The b-chromatic number of cubic graphs

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Abstract

The b-chromatic number of a graph G is the largest integer k such that G admits a proper k-coloring in which every color class contains at least one vertex adjacent to some vertex in all the other color classes. It is proved that with four exceptions, the b-chromatic number of cubic graphs is 4. The exceptions are the Petersen graph, $K_{3,3}$, the prism over K_3, and one more sporadic example on 10 vertices.

Key words: chromatic number; b-chromatic number; cubic graph; Petersen graph.

AMS subject classification (2000): 05C15

1 Introduction

The b-*chromatic number* $\varphi(G)$ of a graph G is the largest integer k such that G admits a proper k-coloring in which every color class has a representative adjacent to at least one vertex in each of the other color classes. Such a coloring is called a b-*coloring*. A b-coloring that uses k colors will be called a k-b-coloring. Let G be a graph and c a b-coloring of G. If $x \in V(G)$ has a neighbor in any other color class we will say that x *realizes color* $c(x)$ and that $c(x)$ is *realized on* x.

Here is a motivation. Let c be an arbitrary proper coloring of G and suppose we want to decrease the number of colors by recoloring all the vertices of a given color class X with other colors. Then this is possible if and only if no vertex of X realizes its color. In other words, one color can be saved by recoloring each vertex of some fixed color class if and only if the coloring c is not a b-coloring.

This concept was introduced in 1999 by Irving and Manlove [13] who proved that determining $\varphi(G)$ is NP-hard in general and polynomial for trees. The complexity issues of b-colorings were further studied in [6, 19]. For instance, determining $\varphi(G)$ is NP-hard already for bipartite graphs [19], while Corteel, Valencia-Pabon, and Vera [6] proved that there is no constant $\epsilon > 0$ for which the b-chromatic number can be approximated within a factor of $120/133$-ϵ in polynomial time (unless P=NP).

The b-chromatic number of the Cartesian product of general graphs was studied in [16, 17] and of complete graphs in [5]. In the series of papers [7, 8, 9], Effantin and Kheddouci studied powers of paths, cycles, complete binary trees, and complete caterpillars. In [12] b-perfect graphs were introduced and studied; paper [18] gives several bounds on $\varphi(G)$ in terms of the clique number and the clique partition number of G; and in [2] the b-spectrum is discussed. More precisely, we wish to determine the set of possible values k from the interval $[\chi(G), \varphi(G)]$ for which a given graph G admits a b-coloring with k colors. Very recently, b-colorings was extensively studied on different families of graphs: Cographs and P_4-sparse graphs are studied in [4], Kneser graphs in [14], vertex-deleted subgraphs in [11], and Mycielskians in [1].

The following problem and related results intrigued our attention. In [10] it was asked whether it is true that every d-regular graph G with girth at least 5 satisfies $\varphi(G) = d + 1$? C_4 ([10]) is a counterexample for $d = 2$, while $K_3 \,\Box\, K_2$ ([10]), and the Petersen graph ([3] and this paper) are counterexamples for $d = 3$. The question was solved in affirmative for (i) d regular graphs with girth at least 6 [15]; (ii) d-regular graphs with girth 5 and no C_6 [10]; (iii) d-regular graphs different from the Petersen graph with girth at least 5 and $d \leq 6$ [3]. Moreover, earlier Kratochvíl, Tuza, and Voigt [19] proved that if a d-regular graph G has at least d^4 vertices, then $\varphi(G) = d + 1$. It follows from their result that for any d, there is only a finite number of d-regular graphs G with $\varphi(G) \leq d$.

