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The chromatic numbers of graph bundles over cycles ☆

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Abstract

Graph bundles generalize the notion of covering graphs and products of graphs. The chromatic numbers of product bundles with respect to the Cartesian, strong and tensor product whose base and fiber are cycles are determined.

1. Introduction

If G is a graph, $V(G)$ and $E(G)$ denote its vertex and edge set, respectively. The *Cartesian product* $G \square H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ and $(a,x)(b,y) \in E(G \square H)$ whenever $ab \in E(G)$ and $x=y$, or $a=b$ and $xy \in E(H)$. The *tensor product* $G \times H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ and $(a,x)(b,y) \in E(G \times H)$ whenever $ab \in E(G)$ and $xy \in E(H)$. The *strong product* $G \boxtimes H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ and $E(G \boxtimes H) = E(G \times H) \cup E(G \square H)$.

Graph bundles [11, 12] generalize the notion of covering graphs and Cartesian products of graphs. The notion follows the definition of fiber bundles and vector bundles that became standard objects in topology [5] as space which locally look like a product. Graph bundles corresponding to arbitrary graph products were introduced in [12, 10]. We refer to [8, 10, 12] for definitions and basic results.

In this paper we will only consider graph bundles over cycles. They can be represented as described below. (Here we take this description as a definition.) Let $\circ$ be a Cartesian, tensor, or a strong product operation. Let C_l, $l \geqslant 3$, be a cycle with consecutive vertices $v_0, v_1, \ldots, v_{l-1}$ and let P_l be the path obtained from C_l by

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removing the edge v_0v_{l-1}. Let F be an arbitrary graph and $\varphi \in Aut(F)$ an automorphism of F. Finally, let $C_l \circ^{\varphi} F$ be the graph obtained from the product $P_l \circ F$ by adding a 'copy' of $K_2 \circ F$ between vertex sets $\{v_0\} \times V(F)$ and $\{v_{l-1}\} \times V(F)$ such that if $V(K_2) = \{1,2\}$ and $(1,x)$ is adjacent to $(2,y)$ in $K_2 \circ F$, then (v_0, x) and $(v_{l-1}, \varphi(y))$ are connected by an edge in $C_l \circ^{\varphi} F$. The graph $C_l \circ^{\varphi} F$ is called the *∘-bundle* with *base* C_l and *fiber* F. The natural projection $p: C_l \circ^{\varphi} F \to C_l$ is called the *bundle projection*. The preimage $p^{-1}(v_i)$ is a *fiber* over v_i. If ∘ is the Cartesian or the strong product, then every fiber is isomorphic to F. In case of the tensor product it is equal to the independent vertex set $\{v_i\} \times V(F)$. As a special case of a bundle we have the product, $C_l \circ^{\mathrm{id}} F = C_l \circ F$. Another special case is when F is discrete (graph without edges). Then we get covering graphs over cycles.

In the rest of the paper, let X be a bundle whose base graph is a cycle C_l, $l \geqslant 3$. We will denote the consecutive vertices of C_l by $v_0, v_1, \ldots, v_{l-1}$. Similarly, if the fiber is the cycle C_s, we let $u_0, u_1, \ldots, u_{s-1}$ be the consecutive vertices on C_s.

Automorphisms of a cycle C_s are of two types. A cyclic shift of the cycle by t elements will be briefly called the *cyclic t-shift*, $0 \leqslant t < s$. It maps u_i to u_{i+t} (index modulo s). As a special case we have the identity ($t=0$). Other automorphisms of C_s are called *reflections*. For example, if φ is a cyclic shift, then the Cartesian bundle $C_l \circ^{\varphi} C_s$ can be represented as a quadrilateral tessellation of the torus (as shown by an example in Fig. 1 with φ a cyclic 1-shift). Similarly, $C_l \circ^{\varphi} C_s$ is a quadrangulation of the Klein bottle when φ is reflection.

Let ∘ be the Cartesian, strong or tensor graph product operation. The following facts will be used several times. The bundle $C_l \circ^{\varphi} C_s$, where φ is a cyclic t-shift, $0 \leqslant t < s$, is isomorphic to the bundle $C_l \circ^{\psi} C_s$, where ψ is the cyclic $(s-t)$-shift. It is also easy to see that all bundles $C_l \circ^{\varphi} C_s$ with φ a reflection are mutually isomorhic if s is odd. If s is even, we have two isomorphism classes of $C_l \circ^{\varphi} C_s$, depending on whether φ has two or no fixed vertices.

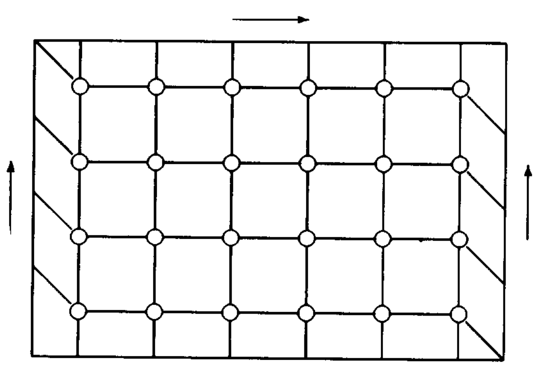

Fig. 1. $C_6 \circ^{\varphi} C_4$ on the torus.

