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On cubic and edge-critical isometric subgraphs of hypercubes

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Abstract

All cubic partial cubes (i.e., cubic isometric subgraphs of hypercubes) up to 30 vertices and all edge-critical partial cubes up to 14 vertices are presented. The lists of graphs were confirmed by computer search to be complete. Non-trivial cubic partial cubes on 36, 42, and 48 vertices are also constructed.

1 Introduction

Partial cubes are, by definition, graphs that admit isometric embeddings into hypercubes. They were introduced by Graham and Pollak [9] and first characterized by

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Djoković [6]. Several additional characterizations followed in [2, 4, 19, 20]. Partial cubes found different applications (see, for example, [5, 7, 12]), while recognition algorithms for these graphs have been developed in [1, 10]. For an extensive presentation of partial cubes we refer the reader to the book [11].

For the (probably) most important subclass of partial cubes, median graphs, Mulder [17] proved that hypercubes are the only regular median graphs. In other words, the only regular median graphs are Cartesian products of copies of K_2. This result has been in [3] extended to the so-called "tree-like" partial cubes. Hence, it is natural to ask which graphs are regular partial cubes. (Regular subgraphs of hypercubes are studied in [18]). Despite the fact that the structure of partial cubes has been well clarified by now, this question seems to be a difficult one.

The Cartesian product of two (regular) partial cubes is a (regular) partial cube. Since even cycles are regular partial cubes, one may wonder whether we get all regular partial cubes as Cartesian products of copies of K_2 and even cycles. In particular, are all cubic partial cubes of the form $C_{2k}\Box K_2$, $k \geq 2$? This was believed to be true for quite a while, until two sporadic examples appeared: the generalized Petersen graph $P(10,3)$ on 20 vertices, cf. [13], and the graph B_1 (see Fig. 1) on 24 vertices from [8], (see also [11]). Calling the graphs $C_{2k}\Box K_2$, $k \geq 2$, *trivial cubic partial cubes*, we have verified that besides these two graphs there is only one other nontrivial cubic partial cube on at most 30 vertices. The third example, denoted B_1' (see Fig. 2), has 30 vertices. It can be obtained from the nontrivial partial cube on 24 vertices by the so-called expansion and was also found by computer search.

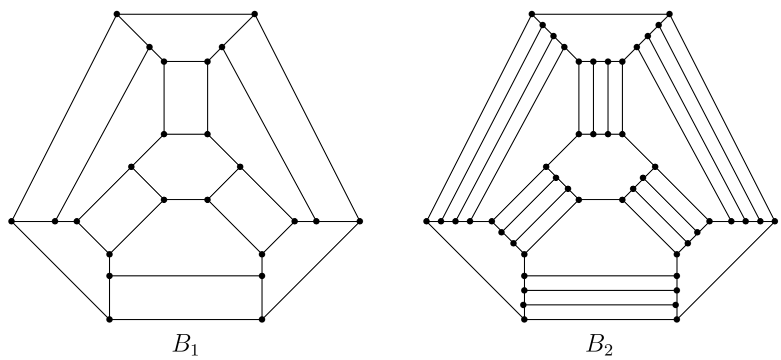

Figure 1: Graphs B_1 and B_2

Edge-critical partial cubes are partial cubes G for which $G - e$ is not a partial cube for all edges e of G. The 3-cube and the subdivision graph of K_4 are the only edge-critical partial cubes on at most 10 vertices [14].

In this note we present all cubic partial cubes up to 30 vertices and all edge-critical partial cubes up to 14 vertices. The lists of graphs were confirmed by computer search to be complete. We also give further larger non-trivial cubic partial cubes on 36, 42, and 48 vertices.

2 Cubic partial cubes

A graph G is called *prime* (with respect to the Cartesian graph product) if $G = G_1 \Box G_2$ implies either G_1 or G_2 is the one-vertex graph K_1.