So what about cubic graphs? If their girth is at least 6 (at least 5 if the Petersen graph is excluded) or have at least 81 vertices, then by the above their b-chromatic number is 4. On the other hand, we checked the prominent example of cubic graphs—the Petersen graph P—and found that $\varphi(P) = 3$. Are there more such exceptions and is it possible to find them all? The answer is affirmative since in this paper we prove the following:

Theorem 1.1 *Let G be a connected cubic graph. Then $\varphi(G) = 4$ unless G is P, $K_3 \,\square\, K_2$ (Fig. 2), $K_{3,3}$, or G_1 (Fig. 1). In these cases, $\varphi(P) = \varphi(K_3 \,\square\, K_2) = \varphi(G_1) = 3$ and $\varphi(K_{3,3}) = 2$.*

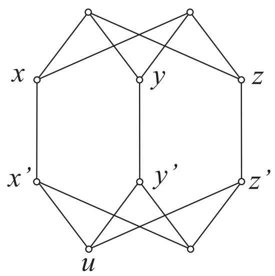

Figure 1: Graph G_1

The proof of Theorem 1.1 is constructive and leads to an algorithm that finds appropriate colorings. In fact, at a later stage we have also verified the result using computer.

The paper is organized as follows. In the next section we determine the b-chromatic number of the four sporadic cases, while in the last section we prove that $\varphi(G) = 4$ for any other cubic graph G. We conclude with three open problems.

2 The four exceptional graphs

In this section we cover the b-chromatic number of the four exceptional graphs from Theorem 1.1. We first note that it is well-known [13] and easy to see that $\varphi(K_{3,3}) = 2$. Consider next the graph $K_3 \,\square\, K_2$, that is, the prism over K_3, see Fig. 2.

Suppose $K_3 \,\square\, K_2$ has a 4-b-coloring. Color a triangle of $K_3 \,\square\, K_2$ with three colors, say 1, 2, and 3. Then the remaining color 4 can be used only once, see Fig. 2 again. Since the vertex colored 4 needs all colors on its neighbors, the remaining two vertices must receive colors 2 and 3. But then none of the colors 2 and 3 is realized, hence

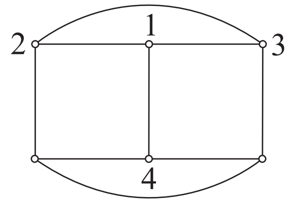

Figure 2: Graph $K_3 \,\square\, K_2$

$\varphi(K_3 \,\square\, K_2) \leq 3$. The chromatic number of $K_3 \,\square\, K_2$ is 3, hence $\varphi(K_3 \,\square\, K_2) = 3$, see also [10].

Proposition 2.1 *Let G_1 be the graph from Fig. 1. Then $\varphi(G_1) = 3$.*

Proof. Clearly, there are only two types of vertices in this graph. Hence it is enough to consider the following two cases.

Suppose first that one of the colors is realized on the vertex u (see Fig. 1). We may without loss of generality assume that u is colored with 1. Then its neighbors are colored with 2, 3, 4. The remaining vertex of the $K_{2,3}$ containing u must then be colored 1. It follows that none of the three vertices already colored with 2, 3, 4 realizes its color. By the symmetry, if some color is realized on a vertex at distance 3 from u, then only two colors are realized. It follows that colors 2, 3, 4 must be realized on the vertices x, y, z, but this is not possible since the two vertices at distance 3 from u must in that case receive color 1.

Assume now that all the colors are realized on vertices x, y, z, x', y', z'. As above we infer that at most two colors can be realized on x', y', z'. Hence, by the symmetry, exactly two are realized on these three vertices. We may assume without loss of generality that these are colors 1 and 2. But then it follows immediately that the colors 1 an 2 must also appear on x, y, z. So at least one of the colors 3 or 4 is not realized. $\square$

The final exception is the Petersen graph. To show that it is not 4-b-colorable, a little more effort is needed. This result also appears in [3]. We do include a proof here because (i) at the time our paper was submitted we were not aware of [3] and (ii) in this way the proof of the main result of this paper will be self-contained.