An *n-coloring* of a graph G is a function f from $V(G)$ to $\{1, 2, \ldots, n\}$, such that $xy \in E(G)$ implies $f(x) \neq f(y)$. The smallest number n for which an n-coloring of G exists is the *chromatic number* $\chi(G)$ of G. Let $\chi(G) = n$. Then G is called *n-critical* if $\chi(G - v) < n$ for every $v \in V(G)$.

Many results are known about the chromatic number of strong products of graphs, see [7, 14–16] and also [2, 3,6]. Several general results about the chromatic numbers of Cartesian, strong and tensor bundles of graphs were obtained in [8]. In this paper we determine the chromatic numbers of product bundles whose base and the fiber are cycles.

At the beginning we establish a simple result which holds for all three considered products.

Lemma 1.1. *For every n, $s \geqslant 3$, $k \geqslant 0$, and $\varphi \in Aut(C_s)$ we have*

$$\chi(C_{s+2k} \circ^{\varphi} C_n) \leqslant \chi(C_s \circ^{\varphi} C_n).$$

Proof. Let c be an m-coloring of the bundle $C_s \circ^{\varphi} C_n$. Then we get an m-coloring of $C_{s+2k} \circ^{\varphi} C_n$, simply by repeating $k-s$ times the coloring of the fibers over v_{s-2} and v_{s-1}, respectively, to the fibers over $v_s, v_{s+1}, \ldots, v_{s+2k-1}$. □

2. Cartesian bundles

In this section we will determine chromatic numbers of Cartesian bundles over cycles whose fiber is a cycle. The results are collected in Table 1, where, for example, 'even/odd' means that we consider Cartesian bundles with even cycle as base and an odd cycle as fiber. '0-refl.', '1-refl.' and '2-refl.' denote a reflection without fixed points, with one fixed point and with two fixed points, respectively.

First two lines of Table 1 are proved in Proposition 2.1. The remaining shifts are demonstrated in Theorem 2.2 and the 1-reflections are settled in Theorem 2.3. At the end of the section we also prove that the latter graphs are χ-critical.

Proposition 2.1. *The first two lines of Table 1 are correct.*

Proof. Let the base graph be an odd cycle and assume $\chi(C_{2k+1} \square^{\varphi} C_{2n}) = 2$. If p is the bundle projection, then we may suppose that $p^{-1}(v_0)$ is colored $1, 2, \ldots, 1, 2$. Hence

Table 1
Chromatic numbers of Cartesian bundles.

Base/fiber\\φ	Even shift	Odd shift	0-refl.	1-refl.	2-refl.
Even/even	2	3	3	–	2
Odd/even	3	2	2	–	3
Even/odd	3	3	–	4	–
Odd/odd	3	3	–	4	–

$p^{-1}(v_1)$ must be colored $2,1,\ldots,2,1$, and by induction, $p^{-1}(v_{2k})$ is colored $1,2,\ldots,1,2$. It is clear that we have a coloring of $C_{2k+1} \square^{\varphi} C_{2n}$ if and only if φ is an odd shift or a reflection without fixed points. Furthermore, if we replace color 1 in $p^{-1}(v_0)$ with color 3 and color 2 in $p^{-1}(v_{2k})$ with 3 we get a 3-coloring when φ is an even shift or a reflection with two fixed points.

The case with an even base is proved analogously. □

We will several times explicitly give colorings of particular graph bundles. Given a graph bundle $C_k \square^{\varphi} C_n$, the vertices of the ith fiber $p^{-1}(v_i)$ will correspond to the ith row of a color matrix and the copies of the vertex u_j of the fiber C_n will be in the jth column. Finally, we will add an additional 'bottom' row with colors corresponding to $p^{-1}(v_0)$ shifted or reflected according to φ. This row will be denoted v_0'. Then the vertical and the horizontal adjacencies can be easily seen.

Theorem 2.2. *Let φ be a cyclic shift and let $n \geqslant 1$. Then for every $k \geqslant 3$, $\chi(C_k \square^{\varphi} C_{2n+1}) = 3$.*

Proof. Sabidussi [13] showed that $\chi(G \square H) = \max\{\chi(G), \chi(H)\}$. Hence, the result holds if φ is trivial. It is therefore enough to prove it for all odd cyclic shifts.

