



On the chromatic number of the lexicographic product and the Cartesian sum of graphs[☆]

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Abstract

Let $G[H]$ be the lexicographic product of graphs G and H and let $G \oplus H$ be their Cartesian sum. It is proved that if G is a nonbipartite graph, then for any graph H , $\chi(G[H]) \geq 2\chi(H) + \lceil \chi(H)/k \rceil$, where $2k+1$ is the length of a shortest odd cycle of G . Chromatic numbers of the Cartesian sum of graphs are also considered. It is shown in particular that for χ -critical and not complete graphs G and H , $\chi(G \oplus H) \leq \chi(G)\chi(H) - 1$. These bounds are used to calculate chromatic numbers of the Cartesian sum of two odd cycles. Finally, a connection of some colorings with hypergraphs is given.

1. Introduction

In the last few years graph products became again a flourishing topic in graph theory. Chromatic numbers of products were investigated as well. Since some of the graph products admit polynomial algorithms for decomposing a given connected graph into its factors (see, for example [2]), chromatic numbers of graph products are interesting for their own sake. The chromatic number is in close connection with graph retracts. Therefore, information on chromatic numbers of graph products helps to understand retracts of products (see, for example [5]).

Graphs considered in this paper are undirected, finite and contain neither loops for multiple edges. An n -coloring of a graph G is a function f from $V(G)$ onto $\mathbb{N}_n = \{1, 2, \dots, n\}$, such that $xy \in E(G)$ implies $f(x) \neq f(y)$. The smallest number n for which an n -coloring exists is the chromatic number $\chi(G)$ of G . G is called χ -critical if $\chi(G-v) < \chi(G)$ for every $v \in V(G)$. Every nontrivial graph contains a χ -critical subgraph with the same chromatic number. A complete graph is a trivial χ -critical graph.

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The size of a largest complete subgraph of a graph G will be denoted by $\omega(G)$ and the size of a largest independent set by $\alpha(G)$. Clearly $\omega(G) \leq \chi(G)$ and $\omega(G) = \alpha(\bar{G})$.

The lexicographic product $G[H]$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ and $(a, x)(b, y) \in E(G[H])$ whenever $ab \in E(G)$, or $a = b$ and $xy \in E(H)$. The Cartesian sum or the disjunction $G \oplus H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ and $(a, x)(b, y) \in E(G \oplus H)$ whenever $ab \in E(G)$ or $xy \in E(H)$. Let $*$ be a graph product. For $x \in V(H)$ set $G_x = G * \{x\}$. Analogously we define H_a for $a \in V(G)$. We call G_x and H_a layer of G and H , respectively. Let f be a coloring of $G * H$. The set of all colors with respect to f in a layer G_x will be briefly denoted by $f(G_x)$.

In the next section we prove a lower bound for the chromatic number of the lexicographic product of graphs: if G is a nonbipartite graph, then for any graph H , $\chi(G[H]) \geq 2\chi(H) + \lceil \chi(H)/k \rceil$, where $2k + 1$ is the length of a shortest odd cycle of G . In Section 3 we consider chromatic numbers of the Cartesian sum of graphs. We prove that for nontrivial χ -critical graphs G and H , $\chi(G \oplus H) \leq \chi(G)\chi(H) - 1$, thus generalizing two results from [10, 12]. In the last section we apply these bounds to show that for $n \geq k \geq 2$.

$$\chi(C_{2k+1} \oplus C_{2n+1}) = \chi(C_{2k+1}[C_{2n+1}]) = \begin{cases} 8, & k=2, \\ 7, & k \geq 3. \end{cases}$$

We finally give a connection of some colorings with hypergraphs.

2. A lower bound for the lexicographic product

Chromatic numbers of the lexicographic product have been investigated in [3, 4, 6, 7, 11]. Geller and Stahl [3] proved that if G has at least one edge, then $\chi(G[H]) \geq \chi(G) + 2\chi(H) - 2$ for any graph H . This bound is the best general lower bound known so far. A short proof of the bound is given in [7]. The main theorem of this section (Theorem 2) implies another lower bound and is essentially proved in [11, Theorem 6]. However, our proof is straightforward and simple. For the proof we need the following lemma.

Lemma 1. *Let X , A and B be any (finite) sets. Then*

$$|A \cap B| + |X| \geq |A \cap X| + |B \cap X|.$$

Theorem 2. *For any graph H and any $k \geq 1$,*

$$\chi(C_{2k+1}[H]) = 2\chi(H) + \lceil \chi(H)/k \rceil.$$

Proof. We first prove the lower bound: $\chi(C_{2k+1}[H]) \geq 2\chi(H) + \lceil \chi(H)/k \rceil$. Let $\chi(H) = n$. It is easy to verify that the bound holds for $k = 1$ and for $k \geq n$. We may hence assume $1 < k < n$.