Proposition 2.2 *Let P be the Petersen graph. Then $\varphi(P) = 3$.*

Proof. Since $3 = \chi(P) \leq \varphi(P) \leq \Delta(P)+1 = 4$, we have to prove that the Petersen graph does not have a 4-b-coloring. Suppose it has. Let $V(P) = X \cup Y$ be a partition of $V(P)$ where $X = \{v_1, v_2, v_3, v_4\}$ is a set of vertices that realize the four colors. We may assume that v_i is colored i and that it is adjacent to at least one vertex colored j, $j \in \{1,2,3,4\} \backslash \{i\}$. Then $Y = V(P) \backslash X$. By abuse of language, X will also denote the subgraph of P induced by X.

Case 1. X is an independent set.
Since Y has six elements there are at least two colors that are used only once in Y, for example colors 1 and 2 (Fig. 3). Then these two vertices in Y form a square with vertices colored 3 and 4 in X, a contradiction.

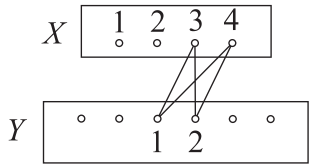

Figure 3: Case 1

Case 2. X contains one edge.
Without lose of generality, assume that K_2 is colored with colors 1 and 2. If colors 3 and 4 are used in Y only once, then one would get a triangle, which is not possible. Colors 3 and 4 must be used twice. But then colors 1 and 2 must be used once in Y. Is this case we get a square as a subgraph of the Petersen graph as seen in the Fig. 4, a contradiction.

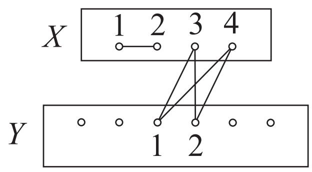

Figure 4: Case 2

Case 3. X induces two independent edges.
If a color is used once in Y we get a triangle. But then colors 1, 2, 3 and 4 must be used at least twice in Y, which is not possible.

Case 4. $X = P_3 + K_1$.

Suppose P_3 is colored with colors 1, 2 and 3. Then color 4 must be used three times in Y or we would get a triangle. The remaining three vertices in Y must receive colors 1, 2 and 3. Every vertex colored 4 in Y must have two neighbors in Y but then we get a square (Fig. 5) as a subgraph.

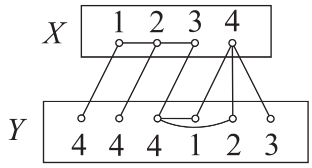

Figure 5: Case 4

Case 5. $X = P_4$.

We may assume that the end vertices of P_4 are colored 1 and 4. If colors 1 or 4 are used once in Y, we get a triangle. But then colors 2 and 3 must be used once. Let y be a vertex colored 4 in Y that has a neighbor in X colored 2. Vertex y can not be adjacent to the vertex colored 3 in Y, because we would get a triangle, and can also not be adjacent to the vertex colored 1 in Y which is adjacent to the vertex colored 3 in X, because we would get a square. Then it must be adjacent to the vertex colored 2 in Y and to the vertex colored 1 in Y which is adjacent to the vertex colored 4 in X (Fig. 6). But then we get another square as a subgraph in the Petersen graph.

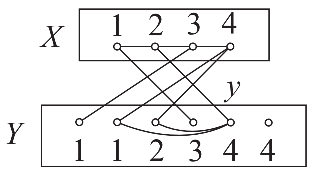

Figure 6: Case 5

Case 6. $X = K_{1,3}$.