Let φ be $(2s+1)$-cyclic shift, $0 \leqslant s < n$. Let c be an assignment of colors to $C_4 \square^{\varphi} C_{2n+1}$ defined by the following color matrix:

	u_0	u_1	u_2	u_3	$\cdots$	u_{2s}	u_{2s+1}	u_{2s+2}	u_{2s+3}	$\cdots$	u_{2n-2}	u_{2n-1}	u_{2n}
v_0	1	2	1	2	$\cdots$	1	2	1	2	$\cdots$	1	2	3
v_1	2	3	2	3	$\cdots$	2	3	2	3	$\cdots$	2	3	1
v_2	1	2	1	2	$\cdots$	1	2	3	1	$\cdots$	3	1	3
v_3	2	1	2	1	$\cdots$	2	3	1	2	$\cdots$	1	2	1
v_0'	1	2	1	2	$\cdots$	3	1	2	1	$\cdots$	2	1	2

Let P be the path on vertices v_0, v_1, v_2 and v_3. Then the subgraph $P \square^{\varphi} C_{2n+1}$ is identical with the Cartesian product $P \square C_{2n+1}$ and is clearly properly colored. Lemma 1.1(ii) completes the proof for the case when k is even.

Let φ be cyclic $(2s+1)$-shift, $0 \leqslant s < n$. Let c be an assignment of colors to $C_3 \square^{\varphi} C_{2n+1}$defined by the following color matrix:

	u_0	u_1	$\cdots$	u_{2s-2}	u_{2s-1}	u_{2s}	u_{2s+1}	u_{2s+2}	$\cdots$	u_{2n-1}	u_{2n}
v_0	1	2	$\cdots$	1	2	1	2	1	$\cdots$	2	3
v_1	3	1	$\cdots$	3	1	3	1	2	$\cdots$	1	2
v_2	2	3	$\cdots$	2	3	1	2	1	$\cdots$	2	3
v_0'	1	2	$\cdots$	1	2	3	1	2	$\cdots$	1	2

(If $s=0$, the coloring in the column of u_{2s} is used for the first column.) This defines a coloring of $C_3 \square^{\varphi} C_{2n+1}$. Lemma 1.1(i) completes the proof for odd values of k. □

Theorem 2.3. *Let φ be a reflection. Then for any $k \geqslant 3$ and $n \geqslant 1$ we have*

$$\chi(C_k \square^{\varphi} C_{2n+1}) = 4.$$

Proof. Let us consider a 3-coloring $c: V(C_{2n+1}) \to \{1,2,3\}$ of the fiber graph C_{2n+1}. We assume that the edges of C_{2n+1} are oriented from v_i to v_{i+1} (index modulo $2n+1$), $i=0,1,\ldots,2n$. Define $h(v_i)=1$ if $c(v_{i+1})=c(v_i)+1$ (mod 3), and let $h(v_i)=-1$ otherwise. Let

$$W(c) = \sum_{i=0}^{2n} h(v_i).$$

We call $W(c)$ the *winding number* of the 3-coloring c. Since $W(c)$ is the sum of $2n+1$ odd numbers (± 1), it is odd and thus non-zero.

Suppose that we have a 3-coloring of $K_2 \square C_{2n+1}$. It induces 3-colorings, c and c', of the two fibers. We claim that $W(c)=W(c')$. This follows easily by the following observation. Consider a vertex v_i with $h(v_i) \neq h'(v_i)$, where h and h' are the h-functions of c and c', respectively. Say, $h(v_i)=1$, $h'(v_i)=-1$. Then $c(v_i)=c'(v_{i+1})$ and $c(v_{i+1})=c'(v_i)$. Now, if $h(v_{i+1})=h'(v_{i+1}),\ldots,h(v_{i+t})=h'(v_{i+t})$, $t \geqslant 0$, and $h(v_{i+t+1}) \neq h'(v_{i+t+1})$ (indices modulo $2n+1$), then $h(v_{i+t+1})=-1$ and $h'(v_{i+t+1})=1$.

Finally, having a 3-coloring of $C_k \square^{\varphi} C_{2n+1}$, and assuming that the induced coloring c on the first fiber has $W(c) \geqslant 0$, all other 3-colorings of fibers have positive winding number. But since φ is reflection, the copy of $K_2 \square C_{2n+1}$ between the last and the first fiber reverses the orientation of C_{2n+1}, thus turning a positive winding number into the negative. This contradicts the observed property that $W(c) \neq 0$.

As it is easy to find 4-colorings of $C_k \square C_{2n+1}$ (cf. the following theorem), the proof is complete. □

Theorem 2.4. *Let φ be a reflection. Then for any $k \geqslant 3$, $n \geqslant 1$ and any vertex $u \in V(C_k \square^{\varphi} C_{2n+1})$,*

$$\chi((C_k \square^{\varphi} C_{2n+1}) \setminus \{u\}) = 3.$$

In other words, $C_k \square^{\varphi} C_{2n+1}$ are 4-critical graphs.

Proof. We will show the theorem for the case when k is even, since the odd case can be treated similarly. By an obvious modification of Lemma 1.1(ii) it is enough to prove our claim for $k=4$. We may, without loss of generality, also assume that $u \in p^{-1}(v_0)$.