Suppose that $\chi(C_{2k+1}[H])=2n+r, r < \lceil n/k \rceil$. Let f be a $(2n+r)$ -coloring of $C_{2k+1}[H]$ and let $\{H_{s-k}, \dots, H_{s-1}, H_s, H_{s+1}, \dots, H_{s+k}\}$ be the H -layers corresponding to consecutive vertices of C_{2k+1} . Write

$$X_j = f(H_{s-2j}) \cap f(H_{s+2j}), \quad j=0, 1, \dots, \lfloor \frac{k}{2} \rfloor.$$

We claim

$$|X_j| \geq n - 2jr, \quad j=0, 1, \dots, \lfloor \frac{k}{2} \rfloor.$$

The claim will be proved by induction on j . It is clearly true for $j=0$.

Suppose now that the claim holds for $i=0, 1, \dots, j$, where $0 \leq j < \lfloor k/2 \rfloor$. Let $|f(H_{s-2j})|=n+t, |f(H_{s+2j})|=n+t'$ and $|X_j|=(n-2jr)+p, p \geq 0$ (note that $t'=t$ when $j=0$). Since $|f(H_{s-2j}) \cup f(H_{s-2j-1})| \geq 2n$, we have

$$|f(H_{s-2(j+1)}) \cap f(H_{s-2j})| \geq n - r + t,$$

and analogously

$$|f(H_{s+2(j+1)}) \cap f(H_{s+2j})| \geq n - r + t'.$$

By the induction hypothesis we obtain

$$|f(H_{s-2(j+1)}) \cap X_j| \geq (n - r + t) - (2jr + t - p) = n - (2j + 1)r + p,$$

and

$$|f(H_{s+2(j+1)}) \cap X_j| \geq (n - r + t') - (2jr + t' - p) = n - (2j + 1)r + p.$$

By Lemma 1 we get

$$\begin{aligned} |X_{j+1}| &\geq 2(n - (2j + 1)r + p) - |X_j| \\ &= 2(n - (2j + 1)r + p) - (n - 2jr + p) \\ &= n - 2(j + 1)r + p \geq n - 2(j + 1)r. \end{aligned}$$

The claim is proved. There are two cases to consider.

Case 1. k is even.

By the claim, $|f(H_{s-k}) \cap f(H_{s+k})| \geq n - kr$. Furthermore, since $r < \lceil n/k \rceil$ and therefore $n > rk$, $|f(H_{s-k}) \cap f(H_{s+k})| > 0$. A contradiction.

Case 2. k is odd.

Again by the claim, and the assumption $r < \lceil n/k \rceil$,

$$|f(H_{s-(k-1)}) \cap f(H_{s+(k-1)})| \geq n - (k-1)r > r.$$

It follows that $|f(H_{s-k}) \cup f(H_{s+k})| < 2n$, another contradiction. The lower bound is proved.

As it is easy to construct a coloring of $C_{2k+1}[H]$ with $2n + \lceil n/k \rceil$ colors, the proof is complete. \square

Another way to prove the upper bound of Theorem 2 is the following. In [6] it is proved that if G is a χ -critical graph, then for any graph H ,

$$\chi(G[H]) \leq \chi(H)(\chi(G) - 1) + \lceil \frac{\chi(H)}{\alpha(G)} \rceil.$$

As odd cycles are χ -critical, $\chi(C_{2k+1}) = 3$ and $\alpha(C_{2k+1}) = k$ we conclude $\chi(C_{2k+1}[H]) \geq 2\chi(H) + \lceil \chi(H)/k \rceil$,

If G is a bipartite graph then for any graph H , $\chi(G[H]) = 2\chi(H)$ (cf. [3]), while for a nonbipartite graph G we have the following corollary.

Corollary 3. *Let G be a nonbipartite graph. Then for any graph H ,*

$$\chi(G[H]) \geq 2\chi(H) + \lceil \frac{\chi(H)}{k} \rceil,$$

where $2k+1$ is the length of a shortest odd cycle in G .

3. Coloring Cartesian sums of graphs

The Cartesian sum (also called the disjunction in [4]) of graphs was introduced by Ore in [9, p. 36]. Some simple observations on the chromatic number of the Cartesian sum of two graphs were first demonstrated in [12, 1]. Much later, the chromatic number of the Cartesian sum turned out to be of interest in [10].