Assume the vertex of X of degree three is colored 2. Then colors 1, 3, and 4 must appear at least twice in Y or else one would get a C_4 as a subgraph. Let x_1 be the vertex colored 1 in X, x_3 the vertex colored 3 in X, and x_4 the vertex colored 4 in X. Then x_1 has two neighbors in Y, y_1 and z_1, colored 3 and 4 respectively, x_3 has two neighbors in Y, y_3 and z_3, colored 1 and 4 respectively, and x_4 has two neighbors in Y, y_4 and z_4, colored 1 and 3 respectively. Vertex z_1 has two more neighbors but x_1. Note that z_1 is adjacent to neither y_1 (because P is triangle-free) nor z_3 (as they are both colored 4). Moreover, z_1 is adjacent to at most one of y_4, z_4 (since P is C_4-free). Hence z_1 must be adjacent to y_3. Similarly we infer that y_1 must be adjacent to y_4 and y_4 to z_3. Now, the only possibility for the third neighbor of z_3 is y_1, but then y_1, z_3, y_4 induce a triangle, a contradiction.

Hence P is not 4-b-colorable and so $\varphi(P) = 3$. □

3 All but four cubic graphs are 4-b-colorable

In this section we demonstrate that all cubic graphs, but the four exceptions treated in the previous section, are 4-b-colorable. The main idea of the proof is to find a partial coloring of a graph that realizes all four colors. Afterwards a coloring can be extended to the whole graph by the standard greedy coloring algorithm. Along the way it will also be clear where the sporadic examples appear.

We will analyze the cases with respect to the girth of a given cubic graph G. Hence let g be its girth and let C be a g-cycle of G. For $i \geq 1$ let

$$D_i = \{v \in V(G) \mid d(v, C) = i\}$$

be the i-th distance level with respect to C.

Case 1: $g = 3$.
In this case C is a triangle. We distinguish subcases with respect to the size of D_1.

Case 1.1: $|D_1| = 1$.
In this subcase $G = K_4$ which is clearly 4-b-colorable.

Case 1.2: $|D_1| = 2$.
Let $D_1 = \{x, y\}$. Note that there is a unique way (up to isomorphism) how x and y are connected with the vertices of C. Assume without loss of generality that x has two neighbors in C (and hence y has one).

Suppose first that x and y are not adjacent. Then x and y have one and two neighbors in D_2, respectively. There are two subcases: either x and y share a common neighbor in D_2 or they don't. The first case is shown on the left-hand side of Fig. 7, the second on its right.

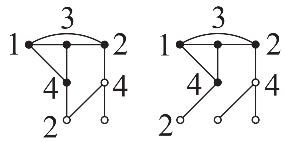

Figure 7: Subgraphs with $g = 3$ and independent set D_1

In both cases a partial 4-b-coloring is given in the figure. Note that in the left coloring there are two vertices colored 2, but since one of them is already of degree 3, the coloring is proper. Analogous arguments apply to color 4 as well as to the coloring on the right. Note that those two vertices are nonadjacent.

Assume next that x and y are adjacent. Then the proof continues as above by considering cases with respect to the structure of the distance levels D_i. The possibilities that need to be considered are shown in Fig. 8.

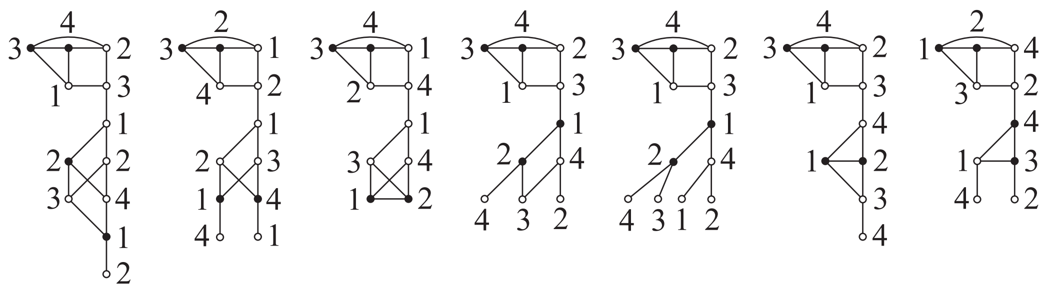

Figure 8: Subgraphs with $g = 3$ and $D_1 = K_2$

Checking the details is straightforward but quite tedious, hence we leave this task (here and later in the proof) to the reader. Since all the cases that have to be considered, together with required local 4-b-colorings, are presented in figures, the checking should not be difficult. Moreover, in each of the colorings the vertices that realize colors are filled. The dashed lines indicate that the corresponding edge is not present in that case.

