x_i and y_i; are adjacent except if $i=j$ (cf. Fig. 2). Clearly, J_n is a split graph with a vertex partition $X+Y$.

The next theorem can be found in [6] and follows essentially from the work of Hammer and Simeone [7].

Theorem 2.1. *Let G be a split graph, $V(G)=K+S$. Exactly one of the following conditions holds:*

(i) $|K|=\omega(G)$ *and* $|S|=\alpha(G)$ *(in this case the partition $K+S$ is unique),*

(ii) $|K|=\omega(G)-1$ *and* $|S|=\alpha(G)$ *(in this case there exists an $y \in S$ such that $K+\{y\}$ is complete),*

(iii) $|K|=\omega(G)$ *and* $|S|=\alpha(G)-1$ *(in this case there exists an $x \in K$ such that $S+\{x\}$ is stable).*

It follows from the theorem that there always exists a partition $K+S$ such that the part K has $\omega(G)$ vertices. We will also use the following corollary to Theorem 2.1.

Corollary 2.2. *Let G be a split graph with a partition $V(G)=K+S$, where $K=\{x_1,x_2,\dots,x_n\}$ and $S=\{y_1,y_2,\dots,y_m\}$. Then this partition is unique if and only if $d_G(x_i)\geqslant n$ for $i=1,2,\dots,n$ and $d_G(y_i)\leqslant n-1$ for $i=1,2,\dots,m$.*

Proof. If $d_G(x_i)=n-1$ for some i, then we have the condition (iii) of Theorem 2.1 and if $d_G(y_i)=n$ for some i, then we have the condition (ii). Conversely, if $d_G(x_i)\geqslant n$ for all i then $\alpha(G)=m$ and if $d_G(y_i)\leqslant n-1$ for all i then $\omega(G)=n$. Hence by the condition (i) of Theorem 2.1 a partition $K+S$ is unique. □

3. Absolute retracts of split graphs

In this section we want to characterize absolute retracts of split graphs and split graphs which are absolute retracts, i.e., absolute retracts in the class of all graphs. Our main results are presented in Theorem 3.2 and Theorem 3.6.

Lemma 3.1. *Let G be a split graph, $V(G)=K+S$, where K is a maximum complete subgraph. Let H be an induced subgraph of G with an induced partition $V(H)=K'+S'$. Then H is a retract of G if and only if $\omega(H)=\omega(G)$ and, if $x\in K-K'$ then $N_G(x)\cap S'=\emptyset$.*

Proof. Let H be a retract of G and let $r:V(G)\to V(H)$ be a retraction. Then $\chi(G)=\chi(H)$ and as split graphs are perfect, $\omega(H)=\omega(G)$. Suppose that there are $y\in S'$ and $x\in K-K'$, such that $(y,x)\in E(G)$. Then $r(y)=y$ and $r(x)\in S'$. But S' is a stable set, hence $(r(y),r(x))\notin E(H)$, a contradiction.

Assume that the conditions of the theorem are fulfilled. H is a split graph, hence by Theorem 2.1, $|K'|=\omega(H)$ or $|K'|=\omega(H)-1$. Let for any $y\in S-S'$, $f(y)$ denote a fixed by arbitrary vertex $x\in K$ such that $(y,x)\notin E(G)$, if such a vertex exists.

Case 1: $|K'|=\omega(H)$. By assumption, $\omega(H)=\omega(G)$ hence $K=K'$. Since K is a maximum complete subgraph, $f(y)$ always exists. Define $r:V(G)\to V(H)$:

$$r(v)=\begin{cases} v, & v\in V(H),\\ f(v), & v\in S-S'.\end{cases}$$

The mapping $r:V(G)\to V(H)$ is well-defined and it is easy to verify that r is also edge-preserving.

