Coloring graph products — A survey ☆

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Abstract

There are four standard products of graphs: the direct product, the Cartesian product, the strong product and the lexicographic product. The chromatic number turned out to be an interesting parameter on all these products, except on the Cartesian one. A survey is given on the results concerning the chromatic number of the three relevant products. Some applications of product colorings are also included.

1. Introduction

Graphs considered in this paper are undirected, finite and contain neither loops nor multiple edges (unless stated otherwise).

A graph product $G * H$ of graphs G and H most commonly means a graph on the vertex set $V(G) \times V(H)$, while its edges are determined by a function on the edges of the factors. There are many such products but only four of them are really important: the *direct product* (known also as the tensor product, the categorical product, the Kronecker product, the cardinal product, the conjunction, the weak direct product, or just the product), the *Cartesian product*, the *strong product* (known also as the strong direct product or the symmetric composition) and the *lexicographic product* (known also as the composition or the substitution).

The direct product, the Cartesian product, the strong product and the lexicographic product of graphs G and H will be denoted by $G \times H$, $G \square H$, $G \boxtimes H$ and $G[H]$, respectively. Let $(a, x), (b, y) \in V(G) \times V(H)$. Then $(a, x)(b, y)$ belongs to

$E(G \times H)$ whenever $ab \in E(G)$ and $xy \in E(H)$;

$E(G \square H)$ whenever $ab \in E(G)$ and $x = y$, or $a = b$ and $xy \in E(H)$;

$E(G \boxtimes H)$ whenever $(a,x)(b,y) \in (E(G \times H) \cup E(G \,\square\, H))$;

$E(G[H])$ whenever $ab \in E(G)$, or $a = b$ and $xy \in E(H)$.

Note that $E(G \times H) \cup E(G \,\square\, H) = E(G \boxtimes H) \subseteq E(G[H])$.

The notation $\times$, $\square$ and $\boxtimes$ is due to Nešetřil. It is a nice notation because $\times$, $\square$ and $\boxtimes$ looks like the direct, the Cartesian and the strong product, respectively, of an edge by itself. We use the (standard) notation $G[H]$ to emphasize that the lexicographic product is noncommutative.

Whenever possible we shall denote the vertices of one factor by a, b, c, ... and the vertices of the other factor by x, y, z, ... just as it is done in the above definitions. Let G and H be graphs and let $*$ be a graph product. For $x \in V(H)$ set $G_x = G * \{x\}$ and for $a \in V(G)$ set $H_a = \{a\} * H$. We call G_x and H_a a *layer* of G and of H, respectively. If $*$ is the Cartesian product, the strong product or the lexicographic product, then G_x is isomorphic to G and H_a to H.

A *homomorphism* $G \to H$ is an edge-preserving map, i.e. a mapping $f : V(G) \to V(H)$ such that $f(x)f(y) \in E(H)$ whenever $xy \in E(G)$. A subgraph R of a graph G is a *retract* of G if there is a homomorphism $r : V(G) \to V(R)$ with $r(x) = x$, for all $x \in V(R)$. The map r is called a *retraction*. An *n-coloring* of a graph G is a function f from $V(G)$ onto $\mathbb{N}_n = \{1, 2, \ldots, n\}$, such that $xy \in E(G)$ implies $f(x) \neq f(y)$. Equivalently, *n-coloring* of G is a homomorphism $G \to K_n$. The smallest number n for which an *n-coloring* exists is the *chromatic number* $\chi(G)$ of G. Note that if there is a homomorphism $G \to H$, then $\chi(G) \leqslant \chi(H)$. The size of a largest complete subgraph of a graph G will be denoted by $\omega(G)$ and the size of a largest independent set by $\alpha(G)$.

The following result, proved in 1957 by Sabidussi [31] and later rediscovered several times, says all about the chromatic number of the Cartesian product.

Proposition 1.1. $\chi(G \,\square\, H) = \max\{\chi(G), \chi(H)\}$.

In the next section we briefly review (new) results on the famous Hedetniemi's conjecture. We also mention the concept of multiplicativity and results concerning the conjecture on infinite graphs. In Section 3 upper and lower bounds are presented for the chromatic number of the strong and the lexicographic product. These bounds then yield to several exact chromatic numbers. In the last section we give three applications of product colorings. Due to the compactness of the paper some proofs will be omitted and some will be sketched only.

2. Hedetniemi's conjecture

Let f be an *n-coloring* of a graph G and let H be a graph. Then $g(a,x) = f(a)$ is an *n-coloring* of the graph $G \times H$. It follows that $\chi(G \times H) \leqslant \min\{\chi(G), \chi(H)\}$. In 1966, Hedetniemi [14] conjectured that for all graphs G and H,

$$\chi(G \times H) = \min\{\chi(G), \chi(H)\}.$$

The conjecture is also known as the Lovász–Hedetniemi's conjecture. In fact, to prove the conjecture it is enough to show that for all graphs G and H, $\chi(G) = \chi(H) = n$ implies $\chi(G \times H) = n$.

Let $G \not\rightarrow H$ denotes that there is no homomorphism $G \rightarrow H$. A graph G is called *multiplicative*, if $G_1 \not\rightarrow G$ and $G_2 \not\rightarrow G$ imply $G_1 \times G_2 \not\rightarrow G$ for all graphs G_1 and G_2. We can now rephrase Hedetniemi's conjecture to: the graphs K_n are multiplicative.

To settle the conjecture is the most tempting problem connected with product colorings. A nice overview of the results on the conjecture was done in 1985 by Duffus et al. [4]. We will therefore briefly give only results which have appeared after their paper.

