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Clique-gated graphs

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Abstract

Clique-gated graphs form an extension of quasi-median graphs. Two characterizations of these graphs are given and some other structural properties are obtained. An $O(nm)$ algorithm is presented which recognizes clique-gated graphs. Here n and m denote the numbers of vertices and edges of a given graph, respectively.

1. Introduction

Quasi-median graphs were introduced by Mulder [15] as a generalization of median graphs. His motivation was to extend median graphs in the way as hypercubes are extended to Hamming graphs. In particular, bipartite quasi-median graphs are median graphs. Both, median and quasi-median graphs form well studied classes of graphs. Median graphs were first investigated by Avann [1] and Nebeský [18]. The more extensive investigation of these graphs was done by Mulder and Bandelt as well as by some other researchers, see for instance [2, 4, 9, 14–17].

A first characterization of quasi-median graphs is due to Mulder [15] and later several different characterizations were discovered. For most of them we refer to the paper [5] of Bandelt et al. where also (relatively) short proofs are given. Besides these characterizations quasi-median graphs can also be described as weak retracts of Hamming graphs [7, 20], as connected subgraphs of Hamming graphs closed under the quasi-median operation [7, 15] and as graphs with finite windex [7].

From the algorithmic point of view, several efficient algorithms concerning these graphs are known. Jha and Slutzky [13] gave an $O(n^2 \log n)$ algorithm for recognizing

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median graphs and it is demonstrated in [11] how to recognise these graphs in $O(n^{3/2} \log n)$ time. For quasi-median graphs an algorithm of the time complexity $O(n^{3/2} \log n + m \log n)$ is developed in [10].

In [20] Wilkeit suggested that it might be interesting to investigate the class of graphs for which every clique is gated. We call such graphs clique-gated. Clique-gated graphs extend the class of quasi-median graphs and contain all bipartite graphs.

In the next section we state the necessary definitions and recall two characterizations of quasi-median graphs. In Section 3 we consider the structure of clique-gated graphs. We characterize them in two different ways and propose two problems. In the last section an $O(nm)$ algorithm is given that recognizes clique-gated graphs, where n and m denote the numbers of vertices and edges of a given graph, respectively.

2. Preliminaries

All graphs considered in this paper are finite undirected graphs without loops or multiple edges. A *clique* is a maximal complete subgraph. $K_4 - e$ is the graph on four vertices with five edges, i.e. the complete graph K_4 with an edge deleted. Note that if a graph contains no $K_4 - e$, then each edge of G belongs to a unique clique.

A subgraph H of a graph G is a *retract* of G, if there is an edge-preserving map r from $V(G)$ to $V(H)$ such that $r(v) = v$ for every $v \in V(H)$. The map r is called a *retraction*. If we allow that r maps an edge of G either to an edge or to a single vertex in H, we call H a *weak retract* of G and r a *weak retraction*.

As usual, the *distance* $d_G(u, v)$ between two vertices u and v of a graph G is the length of a shortest path between u and v. Whenever the graph G will be clear from the context, we will shortly write $d(u, v)$. The distance $d(H, H')$ between two subgraphs of a given graph is defined as $\min\{d(u, v);\ u \in H, v \in H'\}$. A subgraph H of a graph G is an *isometric* subgraph, if $d_H(u, v) = d_G(u, v)$ for all $u, v \in V(H)$. A (weak) retract is necessarily an isometric subgraph.

A set of vertices S is *convex* in G if for any two vertices $u, v \in S$, the interval $I(u, v)$ belongs to S. The *interval* between vertices u and v consists of all vertices on shortest paths between u and v. For an edge $uv \in E(G)$, let U_{uv} be the set of vertices of G which are closer to u than to v and are adjacent to a vertex which is closer to v than to u.

A subgraph H of a graph G is called *gated* in G, if for every $v \in V(G)$ there exists a vertex $x \in V(H)$ such that for every $u \in V(H)$, x lies on a shortest path from v to u. If such a vertex exists it must be unique. We denote the unique vertex x by $k_H(v)$ and call it the *gate* of v in H. We call a graph *clique-gated* if every clique is gated.

A graph G satisfies the *triangle property* if, for any edge uv and a vertex w with $d(u, w) = d(v, w) = k \geqslant 2$, there exists a common neighbour x of u and v with $d(x, w) = k - 1$. A graph G satisfies the *quadrangle property* if, for any vertices u, v, w and z with $d(u, w) = d(v, w) = k = d(z, w) - 1$ and $d(u, v) = 2$ with z a common neighbour of u and v, there exists a common neighbour x of u and v with $d(x, w) = k - 1$. These two properties are schematically shown in Fig. 1.

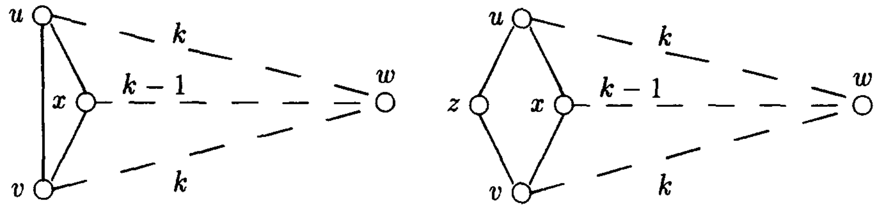

Fig. 1. The triangle and the quadrangle property.