Let $\varphi(u_n)=u_n$ and let $u=u_{n-t}$, $t\geqslant 0$. Suppose that t is odd and consider the following color matrix, where the bullet $\bullet$ stands instead of the vertex u.

$$\begin{array}{cccccccccccccccccccc}
v_0 & \cdots & 3 & 2 & 3 & 2 & \bullet & \cdots & 1 & 2 & 1 & 2 & 1 & \cdots & 2 & 3 & 2 & 3 & 2 & \cdots \\
v_1 & \cdots & 1 & 3 & 1 & 3 & 1 & \cdots & 2 & 1 & 2 & 1 & 2 & \cdots & 3 & 1 & 3 & 1 & 3 & \cdots \\
v_2 & \cdots & 3 & 1 & 3 & 1 & 3 & \cdots & 1 & 2 & 1 & 2 & 1 & \cdots & 2 & 3 & 1 & 3 & 1 & \cdots \\
v_3 & \cdots & 1 & 2 & 1 & 2 & 1 & \cdots & 2 & 1 & 2 & 1 & 2 & \cdots & 3 & 1 & 2 & 1 & 2 & \cdots \\
v'_0 & \cdots & 2 & 3 & 2 & 3 & 2 & \cdots & 1 & 2 & 1 & 2 & 1 & \cdots & \bullet & 2 & 3 & 2 & 3 & \cdots \\
 & \multicolumn{6}{c}{\underbrace{\qquad\qquad\qquad}_{n-t+1}} & \multicolumn{4}{c}{\underbrace{\qquad\quad}_{t-1}} & \multicolumn{4}{c}{\underbrace{\qquad\quad}_{t-1}} & \multicolumn{5}{c}{\underbrace{\qquad\qquad\qquad}_{n-t+1}}
\end{array}$$

It is straightforward to see that $(C_4 \square^{\varphi} C_{2n+1})\setminus\{u\}$ is properly colored. The case when t is even can be treated similarly and is left to the reader. $\square$

3. Strong bundles

Observe first that the subbundle over any edge is isomorphic to the strong product $K_2 \boxtimes C_{2n+1}$. As for $n\geqslant 2$, $\chi(K_2 \boxtimes C_{2n+1})=5$, it follows that $\chi(C_l \boxtimes^{\varphi} C_{2n+1})\geqslant 5$. From [8] we recall the following theorem.

Theorem 3.1. (i) *For any strong bundle* $X=B \boxtimes^{\varphi} F$, $\chi(X)\leqslant\chi(B)\chi(F)$.
(ii) *For any* $k\geqslant 2$ *and any graph* F, $\chi(C_{2k+1} \boxtimes^{\varphi} F)\leqslant 2\chi(F)+\lceil \chi(F)/k \rceil$.

In the rest of the section we will prove the results in Table 2, where all the cycles are of length at least 4.

More precisely, the first and the third line of Table 2 are covered by Propositions 3.2 and 3.3, respectively. The second line is proved in Theorem 3.4 and the last line in Theorem 3.9. Both theorems also specify when the chromatic number is 4 or 5 and 5 or 6, respectively.

Proposition 3.2. *For any* $k\geqslant 2$ *and any* $n\geqslant 2$, $\chi(C_{2k} \boxtimes^{\varphi} C_{2n})=4$.

Proof. The lower bound is trivial while the upper bound follows from Theorem 3.1(i). $\square$

Table 2
Chromatic numbers of strong bundles

Base/fiber\φ	Even shift	Odd shift	0-refl.	1-refl.	2-refl.
Even/even	4	4	4	–	4
Odd/even	4 or 5	5	5	–	5
Even/odd	5	5	–	5	–
Odd/odd	5	5	–	5 or 6	–

Proposition 3.3. *For any $k \geqslant 2$ and any $n \geqslant 2$, $\chi(C_{2k} \boxtimes^{\varphi} C_{2n+1}) = 5$.*

Proof. It is well known that $\chi(K_2 \boxtimes C_{2n+1}) = 5$. Therefore, by Lemma 1.1(ii), it suffices to consider the case $k=2$. Suppose first that $n \geqslant 3$. Define $(2n+1)$-tuples x and y as

$$x=(1,2,1,2,\ldots,1,2,3) \quad \text{and} \quad y=(4,5,3,4,5,4,5,\ldots,4,5).$$

If p is the bundle projection, then color $p^{-1}(v_0)$ and $p^{-1}(v_2)$ by using x and $p^{-1}(v_1)$ by y.

As $n \geqslant 3$, there exists a vertex w in $p^{-1}(v_3)$, which is adjacent to neither of the vertices colored 3 in $p^{-1}(v_0)$ and $p^{-1}(v_2)$. Color w with 3 and the remaining vertices in $p^{-1}(v_3)$ with 4 and 5, to obtain a 5-coloring of $C_{2k} \boxtimes^{\varphi} C_{2n+1}$, $n \geqslant 3$.