It is shown in [12] that $\chi(C_5 \oplus C_5) \leq 8$. This result is extended in [10] to $\chi(C_{2k+1} \oplus C_{2n+1}) \leq 8, k, n \geq 2$. In this section we generalize these two results to χ -critical graphs. But first we summarise some basic observations on chromatic numbers of the Cartesian sum of graphs.

Proposition 4. *Let G and H be any graphs. Then*

- (i) $\chi(G \oplus H) \leq \chi(G)\chi(H)$,
- (ii) $\chi(K_n \oplus H) = n\chi(H)$,
- (iii) If $\chi(G) = \omega(G)$ then $\chi(G \oplus H) = \chi(G)\chi(H)$,
- (iv) $\alpha(G \oplus H) = \alpha(G)\alpha(H)$. Furthermore, if X is a maximum independent set of $G \oplus H$, then $X = G' \times H'$ where G' and H' are maximum independent sets of G and H , respectively.

(v) f is a coloring of $G \oplus H$ if and only if every layer of G and every layer of H is properly colored and in addition, for every edge $ab \in E(G)$, $f(H_a) \cap f(H_b) = \emptyset$ and for every edge $xy \in E(H)$, $f(G_x) \cap f(G_y) = \emptyset$.

Proof. (i) This was first observed in [12] and rediscovered in [1].

(ii) For any H -layer H_a , $|f(H_a)| \geq \chi(H)$. Furthermore, for $a \neq b$, $f(H_a) \cap f(H_b) = \emptyset$.

(iii) Let $\chi(G) = n$. By assumption, K_n is a subgraph of G . It follows that $K_n \oplus H$ is a subgraph of $G \oplus H$. By (ii), $\chi(K_n \oplus H) = n\chi(H)$ and therefore

$$n\chi(H) = \chi(K_n \oplus H) \leq \chi(G \oplus H) \leq \chi(G)\chi(H) = n\chi(H).$$

- (iv) This is shown in [12], the second part only implicitly.
- (v) See [10]. \square

Theorem 5. Let $\chi(G)=n, \chi(H)=m$. Let a and b be nonadjacent vertices of G , and let x and y be nonadjacent vertices of H . If $\chi(G-a) < \chi(G), \chi(G-b) < \chi(G), \chi(H-x) < \chi(H)$ and $\chi(H-y) < \chi(H)$ then $\chi(G \oplus H) \leq nm - 1$.

Proof. Let $G_a = G - a, G_b = G - b, H_x = H - x$ and $H_y = H - y$. By the assumption of the theorem there exist the following colorings:

$$g_a: G_a \rightarrow \mathbb{N}_{n-1}, \quad h_x: H_x \rightarrow \mathbb{N}_{m-1},$$

$$g_b: G_b \rightarrow \mathbb{N}_{n-1}, \quad h_y: H_y \rightarrow \mathbb{N}_{m-1},$$

According to Proposition 4(i), the subgraph $G_a \oplus H_x$ can be colored with $(n-1)(m-1)$ colors. Then we use n new colors for the layer G_x , where the vertex (b, x) is colored with its own color and G_b with the remaining $(n-1)$ colors. Let c be the color of the vertex (a, x) . Next we color the vertex (a, y) with the same color as (b, x) and the color class of (a, x) (in the layer H_a) with the color c . For the rest of the layer H_a we need $m-2$ new colors.

More formally, assume wlog. $g_b(a) = h_y(x) = 1$ and define:

$$f(c, z) = \begin{cases} (g_a(c), h_x(z)), & (c, z) \in V(G_a) \times V(H_x), \\ (g_b(c), m), & g_b(c) > 1 \text{ and } z = x, \\ (n, h_y(z)), & c = a \text{ and } h_y(z) > 1, \\ (1, m), & (g_b(c) = 1 \text{ and } z = x) \text{ or } (c = a \text{ and } h_y(z) = 1), \\ (n, 1), & (c, z) = (a, y) \text{ or } (c, z) = (b, x). \end{cases}$$

The function f is schematically shown in Fig. 1.

Observe that $f: G \oplus H \rightarrow \mathbb{N}_n \times \mathbb{N}_m - \{(n, m)\}$. Furthermore, it is straightforward to verify that f is a coloring of $G \oplus H$, and the proof is complete. \square

Corollary 6. Let G and H be nontrivial χ -critical graphs. Then

$$\chi(G \oplus H) \leq \chi(G)\chi(H) - 1.$$

We conclude this section with a lower bound. It follows from Corollary 3, the fact that $G[H]$ is a subgraph of $G \oplus H$ and the fact that $G \oplus H$ is, roughly speaking, the lexicographic product in both directions.