The *Cartesian product* $G \square H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ and $(a, x)(b, y) \in E(G \square H)$ whenever $ab \in E(G)$ and $x = y$, or $a = b$ and $xy \in E(H)$. For $a \in V(G)$ set $H_a = \{a\} \square H \subseteq G \square H$ and call it an *H-layer*. A *Hamming graph* is the Cartesian product of complete graphs, see [3, 5, 12] and references there.

In the next theorem we recall two characterizations of quasi-median graphs which are relevant to our work.

Theorem 2.1. *For a connected graph G, the following conditions are equivalent:*

(i) *G is a quasi-median graph.*

(ii) *G fulfils the triangle and the quadrangle property, and G does not contain* $K_4 - e$ *or* $K_{2,3}$ *as an induced subgraph.*

(iii) *Every clique of G is gated, and, for every edge uv of G, the set* U_{uv} *is convex.*

Hence the class of clique-gated graphs (properly) contains the class of quasi-median graphs.

3. Two characterizations

On the set of all cliques of a graph G we introduce the relation $\sim$ as follows. For cliques Q and Q' let $Q \sim Q'$ if exactly one of the following holds:

(A) $|Q| = |Q'|$ and the vertices of Q and Q' can be labeled $\{u_1, u_2,, \ldots, u_k\}$ and $\{v_1, v_2, \ldots, v_k\}$, respectively, such that

$$d(u_i, v_j) = \begin{cases} d(Q, Q') & \text{if } i = j, \\ d(Q, Q') + 1 & \text{otherwise.} \end{cases}$$

(B) There exist unique vertices $u \in Q$ and $u' \in Q'$ such that $d(u, u') = d(Q, Q')$ and for $v \in Q \backslash u$ and $v' \in Q' \backslash u'$, $d(u, v') = d(v, u') = d(Q, Q') + 1$ and $d(v, v') = d(Q, Q') + 2$.

The relation $\sim$ is reflexive and symmetric but it is generally not transitive.

Theorem 3.1. *For a connected graph G, the following conditions are equivalent:*

(i) *G is a clique-gated graph.*

(ii) *For any cliques Q and Q′ of G,* $Q \sim Q'$.

(iii) *G does not contain* $K_4 - e$ *as an induced subgraph and G satisfies the triangle property.*

Proof. (i) $\Rightarrow$ (ii). (This part of the proof is essentially done in [7, Lemma 7.5].) Let Q and Q' be arbitrary cliques of G, $Q \neq Q'$. As G is clique-gated, $|Q \cap Q'| \leqslant 1$. Let $X \subseteq Q$ be the set of vertices $u \in Q$ such that $d(u, Q') = d(Q, Q')$ and let $X' \subseteq Q'$ be defined analogously. Since Q and Q' are gated, for any $u \in X$ and any $u' \in X'$, $k_{Q'}(u) \in X'$ and $k_Q(u') \in X$. In addition, the maps $k_{Q'}$ and k_Q are inverses. Therefore if $X = Q$ and $X' = Q'$ then (A) holds. Suppose now that $v' \in Q' \backslash X'$. Then $d(v', Q) = d(v', u) + 1$ for any $u \in X$. As the gate $k_Q(v')$ is unique, $|X| = 1$ and analogously $|X'| = 1$. Hence (B) holds.

(ii) $\Rightarrow$ (iii). G clearly contains no $K_4 - e$ as an induced subgraph. Assume now that for adjacent vertices u and v there exists a vertex w with $d(w, u) = d(w, v) = k \geqslant 2$. Consider a clique Q containing the edge uv and a clique Q' containing w. Since $Q \sim Q'$ then if (A) holds, there is a vertex $x \in Q$ with $d(w, Q) = d(w, x)$ and $d(w, x) = d(w, u) - 1$. And if (B) holds there are unique vertices $x \in Q$ and $x' \in Q'$ with $d(x, x') = d(Q, Q')$. Note that x may be equal to x'. But in any case $x \neq u$ and $x \neq v$, hence G fulfils the triangle property.

(iii) $\Rightarrow$ (i). Assume that a clique Q is not gated, i.e. there are vertices u, v and w such that $uv \in E(Q)$ and $d(w, Q) = d(w, u) = d(w, v) = k \geqslant 1$. If $k = 1$, then since $w \notin Q$, there exists a vertex $x \in Q$, $x \neq u$, $x \neq v$, such that $wx \notin E(G)$. Therefore vertices u, v, w and x induce a $K_4 - e$, a contradiction. Let $k \geqslant 2$. By the triangle property there is a vertex x adjacent to u and v with $d(w, x) = k - 1$. Since $d(w, Q) = k$, we have $x \notin Q$. But then there is a vertex $y \in Q$ such that vertices u, v, y and x induce a $K_4 - e$, another contradiction. $\square$

In light of Theorem 2.1 (ii) and (iii) versus Theorem 3.1 (i) and (iii) one might ask if the property that the set U_{uv} is convex for every edge uv of G, is equivalent to the quadrangle property and being $(K_{2,3})$-free. The answer is negative, consider for example the graph G in Fig. 2. The set U_{uv} is convex for every edge uv of G, yet G does not satisfy the quadrangle property.