Let $n=2$. Then the following color matrices

v_0	1	2	3	4	5	1	2	3	4	5	1	2	1	3	2
v_1	4	5	1	2	3	3	4	5	1	2	3	5	4	5	4
v_2	1	2	3	4	5	5	1	2	3	4	2	1	3	2	1
v_3	3	4	5	1	2	2	3	4	5	1	4	5	4	5	3
v_0'	5	1	2	3	4	4	5	1	2	3	2	3	1	2	1

give 5-colorings for φ being the cyclic 1-shift, the cyclic 2-shift, and the reflection, respectively. For the reflection we have assumed that $\varphi(u_2) = u_2$. The first color matrix (with the changed row of v_0') also gives a 5-coloring in case when φ is the identity. □

Next we consider strong bundles over odd cycles. The first case is when the fiber is an even cycle.

Theorem 3.4. *For $k \geqslant 2$ and $n \geqslant 2$,*

$$\chi(C_{2k+1} \boxtimes^{\varphi} C_{2n}) = \begin{cases} 4, & \varphi \text{ is a cyclic } 2s\text{-shift},\ 1 \leqslant s \leqslant n, \text{ and} \\ & n/\gcd(n,s) \text{ is even}, \\ 5, & \text{otherwise}. \end{cases}$$

Theorem 3.4 will be proved by a series of four lemmas.

Lemma 3.5. *Let $\chi(C_{2k+1} \boxtimes^{\varphi} C_{2n}) = 4$ and let c be a 4-coloring. Denote by p the corresponding bundle projection. Then there exists an index i, $0 \leqslant i \leqslant 2k$, such that $p^{-1}(v_i)$ has at least 3 distinct colors.*

Proof. Assume, on the contrary, that $|c(p^{-1}(v_i))| = 2$, $i = 0, 1, \ldots, 2k$. We may suppose that $p^{-1}(v_0)$ is colored $1,2,\ldots,1,2$. It follows that $p^{-1}(v_{2k})$ must be colored either $1,2,\ldots,1,2$ or $2,1,\ldots,2,1$, a contradiction for any φ. □

Lemma 3.6. *Let φ be a cyclic shift of odd length. Then for any $k \geqslant 2$ and any $n \geqslant 2$, $\chi(C_{2k+1} \boxtimes^{\varphi} C_{2n}) = 5$.*

Proof. The upper bound follows from Theorem 3.1(ii).

To prove the lower bound, assume that $\chi(C_{2k+1} \boxtimes^{\varphi} C_{2n})=4$ and let c be a 4-coloring. Let $w_0, w_1, \ldots, w_{2n-1}$ be consecutive vertices of the fiber $p^{-1}(v_0)$. According to Lemma 3.5, we may assume that $|c(p^{-1}(v_0))| \geqslant 3$. Hence, there are 3 consecutive vertices in $p^{-1}(v_0)$ with pairwise different colors. Assume, without loss of generality, $c(w_0)=1$, $c(w_1)=2$ and $c(w_2)=3$. Then the vertical layers corresponding to u_0, u_1 and u_2 must be colored $1,3,1,3,\ldots,3,1$; $2,4,2,4,\ldots,4,2$ and $3,1,\ldots,1,3$, respectively. It follows by induction that in the vertical layer corresponding to u_{s+1} there are only colors 1 and 3. Here s is the length of the cyclic shift φ. Consider the vertex w in the fiber $p^{-1}(v_{2k})$ which corresponds to u_{s+1}. Then $c(w) \in \{1,3\}$. However, w is adjacent to w_0 and w_2, a contradiction. $\square$

Lemma 3.7 *Let φ be reflection. Then $\chi(C_{2k+1} \boxtimes^{\varphi} C_{2n})=5$, for every $k, n \geqslant 2$.*

Proof. The upper bound follows from Theorem 3.1(ii).

As the fiber is an even cycle, there are two non-isomorphic bundles with φ a reflection. If φ has no fixed points on C_{2n}, then it interchanges two of the adjacent vertices of C_{2n}. The corresponding layers induce a subgraph isomorphic to $K_2 \boxtimes C_{2k+1}$ whose chromatic number is equal to 5. Hence the lower bound. Suppose φ has fixed points. Then using Lemma 3.5 and arguments from the proof of Lemma 3.6 we see that the bundle cannot be 4-colored. $\square$

The next lemma will complete the proof of Theorem 3.4.

Lemma 3.8. *Let φ be a 2s-shift, $1 \leqslant s \leqslant n$. Then for any $k \geqslant 2$ and any $n \geqslant 2$,*

$$\chi(C_{2k+1} \boxtimes^{\varphi} C_{2n}) = \begin{cases} 4, & n/\gcd(n,s) \text{ is even,} \\ 5, & \text{otherwise.} \end{cases}$$