The Cartesian product of two regular partial cubes is a regular partial cube. Therefore the problem of characterizing regular partial cubes reduces to prime (with respect to the Cartesian product) partial cubes. For the cubic case, this fact leads to the following observation:

Proposition 2.1 *Let G be a cubic partial cube. Then either $G = C_{2n} \Box K_2$ for some $n \geq 2$ or G is a prime graph.*

Proof. Assume $G = G_1 \Box G_2$, where $G_1, G_2 \neq K_1$. As G is connected, then so are G_1 and G_2. Since G is cubic and the degree of $(u, v) \in V(G_1 \Box G_2)$ is the sum of the degrees of $u \in G_1$ and $v \in G_2$, then one of the factors, say G_2, contains only vertices of degree one or less. Therefore $G_2 = K_2$. Furthermore, G_1 must be 2-regular, and hence a cycle. Moreover, it is an even cycle since partial cubes are bipartite graphs. □

We now construct the nontrivial cubic partial cubes B_1', B_1'' , B_1''', and B_2 on 30, 36, 42, and 48 vertices, respectively. The last graph is shown in Fig. 1, while the others are given in in Fig. 2. These graphs can be constructed by expansions from B_1, and hence we first introduce the concept of expansion.

Let G' be a connected graph. A *proper cover* consists of two isometric subgraphs G_1', G_2' of G' such that $G' = G_1' \cup G_2'$, $G_0' = G_1' \cap G_2'$ is a nonempty subgraph, and there are no edges between $G_1' \setminus G_2'$ and $G_2' \setminus G_1'$. (The subgraph G_0' is called the *intersection* of the cover.) The *expansion* of G' with respect to G_1', G_2' is the graph G constructed as follows: Let G_i be an isomorphic copy of G_i', for $i = 1, 2$, and, for any vertex u' in G_0', let u_i be the corresponding vertex in G_i, for $i = 1, 2$. Then G is obtained from the disjoint union $G_1 \cup G_2$, where for each u' in G_0' the vertices u_1 and u_2 are joined by an edge.

Chepoi [4] proved that a graph is a partial cube if and only if it can be obtained from K_1 by a sequence of expansions. This result was later independently obtained in [7] and is analogous to the convex expansion theorem for median graphs [16].

An expansion is called *peripheral* if at least one of the graphs G_1' or G_2' is equal to G. In this situation the other graph equals the intersection, and is necessarily isometric in G. We recall from [3] that a regular, prime partial cube on at least three vertices can not be obtained by peripheral expansion from some partial cube.

For the proof of the next result we also need the following concept of isometric dimension. Two edges $e = xy$ and $f = uv$ of a graph G are in the Djoković-Winkler [6, 20] relation Θ if $d_G(x, u) + d_G(y, v) \neq d_G(x, v) + d_G(y, u)$. Winkler [20] showed that a bipartite graph is a partial cube if and only if $\Theta = \Theta^*$ (where Θ^* denotes the transitive closure of Θ). Thus Θ defines an equivalence relation on the edges of a partial cube. The *isometric dimension*, $\mathrm{idim}(G)$, of a partial cube G is defined as the number of its Θ-classes.