Case 1.3: $|D_1| = 3$.

In this subcase each of the vertices from C has its own private neighbor in D_1. If D_1 induces an independent set then we color the triangle with 1, 2, 3 and all the vertices of D_1 with 4 and we are done.

Suppose next that D_1 induced a graph with one edge. The case analysis is done in Fig. 9.

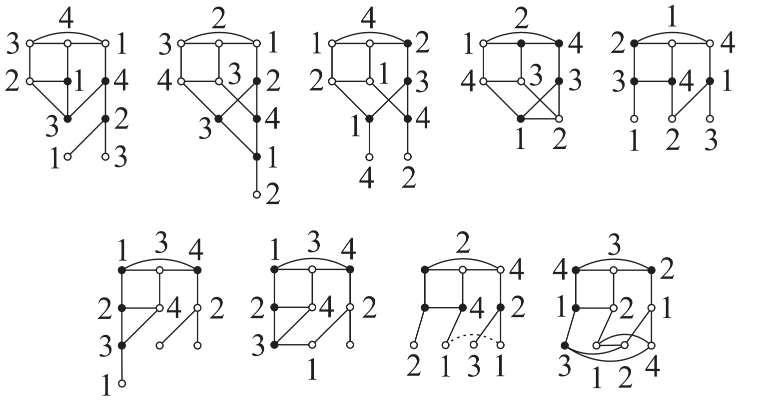

Figure 9: Subgraphs with $g = 3$ and $D_1 = K_2 + K_1$

Assume D_1 induced P_3. The cases are shown in Fig. 10.

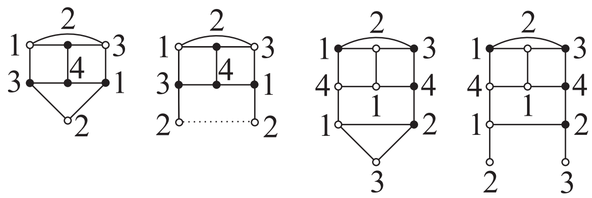

Figure 10: Subgraphs with $g = 3$ and $D_1 = P_3$

Finally, if D_1 induced a triangle, we get the prism over K_3. This settles the case $g = 3$.

Case 2: $g = 4$.

Now C is a square. Once more we distinguish subcases with respect to the size of D_1. It is obvious that the case $|D_1| = 1$ is not possible.

Case 2.1: $|D_1| = 2$.

The cases when the vertices from D_1 are not adjacent are presented in Fig. 11.

When $D_1 = K_2$ we get $K_{3,3}$, see Fig. 12.

Case 2.2: $|D_1| = 3$.

Let D_1 be an independent set. The cases to be analyzed are drawn on Fig. 13. Note that as one of the subcases the sporadic example G_1 appears.

When $D_1 = K_2 + K_1$ we get cases as shown in Fig. 14.

For the subcase when $D_1 = P_3$ see Fig. 15.