Proof. Assume $\chi(C_{2k+1} \boxtimes^{\varphi} C_{2n})=4$. Let c and $w_0, w_1, \ldots, w_{2n-1}$ be as in the proof of Lemma 3.6, and let $c(w_0)=1$, $c(w_1)=2$ and $c(w_2)=3$. It follows that $c(w_{2s})=3$, $c(w_{2s+1})=4$ and $c(w_{2s+2})=1$, as well as $c(w_{4s})=1$, $c(w_{4s+1})=2$ and $c(w_{4s+2})=3$. Throughout all of the proof indices are modulo $2n$. In general, for an odd $k \geqslant 1$, $c(w_{2sk})=3$, $c(w_{2sk+1})=4$ and $c(w_{2sk+2})=1$ while for an even $k \geqslant 0$, $c(w_{2sk})=1$, $c(w_{2sk+1})=2$ and $c(w_{2sk+2})=3$. Let $k'>0$ be the smallest number such that $2sk' \equiv 0 \pmod{2n}$. It is not hard to see (cf., e.g., [1, Theorem 13.4]) that

$$k' = \frac{\operatorname{lcm}(2n,2s)}{2s} = \frac{\operatorname{lcm}(n,s)}{s} = \frac{n}{\gcd(n,s)}.$$

If k' is odd, then $c(w_{2sk'})=c(w_0)=3$, a contradiction.

Assume k' is even. It follows that n must be even. Let $s=2^p r$, where r is odd and $p \geqslant 0$. As k' is even,

$$2s=2^{p+1}r \quad \text{and} \quad 2n=2^{p+2}t$$

for some $t \geqslant 0$. For $p \geqslant 0$ we now define the color matrix A of dimension $(2k+1) \times 2^{p+2}$ in the following way:

$$\underbrace{\begin{matrix} 1 & 2 & 3 & 4 & \cdots & 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 & \cdots & 3 & 4 & 1 & 2 \\ 1 & 2 & 3 & 4 & \cdots & 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 & \cdots & 3 & 4 & 1 & 2 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 2 & 3 & 4 & \cdots & 1 & 2 & 3 & 4 \end{matrix}}_{2^{p+1}} \quad \underbrace{\begin{matrix} 3 & 4 & 1 & 2 & \cdots & 3 & 4 & 1 & 2 \\ 1 & 2 & 3 & 4 & \cdots & 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 & \cdots & 3 & 4 & 1 & 2 \\ 1 & 2 & 3 & 4 & \cdots & 1 & 2 & 3 & 4 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ 3 & 4 & 1 & 2 & \cdots & 3 & 4 & 1 & 2 \end{matrix}}_{2^{p+1}}$$

(For $p=0$ take the first two columns in each half). Let c be an assignment of colors to $C_{2k+1} \boxtimes^{\varphi} C_{2^{p+2}t}$, defined by the horizontal sequence of t color matrices A. As we consider a $(2^{p+1}r)$-shift and r is odd, c is a 4-coloring. □

The final case to consider when the base and the fiber are odd.

Theorem 3.9. *For $k \geqslant 2$ and $n \geqslant 2$,*

$$\chi(C_{2k+1} \boxtimes^{\varphi} C_{2n+1}) = \begin{cases} 6, & n=2 \text{ and } \varphi \text{ is a reflection;} \\ 6, & n=3,\ k=2, \text{ and } \varphi \text{ is a reflection;} \\ 5, & \text{otherwise.} \end{cases}$$

The theorem will be proved by the next three lemmas.

Lemma 3.10. *Let φ be a reflection. Then*

(i) *for* $k \geqslant 2$, $\chi(C_{2k+1} \boxtimes^{\varphi} C_5) = 6$,
(ii) $\chi(C_5 \boxtimes^{\varphi} C_7) = 6$,
(iii) *for* $n \geqslant 4$, $\chi(C_5 \boxtimes^{\varphi} C_{2n+1}) = 5$,
(iv) *for* $k, n \geqslant 3$, $\chi(C_{2k+1} \boxtimes^{\varphi} C_{2n+1}) = 5$.

Proof. (i) Assume that $\chi(C_{2k+1} \boxtimes^{\varphi} C_5) = 5$ and let c be a 5-coloring. Let p be the bundle projection. We distinguish three cases.

Case 1: There is a fiber which uses only 3 colors. We may assume that $c(p^{-1}(v_0)) = \{1,2,3\}$ and that the color 3 occurs exactly once. It follows that $c(p^{-1}(v_1))$ has neither color 1 nor color 2 and that 3 occurs exactly once. Hence $c(p^{-1}(v_1)) = \{3,4,5\}$, where the color 3 occurs exactly once. By induction, $c(p^{-1}(v_{2k})) = \{1,2,3\}$, a contradiction.

Case 2: There is a fiber which uses 4 colors. We may assume that $c(p^{-1}(v_0)) = \{1,2,3,4\}$ and that the color 4 occurs twice. Then $c(p^{-1}(v_1)) = \{1,2,3,5\}$, where 5 occurs twice. By induction, $c(p^{-1}(v_{2k})) = \{1,2,3,4\}$, a contradiction.

Case 3: Each fiber uses 5 colors. We may assume that $c(p^{-1}(v_0)) = \{1,2,3,4,5\}$. It is easy to see that there are only two ways to color $p^{-1}(v_1)$: the cyclic shift of colors by 2, or by 3. Therefore, $p^{-1}(v_{2k})$ must be colored by a cyclic shift of colors from $p^{-1}(v_0)$. Since φ is a reflection, this is not the coloring of the bundle.