Case 2: $|K'|=\omega(H)-1$. Let x be the unique vertex from $K-K'$. Since $\omega(H)=\omega(G)$, there exists a vertex $w\in S'$ such that the vertex set $K'+\{w\}$ induces a complete subgraph of H. Define $r:V(G)\to V(H)$ as follows:

$$r(v)=\begin{cases} v, & v\in V(H),\\ f(v), & v\in S-S',\quad (v,x)\in E(G),\\ w, & v\in S-S',\quad (v,x)\notin E(G)\ \text{or}\ v=x.\end{cases}$$

Because K is a maximum complete subgraph, $f(y)$ exists whenever $(y,x)\in E(G)$. It follows that r is well-defined. Furthermore, as $N_G(x)\cap S'=\emptyset$ it is straightforward to verify that r preserves adjacency, hence it is a retraction. □

Since the maximal complete subgraphs of a chordal graph can be calculated in linear time (see [6, Theorem 4.17]), Lemma 3.1 provides us a linear time characterization of the retracts of a split graph.

Theorem 3.2. *A split graph is an absolute retract of split graphs if and only if a partition of its vertex set into a stable set and a complete set is unique or it is a complete split graph.*

Proof. Suppose first that a partition of $V(G)$ is unique. Let G be an isometric and isochromatic subgraph of a split graph H with $V(H)=K+S$. Let $V(G)=K'+S'$ be an induced partition. As $V(G)$ has a unique partition it follows from Theorem 2.1 that $|K'|=\omega(G)$. Furthermore, G is isochromatic in H and $K'\subseteq K$, hence $K=K'$. It follows from Lemma 3.1 that G is a retract of H. Since H was arbitrary, G is an absolute retract of split graphs.

Suppose next that a partition of $V(G)$ is not unique. Let $V(G)=K+S$, $K=\{x_1,x_2,\dots,x_n\}$, $S=\{y_1,y_2,\dots,y_m\}$, where $\omega(G)=n$. It follows from Corollary 2.2 that at least one vertex from K is of degree $n-1$. We may suppose that $d_G(x_1)=n-1$. We distinguish two cases.

Case 1: At least one vertex from S is of degree $\leqslant n-2$. Assume $d_G(y_1)\leqslant n-2$ and define a graph H as follows:

$$V(H)=V(G)\cup\{z\},$$

$$E(H)=E(G)\cup\{(z,x_i),\ i=2,3,\dots,n\}\cup\{(z,y_1)\}.$$

Since x_1 is not adjacent to any of the y_i, the graph H is a split graph with a vertex partition $V(H)=K'+S'$, where $K'=\{z,x_2,x_3,\dots,x_n\}$ and $S'=V(H)-K'$. Let $c: V(G)\to K_n$ be an n-colouring of G and let t be any colour not in the set $\{c(x_1)\}\cup\{c(x_i);\ x_i \text{ is adjacent to } y_1\}$. Since $d_G(y_1)\leqslant n-2$, we can always choose t from K_n. Then $c': V(H)\to K_n$

$$c'(v)=\begin{cases} c(x_1), & v=z,\\ t, & v=y_1,\\ c(v), & \text{otherwise.}\end{cases}$$

is an n-colouring of H, therefore G is an isochromatic subgraph of H. Since G is connected it is also straightforward to verify that G is isometric in H. Hence G is an isometric and isochromatic subgraph of H, but by Lemma 3.1 G is not a retract of H. It follows that G is not an absolute retract of split graphs.

Case 2: All the vertices from S are of degree $n-1$. As $d_G(x_1)=n-1$, y_i is adjacent to $x_2,x_3,\dots,x_n$ for all $i=1,2,\dots,m$. Hence G is a complete split graph with a vertex partition $\{x_2, x_3, \ldots, x_n\} + \{x_1, y_1, y_2, \ldots, y_m\}$. Let G be an isometric and isochromatic subgraph of H with $V(H) = K + S$, $|K| = n$, and let $V(G) = K' + S'$ be the induced partition. If $K = K'$ then G is a retract of H. Otherwise, $|K - K'| = 1$ and let $z \in K - K'$. If for some i, $i \neq 1$, the vertex x_i belongs to S' then H is not a split graph, hence $K = \{x_2, x_3, \ldots, x_n, z\}$ and $x_1 \in S'$. But then z is not adjacent to any of y_i, for otherwise $\omega(H) \geq n+1$. It follows, using Lemma 3.1 again, that G is a retract of H. □