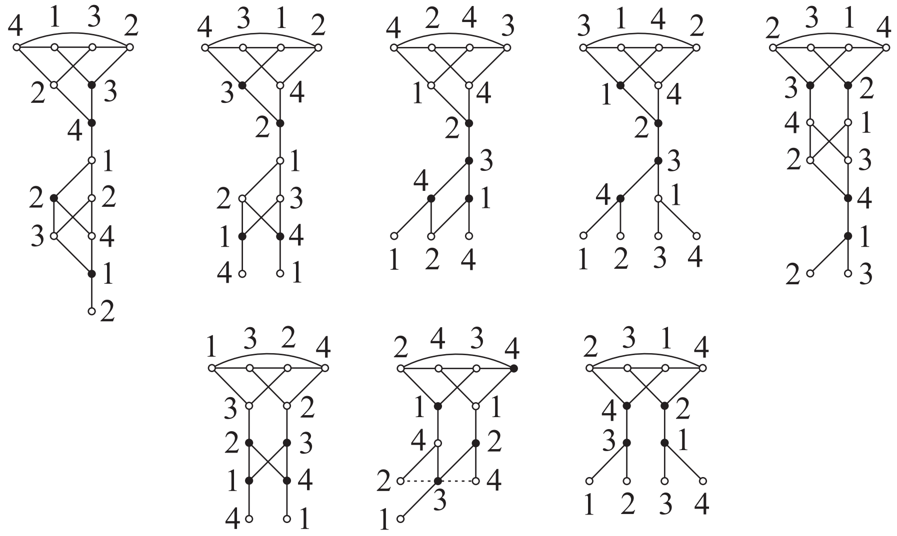

Figure 11: Subgraphs with $g = 4$ and independent set D_1

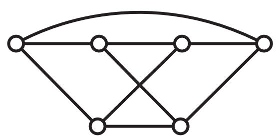

Figure 12: Graph $K_{3,3}$

Case 2.3: $|D_1| = 4$.

This last subcase is easy to handle since we get a subgraph shown in Fig. 16 that can be easily 4-b-colored.

Case 3: $g = 5$.

Let $C = x_1x_2x_3x_4x_5$. Since G is cubic (of girth 5), any x_i $(1 \leq i \leq 5)$ has a private neighbor y_i, see the left-hand side of Fig. 17.

Note that y_1 cannot be adjacent to y_2 or y_5 since the girth is 5. If y_1 is not adjacent to both y_3 and y_4, we get one of the subgraphs shown in Fig. 17. The given coloring shows that $\varphi(G) = 4$ in this case. (For instance, the two vertices of degree one and color 1 are not adjacent by the girth assumption.) Note that the argument applies to any of the vertices y_i. Hence the only remaining case is when any y_i has neighbors y_{i+2} (mod5) and y_{i+3} (mod5). This gives the Petersen graph.

Case 4: $g \geq 6$.

Every such graph contains a path on 6 vertices $x_1x_2x_3x_4x_5x_6$ (for $n = 6$, it may in-