Finally, the following matrix of colors determines a 6-coloring of $C_5 \boxtimes^{\varphi} C_5$, when φ is a reflection. We assume that $\varphi(u_2)=u_2$.

v_0	3	5	2	5	6
v_1	1	6	3	1	4
v_2	5	2	4	2	6
v_3	3	6	5	3	1
v_4	2	4	1	6	4
v_0'	6	5	2	5	3

Lemma 1.1(i) completes the proof of (i).

(ii) Let $\varphi(u_3)=u_3$. Assume that there is a 5-coloring of $C_5 \boxtimes^{\varphi} C_7$ which uses 5 colors in the u_3-layer. Then in the u_2-layer, only one color can occur twice. But if it does, we cannot color the u_4-layer. It follows that in the u_2-layer and in the u_4-layer all 5 colors are present. Furthermore, both colorings must be equal. Analogously, the layers u_0 and u_{2n} have the same coloring, a contradiction.

The other cases can be checked similarly, but many of them have a lot of subcases, hence we will not do it here. All cases have been checked by hand and computer.

(iii) Since there is only one non-isomorphic strong bundle when φ is a reflection, we may assume again $\varphi(u_n)=u_n$. Then the color matrix

v_0	2	1	2	1	2	5	2	3	5
v_1	3	5	3	5	3	4	1	4	1
v_2	4	1	4	1	2	5	3	5	2
v_3	5	2	3	5	3	1	2	4	1
v_4	4	1	4	1	4	5	3	5	3
v_0'	5	3	2	5	2	1	2	1	2

realizes a 5-coloring of $C_5 \boxtimes^{\varphi} C_9$. To get the color matrix for the general case we just extend the above matrix by adding the last color column on the left, and the first color column on the right, and so on. This proves (iii).

(iv) Let $\varphi(u_n)=u_n$. Then the color matrix

v_0	1	3	1	5	4	5	3
v_1	2	5	4	3	2	1	4
v_2	3	1	2	1	4	3	5
v_3	2	4	5	3	5	1	4
v_4	1	3	2	1	4	2	5
v_5	2	5	4	5	3	1	3
v_6	4	1	3	2	4	2	5
v_0'	3	5	4	5	1	3	1

gives a 5-coloring of $C_7 \boxtimes^{\varphi} C_7$. To get the color matrix for the general case we just extend the above matrix by adding the last color column on the left, and the first color column on the right, and so on to get $2n+1$ rows. Finally, we replicate $k-3$ times the two rows corresponding to v_5 and v_6 in order to get the color matrix for $C_{2k+1} \boxtimes^{\varphi} C_{2n+1}$. □

Lemma 3.11. *Let φ be a cyclic shift. Then $\chi(C_{2k+1} \boxtimes^{\varphi} C_5)=5$, for every $k \geqslant 2$.*

Proof. By Lemma 1.1(i), it suffices to consider the case when $k=2$. It is enough to exhibit 5-colorings when φ is a cyclic s-shift, $0 \leqslant s \leqslant 2$. In the table below we give two colorings. The first is a coloring for $s=0$ and $s=1$ (with appropriate change of the last row). The second is for $s=2$.

v_0	1	2	3	4	5		1	2	3	4	5
v_1	4	5	1	2	3		4	5	1	2	3
v_2	2	3	4	5	1		1	2	3	4	5
v_3	5	1	2	3	4		4	5	1	2	3
v_4	3	4	5	1	2		1	2	3	4	5
v_0'	1	2	3	4	5		4	5	1	2	3

□

Lemma 3.12. *Let φ be a cyclic shift. Then $\chi(C_{2k+1} \boxtimes^{\varphi} C_{2n+1})=5$, for every $k \geqslant 2$ and every $n \geqslant 3$.*

Proof. By Lemma 1.1(i) we may restrict to the case $k=2$. To get a 5-coloring for $C_5 \boxtimes^{\varphi} C_{2n+1}$, where φ is identity or the cyclic 1-shift, use the left matrix from Lemma 3.11 and repeat the last two columns $(n-2)$ times. We may now assume that φ is a cyclic $(2s+1)$-shift, $s=1,2,\ldots,n-1$.

Let A and B be the following color matrices, respectively:

1	2	3	4	5		4	5
4	5	1	2	3		1	3
1	2	3	4	5		2	5
4	5	1	2	3		4	3
1	2	3	4	5		1	2

Let c be an assignment of colors to $C_5 \boxtimes^{\varphi} C_{2n+1}$, defined by the horizontal sequence of color matrices $A, B, B, \ldots, B$. Hence, the first 5 layers are colored with A and the remaining $2n-4$ layers with copies of B. Note first that c is a coloring when φ is the cyclic 3-shift.