Corollary 3.3. *Let G be a split graph and let both G and its complement $\bar{G}$ be connected. Then G is an absolute retract of split graphs if and only if $\bar{G}$ is an absolute retract of split graphs.*

Proof. Recall that a graph G is a split graph if and only if its complement $\bar{G}$ is a split graph.

Let G and $\bar{G}$ be connected and let G be an absolute retract of split graphs. Then G is not a complete split graph for otherwise $\bar{G}$ would be disconnected. It follows from Theorem 3.2 that G has a unique vertex partition $V(G) = K + S$, $K = \{x_1, x_2, \ldots, x_n\}$, $S = \{y_1, y_2, \ldots, y_m\}$. Then by Corollary 2.2 $d_G(x_i) \geq n$ for $i = 1, 2, \ldots, n$ and $d_G(y_i) \leq n-1$ for $i = 1, 2, \ldots, m$. Clearly, $\bar{G}$ is a split graph with a vertex partition $K + S$, where K is a stable set and S a complete set. Furthermore, $d_{\bar{G}}(x_i) \leq m-1$ for all i and $d_{\bar{G}}(y_i) \geq m$ for all i. Using Corollary 2.2 again, $\bar{G}$ has a unique partition, hence it is an absolute retract of split graphs. □

In the proof of our next theorem we will use the following two results.

Theorem 3.4 (Pesch [16, p. 42]). *Let G be an absolute retract, $\chi(G) = n$. Then G is $(n-1)$-connected.*

Theorem 3.5 (Pesch and Poguntke [17]). *Let G be an n-chromatic graph. The following is equivalent:*

(i) *G is an absolute retract, and* $\mathrm{diam}(G) \leq 3$ *if* $n = 2$, $\mathrm{diam}(G) \leq 2$ *if* $n \geq 3$.
(ii) *For each colouring $c: G \to K_n$ and for each $i \in V(K_n)$, there is a $z_i \in V(G)$ with $z_i v \in E(G)$ for all $v \in V(G)$ with $c(v) \neq i$.*

We now characterize split graphs which are absolute retracts.

Theorem 3.6. *For a split graph G with $\omega(G) = n$, the following conditions are equivalent:*

(i) *G is $(n-1)$-connected.*
(ii) *There is a unique n-colouring of G.*
(iii) *$d_G(x) \geq \omega(G) - 1$, for all $x \in V(G)$.*
(iv) *G is an absolute retract.*

Proof. Let G be a split graph, $V(G) = K + S$, $K = \{x_1, x_2, \ldots, x_n\}$, $S = \{y_1, y_2, \ldots, y_m\}$, where $\omega(G) = n$.

(i)⇒(ii). Assume that there is no unique n-colouring of G. Since the complete set K is uniquely colourable, there is a vertex $y \in S$ for which $|N_G(y)| \leqslant n-2$. It follows that $N_G(y)$ separates G, a contradiction.

(ii) ⇒(iii). Clearly, $d_G(x_i) \geqslant n-1$, for $i=1,2,\ldots,n$. As y_i is adjacent only to vertices from K, y_i must be adjacent to exactly $n-1$ vertices from K, for otherwise G is not uniquely colourable.

(iii)⇒(iv). Let $c: G \to K_n$ be a given n-colouring of G and let $i \in V(K_n)$. We may suppose $c(x_i)=i$. We claim that the vertex x_i satisfies the condition of Theorem 3.5 for for colour i. Clearly, x_i is adjacent to every $x_j, j \neq i$. Let $y \in S$ be any vertex with $c(y) \neq i$. As $d_G(y)=n-1$ and $(y, x_j) \notin E(G)$, $c(y)=j$, it follows that $(y, x_i) \in E(G)$. This proves the claim, hence G is an absolute retract. Note also that if $n \geqslant 3$ then every pair of vertices from S has a common neighbour in K, hence $\mathrm{diam}(G)=2$ and if G is bipartite then $\mathrm{diam}(G)=3$.