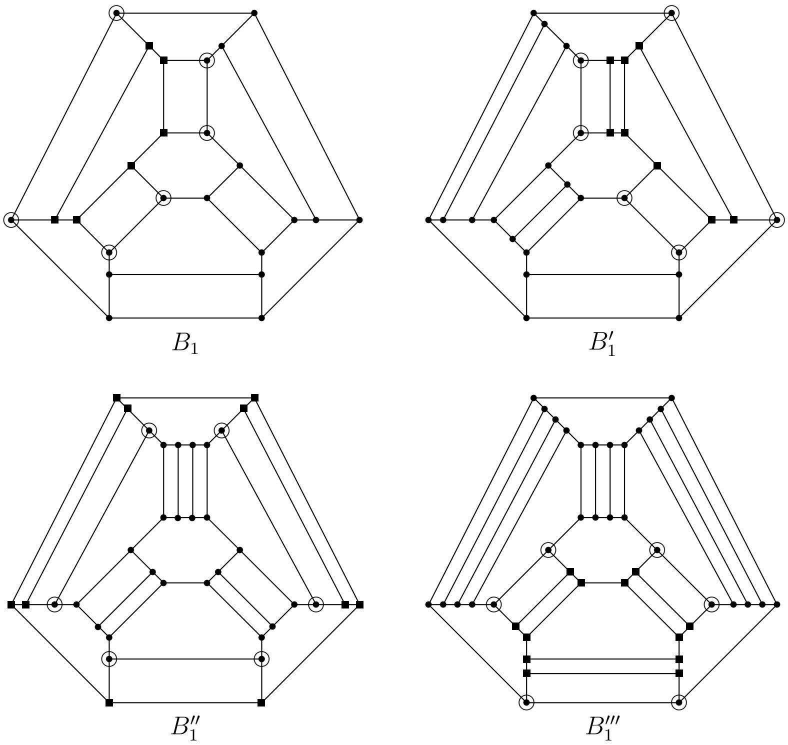

Figure 2: Graphs B_1, B_1', B_1'', and B_1'''

Theorem 2.2 *Graphs B_1', B_1'', B_1''', and B_2 are cubic prime partial cubes.*

Proof. We know already that B_1 is a partial cube. Now, B_1', B_1'', B_1''', and B_2 can be obtained from B_1, B_1', B_1'', and B_1''', respectively, by an expansion. These expansions are schematically explained in Fig. 2 in the following way. A proper cover in each expansion is chosen as follows: G_1' is induced by the vertices denoted by filled circles, G_2' is induced by the vertices denoted by filled squares and their intersection is formed by the remaining vertices; that is, the vertices denoted by filled circles surrounded by another circle. It is easy to verify that in this way we really obtain a proper cover; that is, G_1' and G_2' are isometric subgraphs of the corresponding graphs B_1, B_1', B_1'', and B_1''', and there are no edges between $G_1' \setminus G_2'$ and $G_2' \setminus G_1'$. Hence, by the theorem of Chepoi the obtained graphs are partial cubes. Clearly, they are cubic.

We now show that these four graphs are prime. Observe first that $\mathrm{idim}(B_1) = 6$ and therefore $\mathrm{idim}(B_1') = 7$, $\mathrm{idim}(B_1'') = 8$, $\mathrm{idim}(B_1''') = 9$, and $\mathrm{idim}(B_2) = 10$. If any of these four graphs were not prime, then by Proposition 2.1 it would be isomorphic to $C_{15} \Box K_2$, $C_{18} \Box K_2$, $C_{21} \Box K_2$, and $C_{24} \Box K_2$, respectively. Two of these graphs are not bipartite, while the isometric dimensions of the other two; that is, of $C_{18} \Box K_2$,

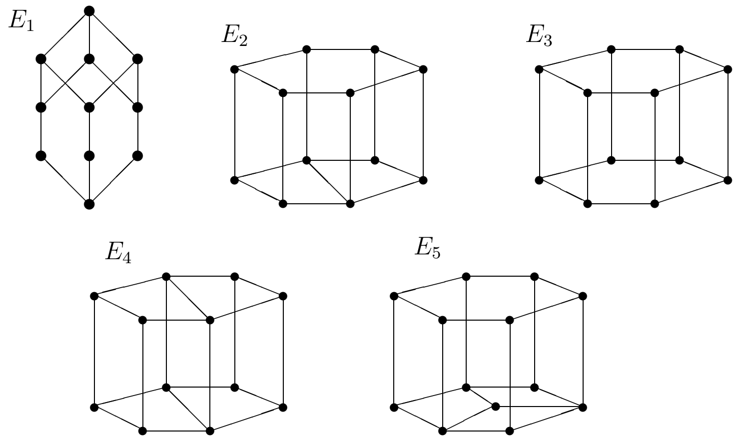

Figure 3: Edge-critical partial cubes on 11, 12, and 13 vertices

and $C_{24}\square K_2$, are 10 and 13. □

It seems tempting to continue the expansion procedure with B_2 to obtain new cubic partial cubes. However, we were not able to obtain more examples in this way. In particular, the graph that is constructed from B_2 analogously as B_2 is constructed from B_1 is not a partial cube.