Assume now that φ is a cyclic $(2s+1)$-shift, $2 \leqslant s \leqslant n-1$. We are going to change c in such a way that it will become a coloring. In $p^{-1}(v_0)$, the colors 1 and 2 occur only once, and in $p^{-1}(v_4)$ the colors 4 and 5 occur only once. Hence, it is sufficient to change the coloring of the vertical layers corresponding to the vertices u_{2s+1}, u_{2s+2} and $u_{2n-2s+3}$, $u_{2n-2s+4}$ of the fiber. There are two cases to consider.

Case 1: $2s+1 \neq 2n-2s+3$. In this case we replace the coloring of the layers of vertices u_{2s+1} and u_{2s+2} and the coloring of the layers $2n-2s+3$ and $2n-2s+4$ with the following matrices, respectively:

$$\begin{matrix} 4 & 5 \\ 1 & 3 \\ 2 & 5 \\ 1 & 3 \\ 4 & 5 \end{matrix} \qquad \begin{matrix} 1 & 2 \\ 4 & 3 \\ 2 & 5 \\ 4 & 3 \\ 1 & 2 \end{matrix}$$

Case 2: $2s+1 = 2n-2s+3$. In this case we replace the coloring of the layers of u_{2s+1} and u_{2s+2} with the following matrix:

$$\begin{matrix} 1 & 2 \\ 4 & 3 \\ 2 & 5 \\ 1 & 3 \\ 4 & 5 \end{matrix} \qquad \square$$

Note that we did not consider the case in which the fiber or the base is the 3-cycle C_3. If the fiber is C_3, then $C_n \boxtimes^{\varphi} C_3$ is isomorphic to $C_n \boxtimes C_3$. More generally, if the fiber is K_k, $k \geqslant 3$, then $C_n \boxtimes^{\varphi} K_k$ is isomorphic to $C_n \boxtimes K_k$. Hence, $\chi(C_{2m} \boxtimes^{\varphi} K_k) = 2k$ (trivial) and if $n = 2m+1$, then $\chi(C_{2m+1} \boxtimes^{\varphi} K_k) = 2k + \lceil k/m \rceil$ (see [14, Theorem 6]). In particular,

$$\chi(C_{2m+1} \boxtimes^{\varphi} C_3) = \begin{cases} 9, & m=1 \\ 8, & m=2, \\ 7, & m \geqslant 3. \end{cases}$$

We cannot argue similarly if the base is C_3. Anyhow, the case $C_3 \boxtimes^{\varphi} C_n$ is a special case of $K_k \boxtimes^{\varphi} C_n$, $k \geqslant 3$. We believe that it would be also interesting to investigate chromatic numbers of $K_k \boxtimes^{\varphi} C_n$.

4. Tensor bundles

From [8] we recall the following proposition.

Proposition 4.1. *For any tensor bundle* $X = B \times^{\varphi} F$, $\chi(X) \leqslant \chi(B)$.

Then we have the following theorem.

Theorem 4.2. *For any $m \geqslant 3$ and any $n \geqslant 3$,*

$$\chi(C_m \times^{\varphi} C_n) = \begin{cases} 3, & m \text{ odd, } n \text{ odd, or} \\ & m \text{ odd, } n \text{ even, } \varphi \text{ odd shift, or} \\ & m \text{ odd, } n \text{ even, } \varphi \text{ reflection without fixed points;} \\ 2, & \text{otherwise.} \end{cases}$$

Proof. Let $m=2k$. Then by Proposition 4.1, $\chi(C_{2k} \times^{\varphi} C_n) \leqslant 2$ and therefore $\chi(C_{2k} \times^{\varphi} C_n)=2$.

Let $m=2k+1$. Again by Proposition 4.1, $\chi(C_{2k+1} \times^{\varphi} C_n) \leqslant 3$. Assume the bundle is bipartite. Suppose first that n is odd. Then $p^{-1}(v_0)$ and $p^{-1}(v_1)$ induce the cycle of length $2n$. Hence, all the vertices in $p^{-1}(v_0)$ must posses the same color. It follows that the vertices in $p^{-1}(v_{2k})$ must be colored with the same color, a contradiction.

Assume that n is even. Then $p^{-1}(v_0)$ and $p^{-1}(v_1)$ induce two disjoint cycles of length n. If we assign to all the vertices in $p^{-1}(v_0)$ the same color, we have a contradiction as above. Therefore, all the fiber layers must possess; the same color pattern, say $1,2,\ldots,1,2$. We separately consider all possibilities for φ.

If φ is a shift it is straightforward that we have a coloring of $C_{2k+1} \times^{\varphi} C_n$ if and only if φ is an even shift.

Let φ be a reflection without fixed points. Then there are two consecutive vertices of the cycle $p^{-1}(v_0)$ that are exchanged by φ. But then these two vertices are adjacent to the corresponding vertices in $p^{-1}(v_{2k})$, a contradiction.

Let φ be a reflection with two fixed points. Consider arbitrary vertex u in $p^{-1}(v_0)$. Then φ maps u to a vertex in $p^{-1}(v_{2k})$ which is on an even distance from u in C_n. Therefore, u is adjacent to two vertices in $p^{-1}(v_{2k})$ both on an odd distance from it. It follows that we have a 2-coloring. □

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