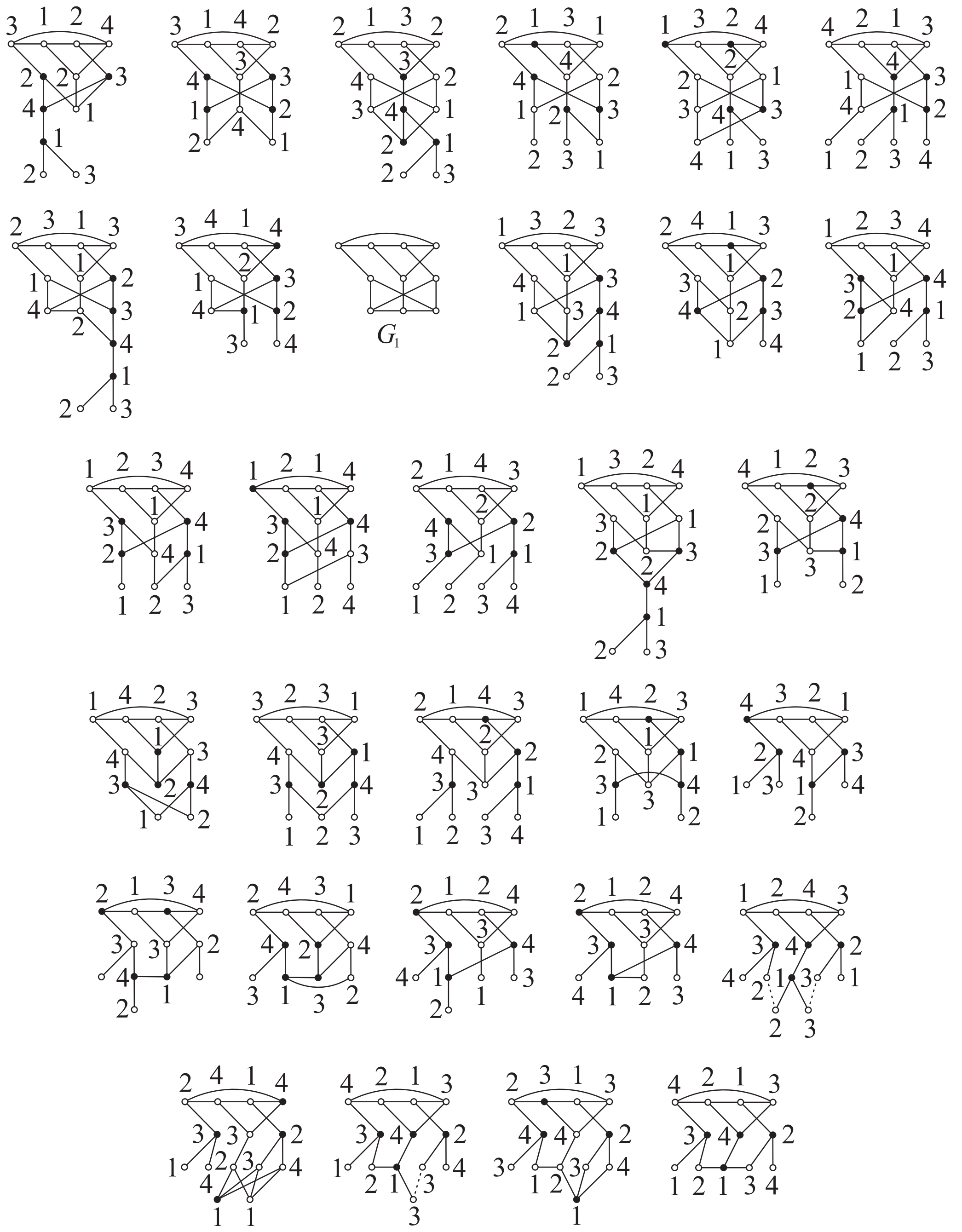

Figure 13: Subgraphs with $g = 4$ and with independent D_1

duce a cycle) with additional vertices $y_1, y_2, y_3, y_4, y_5, y_6$, such that any x_i is adjacent to y_i, see Fig. 18.