(iv)⇒(i). This follows from Theorem 3.4. □

Corollary 3.7. *Every bipartite split graph is an absolute retract.*

Proof. A bipartite split graph is a tree with a diameter less than four (see Fig. 3). Clearly $d_G(x) \geqslant \omega(G)-1=1$ for all $x \in V(G)$. □

The above corollary is well-known, since every tree is an absolute retract, as was first observed by Hell [8]. Furthermore, Hell [8, Proposition 6.3.7] showed also that a bipartite graph G of diameter three is an absolute retract if and only if there is some edge $(x, y) \in E(G)$ such that every vertex of G is adjacent to either x or y.

4. Absolute retracts of reflexive split graphs

For the sake of completeness we give a characterization of the absolute retracts of reflexive split graphs. If fact, one can deduce this characterization also by using Theorem 2.2 from [3] (which was discovered independently by Martin Farber (unpublished)). However, our result is straightforward and self-contained hence we include it here. Note that the situation differs from the irreflexive case in that for split graphs the absolute retracts are the same in the class of all graphs and the class of split graphs.

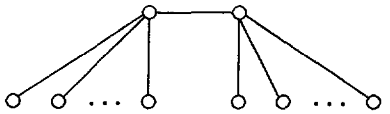

Fig. 3. A bipartite split graph.

Theorem 4.1. *For a reflexive split graph G, the following conditions are equivalent:*

(i) *G is an absolute retract of reflexive split graphs.*

(ii) *G is an absolute reflexive retract.*

(iii) *G has no retract isomorphic to some J_n, $n \geqslant 3$.*

A split graph is a chordal graph, i.e. it does not contain an induced subgraph isomorphic to C_n, $n \geqslant 4$. It follows from Theorem 4.1 that the graphs J_n, $n \geqslant 3$ are not absolute retracts. It was observed by Bandelt and Pesch in [2] that the graph J_3 is the smallest chordal graph (and in particular, split graph) which is not an absolute retract.

Every retract is necessarily an isometric subgraph. A generalization of this condition to 'holes' (called 'gaps' in [13]) was first introduced by Nowakowski and Rival in [13]. A *hole* of a graph G is a pair (K, δ), where K is a nonempty set of vertices of G and δ a function from K to nonnegative integers such that no $x \in V(G)$ has $d_G(x, y) \leqslant \delta(y)$ for all $y \in K$. We also require that if (K, δ) is a hole, then K has no proper subset K' such that $(K', \delta | K')$ is also a hole. An *m-hole* is a hole (K, δ) with $|K| = m$. For more information on the concept of hole, motivations and examples we refer to [11].

A hole (K, δ) of a subgraph H of G is *separated* in G if (K, δ) is also a hole of G. Being an isometric subgraph is equivalent to having all 2-holes separated. Having all holes separated is another necessary condition for being a retract, cf. [11–13]. As we see in Lemma 4.2 this is also sufficient for reflexive split graphs.

Let (K, δ) be a hole of G with $\delta(x) = 1$ for every $x \in K$. Such a hole will be called a $\vec{1}$-*hole*.

Lemma 4.2. *Let H be a reflexive split graph and let H be an isometric subgraph of a reflexive graph G. Then H is a retract of G if and only if every $\vec{1}$-hole of H is separated in G.*

Proof. We must show that if every $\vec{1}$-hole of H is separated in G then H is a retract of G.

Let H be a reflexive split graph, $V(H) = K + S$. We claim that there is a retraction $f: V(G) \to V(H)$ which maps every vertex $x \in V(G) - V(H)$ onto K.