Corollary 7. Let G and H be nonbipartite graphs. Then

$$\chi(G \oplus H) \geq \max \left\{ 2\chi(H) + \left\lceil \frac{\chi(H)}{k} \right\rceil, 2\chi(G) + \left\lceil \frac{\chi(H)}{n} \right\rceil \right\},$$

where $2k + 1$ and $2n + 1$ are the lengths of shortest odd cycles of G and H , respectively.

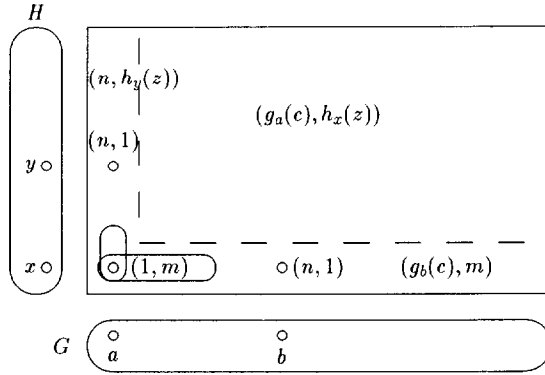


Fig. 1. The function f .

4. Products of odd cycles

It is well known that $\chi(C_{2k+1} \times C_{2n+1})=3$, $\chi(C_{2k+1} \square C_{2n+1})=3$ and $\chi(C_{2k+1} \boxtimes C_{2n+1})=5$, where \times , \square and \boxtimes denote the categorical, the Cartesian and the strong product of graphs, respectively. The result for the lexicographic product is contained in Theorem 2. In this section we add to these results the chromatic numbers for the Cartesian sum of two odd cycles.

Throughout this section we will use the facts that a lower bound for the lexicographic product is a lower bound for the Cartesian sum and that an upper bound for the Cartesian sum is an upper bound for the lexicographic product.

Corollary 8. For $n \geq 2$, $\chi(C_5 \oplus C_{2n+1}) = \chi(C_5[C_{2n+1}]) = 8$.

Proof. Follows from Theorem 2 and Corollary 6. \square

Let us prove that $\chi(C_5 \oplus C_5) \geq 8$ holds also by the following alternative argument. Since $\chi(C_5 \oplus C_5) > 6$, suppose $\chi(C_5 \oplus C_5) = 7$. By Proposition 4(iv), $\alpha(C_5 \oplus C_5) = 4$. Note first that when we color a maximum independent set of $C_5 \oplus C_5$, the number of uncolored vertices in any layer remains odd. As $|V(C_5 \oplus C_5)| = 25$, in a 7-coloring of $C_5 \oplus C_5$ there are at least 4 color classes of size 4. Suppose that the fifth class is of size 4 as well. Then the remaining 5 vertices are all in different layers of the product. But these vertices cannot be colored by 2 colors. It follows that in a 7-coloring there are 4 color classes of size 4 and 3 classes of size 3. Consider now any configuration of 4 color classes of size 4 and 2 classes of size 3. It is easy to see that the remaining 3 vertices belong to 3 different layers of the product, hence they cannot be colored by a single color.

We continue the investigation of the Cartesian sum of two odd cycles.

Lemma 9. $\chi(C_7 \oplus C_7) = \chi(C_7[C_7]) = 7$.

Proof. Using Proposition 4(v) it is easy to check that the following matrix determine a coloring of $C_7 \oplus C_7$:

$$\begin{matrix} 2 & 1 & 2 & 3 & 1 & 2 & 3 \\ 4 & 5 & 6 & 5 & 4 & 5 & 6 \\ 7 & 1 & 2 & 7 & 1 & 2 & 1 \\ 4 & 5 & 4 & 3 & 4 & 5 & 3 \\ 7 & 1 & 6 & 7 & 1 & 7 & 6 \\ 4 & 3 & 2 & 3 & 4 & 2 & 3 \\ 7 & 5 & 6 & 7 & 6 & 5 & 6 \end{matrix}$$

The lower bound follows from Theorem 2. \square

Theorem 10. For $n \geq k \geq 2$,

$$\chi(C_{2k+1} \oplus C_{2n+1}) = \chi(C_{2k+1}[C_{2n+1}]) = \begin{cases} 8, & k=2, \\ 7, & k \geq 3. \end{cases}$$