We next show how to obtain new clique-gated graphs from known such graphs.

Proposition 3.2. (i) *A weak retract of a clique-gated graph is clique-gated.*

(ii) *Let G and H be connected graphs. Then $G \square H$ is clique-gated if and only if G and H are clique-gated.*

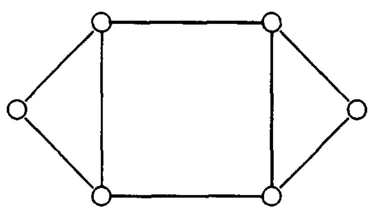

Fig. 2. The graph G.

Proof. (i) Let G be a clique-gated graph and let H be a weak retract of G. Since H is an induced subgraph of G, H contains no $K_4 - e$ as an induced subgraph. By Theorem 3.1 (iii) it remains to show that H satisfies the triangle property.

Assume that for $uv \in E(H)$ there is a vertex $w \in V(H)$ such that $d_H(w, u) = d_H(w, v) = k \geqslant 2$. As G fulfils the triangle property, there exists a vertex $x \in V(G)$ adjacent to u and v and $d_G(x, w) = k - 1$. Let $v: V(G) \to V(H)$ be a weak retraction map. As r is non-expanding we have:

$$k - 1 = d_G(x, w) \geqslant d_H(r(x), r(w)) = d_H(r(x), w)).$$

If follows that $r(x) \neq u$ and $r(x) \neq v$. Furthermore, if $r(x) = y$, then $yu \in E(H)$, $yv \in E(H)$ and $d(w, y) = k - 1$. Thus H satisfies the triangle property.

(ii) Straightforward. □

Since a graph is bipartite if and only if any of its edges is gated (cf. [20]), Proposition 3.2 (ii) in particular implies that $G \square H$ is bipartite if and only if G and H are bipartite.

To conclude this section we propose two problems.

Problem 1. Can a clique-gated graph include any isometric odd cycle of length at least 5?

In fact, if we assume that a clique-gated graph is $(K_{2,3})$-free, then it is not difficult to prove the next proposition, the proof of which is left to the reader.

Proposition 3.3. Let G be a clique-gated graph without an induced $K_{2,3}$. Then G contains no C_5 as an induced subgraph.

It follows in particular from Proposition 3.3. and Theorems 2.1 (ii) and 3.1 (iii) that quasi-median graphs contain no C_5 as an induced subgraph. We therefore also ask:

Problem 2. Can a clique-gated graph include any induced C_5?

4. Recognizing clique-gated graphs

In this section we propose an $O(nm)$ algorithm which recognizes clique-gated graphs. Note that Theorem 3.1 (iii) immediately implies that clique-gated graphs can be recognized in polynomial time. For the first condition one has to check every quadruple of vertices and for the second condition one has to check triples of vertices. Thus a straightforward implemetation would yield to an $O(n^4)$ algorithm.

Let R be a relation defined on the edge set of a graph in the following way. Edges e and e' are in the relation R if they are edges of a common triangle. Let R^* denote the reflexive and transitive closure of R. Then we have:

Lemma 4.1. *A graph G is clique-gated if and only if*

(P) *the edge set of each equivalence class induced by R^* is the edge set of a gated clique.*

Proof. If G is clique-gated then (P) holds because each edge is in a unique clique.

Assume now that (P) holds and let Q be a clique of G. We want to show that Q is gated. It is enough to prove that the edge set of Q is the edge set of an equivalence class. Let $e = uv$ and $e' = u'v'$ be arbitrary edges of Q. Then uu', uv', vu' *and* vv' are edges of G and by the transitivity of R^*, uv *and* $u'v'$ belong to the same equivalence class. It follows that no other clique exists but those for which the edge set is the edge set of an equivalence class. □

In order to obtain an efficient algorithm, condition (P) must be tested and carefully implemented. This is done by the following algorithm:

1. Compute R^*.
2. Test whether the edges of each equivalence class are edges of a clique.
3. Test whether each such clique is gated.

Theorem 4.2. *One can determine in* $O(nm)$ *time and* $O(m)$ *space whether a given graph on n vertices and m edges is clique-gated or not.*

Proof. Step 1 can be done taking an edge uv and going through the sorted adjacency list of both u and v. If u and v have a common neighbour w then the edges uv, uw and vw are made equivalent using the Union–Find data structure.

Step 2 is done by counting the number of different vertices and edges of a class. As a result we obtain cliques and the list of vertices of each clique.

Finally, in Step 3, compute first for each vertex u the distance to all other vertices. For each clique exactly one vertex must be closer to u than the other vertices.

All three steps can be obviously computed in $O(nm)$ time. The space bound is $O(m)$, since each vertex v is in at most *degree*(v) different lists computed in Step 2. □

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