For $x \in V(G) - V(H)$ let $S_x = N_G(x) \cap S$. If $S_x = \emptyset$ let $f(x)$ be any one fixed vertex of K. Suppose next that S_x is nonempty and set $\delta(y) = 1$ for every $y \in S_x$. Note first that (S_x, δ) is not a $\vec{1}$-hole of H, for otherwise this hole would not be separated in G. Then either there is a vertex $u \in K$ adjacent to every vertex of S_x or there exists a $\vec{1}$-hole $(X, \delta | X)$ of H, $X \subset S_x$. The former case is impossible as every $\vec{1}$-hole of H is separated in G. Hence we may define $f(x) = u$. It is easy to see that $f: V(G) \to V(H)$ is a retraction. □

Lemma 4.3. *Let G be a reflexive split graph with a vertex partition $K + S$. If (X, δ) is a $\vec{1}$-hole of G then $X \subseteq S$.*

Proof. Let $X' = S \cap X$ and assume on the contrary $X' \subset X$. Since G is connected, each $v \in S$ is adjacent to some vertex of K. If $|X'| \leqslant 1$ then there is a vertex $x \in K$ adjacent to every vertex from X, thus (X, δ) is not a $\vec{1}$-hole. Let $|X'| \geqslant 2$. By the definition of a hole $(X', \delta | X')$ is not a hole, hence there exists a vertex $x \in V(G)$ adjacent to every vertex from X'. Clearly, $x \in K$. It follows that x is adjacent to every vertex from X hence (K, δ) is not a $\vec{1}$-hole of G.

Proof of Theorem 4.1. (i)$\Rightarrow$(iii). Suppose that G is an absolute retract of reflexive split graphs, with partition $K+S$, and assume that it has a retract isomorphic to some J_n, $n \geqslant 3$. Let $f: V(G) \to V(J_n)$ be a retraction. According to Corollary 2.2 a partition $V(J_n) = X + Y$ of the vertex set $V(J_n)$ is unique. It follows that $X \subseteq K$ and $Y \subseteq S$ for otherwise the induced partition of $V(J_n)$ in G would induce another partition of $V(J_n)$. Define the graph H as follows. Let $V(H) = V(G) \cup \{x\}$ and let $E(H) = E(G) \cup \{(x, y) | y \in K \cup Y\}$. As every pair of vertices from Y has a common adjacent vertex in X it easily follows that G is an isometric subgraph of H. As H is also a split graph and G is an absolute retract of reflexive split graphs, there exists a retraction $g: V(H) \to V(G)$. Then $f \circ g$ is a retraction $V(H) \to V(J_n)$ and J_n is a retract of H. But (Y, δ) is a $\vec{1}$-hole of J_n which is not separated in H, a contradiction.

(iii)$\Rightarrow$(ii). Suppose that the reflexive split graph G, $V(G) = K + S$, is not an absolute reflexive retract. Then there exists a reflexive graph H such that G is an isometric subgraph of H but G is not a retract of H. By Lemma 4.2 there exists a $\vec{1}$-hole (Y, δ) of G which is not separated in H. Since G is isometric in H all the 2-holes are separated, hence $|Y| \geqslant 3$. By Lemma 4.3, $Y \subseteq S$. By the definition of a hole, for every vertex $y \in Y$ there exists a vertex $x_y \in K$ which is nonadjacent to y and adjacent to every vertex from $Y - \{y\}$. Clearly, if $y, y' \in Y$ and $y \neq y'$ then $x_y \neq x_{y'}$. It follows that $Y' = Y + \{x_y | y \in Y\}$ induce a graph J_n, $n \geqslant 3$. Furthermore, since (Y, δ) is a $\vec{1}$-hole of G there is no vertex of G adjacent to every $y \in Y$. Hence if $x \in V(G) - Y'$ and $(x, y) \notin E(G)$, where $y \in Y$, we map x to x_y. It now easily follows that this is retraction, a contradiction.

(ii)$\Rightarrow$(i). This part of the proof is trivial. $\square$

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