Proof. It remains to prove the theorem for $n \geq k \geq 3$. By Lemma 9, $\chi(C_7 \oplus C_7) = 7$ and let $f: V(C_7) \times V(C_7) \rightarrow \mathbb{N}_7$ be a 7-coloring of $C_7 \oplus C_7$. Define $g: V(C_{2k+1}) \times V(C_{2n+1}) \rightarrow \mathbb{N}_7$ in the following way:

$$g(v_i, v_{i'}) = f(v_j, v_{j'}),$$

where

$$j = \begin{cases} i, & j \leq 7, \\ (i \bmod 2) + 1, & j > 7 \end{cases}$$

and

$$j' = \begin{cases} i', & j' \leq 7, \\ (i' \bmod 2) + 1, & j' > 7. \end{cases}$$

The mapping g repeats the color pattern of the first two layers in every direction of the product. It is easy to check that g is a 7-coloring, therefore $\chi(C_{2k+1} \oplus C_{2n+1}) \leq 7$. As the lower bound follows from Theorem 2, the proof is complete. \square

We conclude the paper by showing that some colorings give us 3-regular 3-uniform hypergraphs.

Proposition 11. Let G and H be graphs isomorphic to C_7 . Let f be a 7-coloring of $G \oplus H$. Let $F_G = \{f(G_x) \mid x \in V(H)\}$ and let $F_H = \{f(H_a) \mid a \in V(G)\}$. Then (\mathbb{N}_7, F_G) and (\mathbb{N}_7, F_H) are 3-regular 3-uniform hypergraphs.

Proof. It is enough to prove the proposition for (\mathbb{N}_7, F_G) . Let $V(H) = \{v_1, v_2, \dots, v_7\}$. We claim that $|f(G_{v_i})| = 3$, $i \in \{1, 2, \dots, 7\}$. Clearly, $|f(G_{v_i})| \leq 4$, for otherwise $|f(G_{v_i}) \cup f(G_{v_{i+1}})| \geq 8$. Suppose now that $|f(G_{v_1})| = 4$. Then $f(G_{v_2}) = f(G_{v_7})$ and $|f(G_{v_2})| = 3$. Hence $f(G_{v_3}) \subseteq f(G_{v_1})$ and $f(G_{v_6}) \subseteq f(G_{v_1})$. Therefore, $|f(G_{v_3}) \cup f(G_{v_6})| \geq 2$. But then there are at most 5 colors left to color the layers G_{v_4} and G_{v_5} . This contradiction proves the claim. It follows that (\mathbb{N}_7, F_G) is 3-uniform. Furthermore, as $|F_G| = |\mathbb{N}_7| = 7$ we conclude that (\mathbb{N}_7, F_G) is a 3-regular hypergraph. \square

References

- [1] M. Borowiecki, On chromatic number of products of two graphs, *Colloq. Math.* 25 (1972) 49–52.
- [2] J. Feigenbaum and A.A. Schäffer, Finding the prime factors of strong direct product graphs in polynomial time, *Discrete Math.* 109 (1992) 77–102.
- [3] D. Geller and S. Stahl, The chromatic number and other parameters of the lexicographic product, *J. Combin. Theory Ser. B* 19 (1975) 87–95.
- [4] P. Hell and F. Roberts, Analogues of the Shannon capacity of a graph, *Ann. Discrete Math.* 12 (1982) 155–168.
- [5] S. Klavžar, Two remarks on retracts of graph products, *Discrete Math.* 109 (1992) 155–160.
- [6] S. Klavžar, Strong products of χ -critical graphs. *Aequationes Math.* 45 (1993) 153–162.
- [7] S. Klavžar and U. Milutinović, Strong products of Kneser graphs, *Discrete Math.* 133 (1994) 297–300.
- [8] J. Nešetřil and V. Rödl, Products of graphs and their applications, *Lecture Notes in Mathematics* Vol. 1018 (Springer, Berlin, 1983) 151–160.
- [9] O. Ore, *Theory of graphs*, Vol. 38 (Amer. Math. Soc. Colloq. Publ., Providence, RI, 1962).
- [10] V. Puš, Chromatic number of products of graphs, *Comment. Math. Univ. Carolin.* 29 (1988) 457–463.
- [11] S. Stahl, n -tuple colorings and associated graphs, *J. Combin. Theory Ser. B* 20 (1976) 185–203.
- [12] K.W. Yang, Chromatic number of Cartesian sum of two graphs, *Proc. Amer. Math. Soc.* 19 (1968) 617–618.