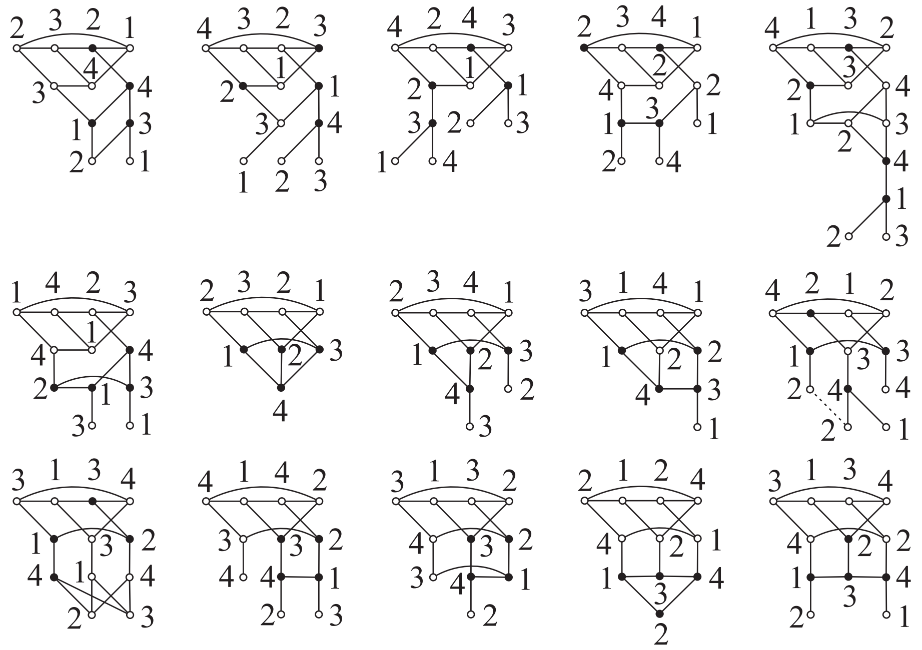

Figure 14: Subgraphs with $g = 4$ and $D_1 = K_2 + K_1$

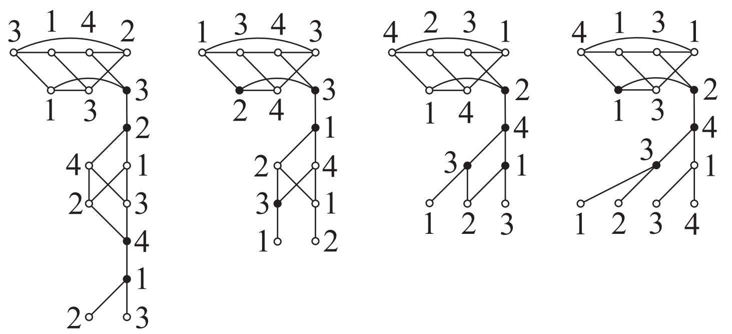

Figure 15: Subgraphs with $g = 4$ and $D_1 = P_3$

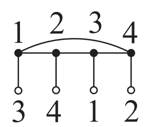

Figure 16: Subgraph with $g = 4$ and $|D_1| = 4$

Furthermore, since $g \geq 6$, y_1 can only be adjacent to y_4. This subgraph can be b-colored with 4 colors as shown in Fig. 18. This completes the argument. $\square$

We add that Cases 3 and 4 were also proved in [3, 15].

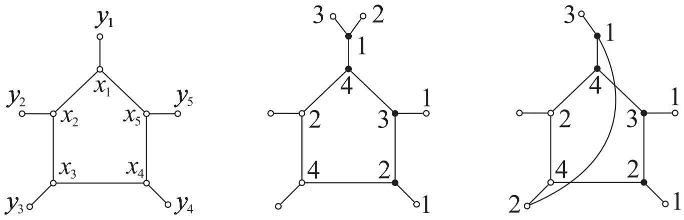

Figure 17: 4-b-colorings of two girth size 5 subgraphs

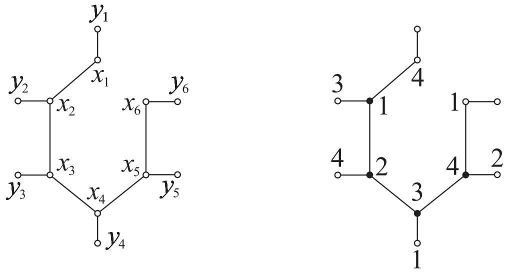

Figure 18: 4-b-coloring of a girth size 6 subgraph

4 Concluding remarks

As we have noticed, a result of Kratochvíl, Tuza, and Voigt from [19] implies that for any d there is only a finite number of d-regular graphs G with $\varphi(G) \leq d$. In this paper we have proved that for $d = 3$ this number is 4. It is hence natural to ask if it would be possible to make the list of 4-regular (or higher regular) graphs G with $\varphi(G) \leq 4$.

Another related question is the following. Let

$$f(d) = |\{G \mid G \text{ is } d\text{−regular}, \ \varphi(G) \leq d\}| \,.$$

What is the growth rate of f?

To conclude we ask whether the bound d^4 from [19] that guarantees that for a d-regular graph G on at least d^4 vertices we have $\varphi(G) = d + 1$ could be lowered?

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