3 Computer search for cubic and edge-critical partial cubes

Using the Djoković-Winkler relation, we have implemented a recognition algorithm for partial cubes and applied it to all connected bipartite cubic graphs up to 30 vertices. (These graphs were constructed using Brendan McKay's Nauty program [15].) The examination of the entire set of graphs was run concurrently on a cluster of 16 pentium-class machines, and doubled-checked on an 8 processor Sun Sparc server. The obtained results are summarized in the following table:

n		n		
< 8	-	20	$C_{10}\square K_2$	$P(10,3)$
8	$C_4\square K_2$	22	-	-
10	-	24	$C_{12}\square K_2$	B_1
12	$C_6\square K_2$	26	-	-
14	-	28	$C_{14}\square K_2$	-
16	$C_8\square K_2$	30	-	B_1'
18	-			

The above table shows that, up to 30 vertices, there are only 3 nontrivial cubic partial cubes.

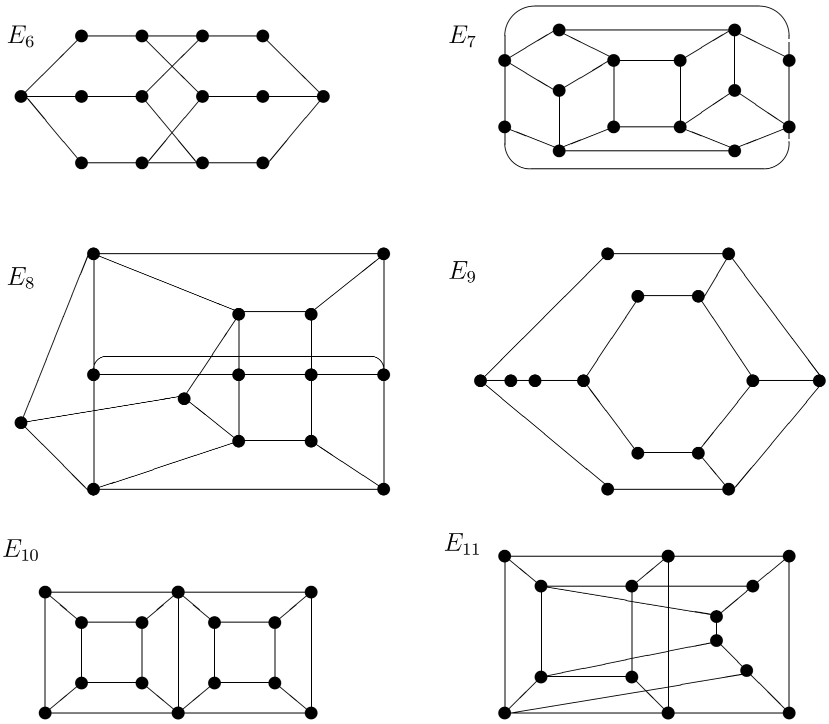

Figure 4: Edge-critical partial cubes on 14 vertices.

Recall that a partial cube G is called edge-critical if for any edge e of G, $G - e$ is not a partial cube. In [14] it was shown that the 3-cube Q_3 and the subdivision graph of K_4, $S(K_4)$, are the only edge-critical partial cubes on at most 10 vertices. Moreover, two such graphs on 12 vertices and one on 13 vertices are listed. We have now searched for all edge-critical partial cubes on at most 14 vertices and established the following complete list of edge-critical partial cubes. The computation is a variant of that used for cubic partial cubes; for each connected bipartite graph G on at most 14 vertices, if G is determined to be a partial cube, then all of the non-isomorphic graphs obtained by deleting a single edge from G are tested. Brendan McKay's Nauty program [15] is used to also filter isomorphic graphs from the edge deletions. The results are summarized in the following table.

n	< 8	8	9	10	11	12	13	14
	–	Q_3	–	$S(K_4)$	E_1	E_2, E_3, E_4	E_5	$E_6, \ldots, E_{11}$

We note that there is one previously undiscovered graph on each of 11 and 12 vertices (E_1 and E_2 respectively), and six on 14 vertices. These are given in Fig. 3 and 4.

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