In [29] Poljak and Rödl introduced the function

$$f(n) = \min\{\chi(G \times H) \mid \chi(G) = \chi(H) = n\}.$$

It is somehow surprising that it is not even known whether $f(n)$ tends to infinity for $n \rightarrow \infty$. It is proved in [29] that if f is bounded then $f(n) \leqslant 16$ for all n. Poljak [28] improved the result from 16 to 9:

Theorem 2.1. *The minimum chromatic number of a direct product of two n-chromatic graphs is either bounded by 9, or tends to infinity.*

We have already mentioned that the Hedetniemi's conjecture can be formulated using the concept of multiplicativity. This was the motivation to Häggkvist, et al. [11] to gain insights for the eventual proof or disproof of the conjecture. However, the concept of multiplicativity is not a new one, cf., for example, [27] (where multiplicativity is called "productivity"). In [11] it is proved among others:

Theorem 2.2. (i) *A directed cycle C_n is multiplicative if and only if n is a prime power.*

(ii) *Each cycle C_n is multiplicative.*

Zhou [43,44] obtained new classes of multiplicative graphs and digraphs, and some classes of nonmultiplicative digraphs. He also introduced weak multiplicativity and very weak multiplicativity. A connected graph G is *weakly multiplicative* if the following holds for all graphs H_1 and H_2: if G is not a retract of H_1 and G is not a retract of H_2, then G is not a retract of $H_1 \times H_2$. Since the theory of multiplicativity has no direct connection to the Hedetniemi's conjecture, we mention here just one result of Zhou.

Theorem 2.3. *An oriented path P is multiplicative if and only if P contains a directed path P_n as its retract.*

The concept of multiplicativity was introduced to ordered sets by Sauer and Zhu [32]. Anyhow, it remains to wait whether the theory of multiplicativity can help to solve the Hedetniemi's conjecture.

Duffus et al. [4] proposed another approach to the conjecture, which is based on a result of Hajós [13]. Their idea was further developed by Sauer and Zhu [33].

The *Hajós sum* of graphs G and H, with respect to edges $ab \in E(G)$ and $xy \in E(H)$, is the graph obtained from the disjoint union of G and H by contracting the vertices a and x, removing the edges ab and xy, and joining b with y. Hajós [13] proved that every graph G with $\chi(G) > n$ can be constructed from copies of K_{n+1} by the following three operations: Hajós sum, adding vertices and edges, contracting nonadjacent vertices. For abbreviation let us call these operations the *three operations*. It is easy to see that none of the three operations decrease the chromatic number.

Fix an integer n. Call a graph G, $\chi(G) > n$, *persistent*, if $\chi(G \times H) = n + 1$, for any graph H with $\chi(H) = n + 1$. Clearly, Hedetniemi's conjecture (for the fixed n) is equivalent to asserting that every graph G, $\chi(G) > n$, is persistent. In view of the Hajós theorem it is enough to prove that every graph constructed by the three operations is persistent. Furthermore, it is easy to see that Hedetniemi's conjecture is essentially equivalent to the statement: the Hajós sum of persistent graphs is persistent.

Theorem 2.4 (Duffus et al. [4]). *Let G be constructed from copies of K_{n+1} by performing the three operations in such a way that any contractions of nonadjacent vertices are performed after all other operations. Then G is persistent.*

Call a graph G *strongly persistent* if G is persistent and the Hajós sum with any other persistent graph is again persistent.

Theorem 2.5 (Sauer and Zhu [33]). *Let G be constructed from copies of K_{n+1} by performing the three operations where at most one contraction is performed. Then G is strongly persistent. Furthermore, the Hajós sum of two strongly persistent graphs is a strongly persistent graph.*

In [10,12,25,36] direct products of infinite graphs and direct products of infinite number of factors are considered. Miller [25] has shown that the direct product of infinitely many odd cycles is a bipartite graph. Hence the Hedetniemi's conjecture does not hold for the direct product of infinitely many graphs. Greenwell and Lovász [10] considered infinite products of complete graphs. They proved that each n-coloring of such a product is induced by n-colorings of the factors and a measure μ of the index set. Hajnal [12] and Soukup [36] proved several results on the direct product of infinite graphs.

3. Bounds for the strong and the lexicographic product

In this section we survey lower and upper bounds for the chromatic number of the strong and the lexicographic product. Bounds are given in terms of different parameters of factors. A recent result of Feigenbaum and Schäffer [6], that the strong product

admits a polynomial algorithm for decomposing a given connected graph into its factors, makes such results important also from the algorithmic point of view.

Clearly, any upper bound for the lexicographic product is also an upper bound for the strong product and any lower bound for the strong product is a lower bound for the lexicographic product. The lexicographic product is not "far away" from the strong product, for example $G[K_n] \simeq G \boxtimes K_n$, $n \geqslant 1$. Hence it is no surprise that bounds are similar for these two products.

Clearly, $\chi(G[H]) \leqslant \chi(G)\chi(H)$. This trivial upper bound is attained for any G and H with $\chi(G) = \omega(G)$ and $\chi(H) = \omega(H)$. But it was shown by Puš [30], that there is no graph product $*$ for which $\chi(G * H) = \chi(G)\chi(H)$ holds for all graphs G and H.

The following result of Geller and Stahl [9] says that it is enough to consider lexicographic products with the second factor being complete.

Theorem 3.1. *If $\chi(H) = n$, then $\chi(G[H]) = \chi(G[K_n])$ holds for any graph G.*

Proof. By the assumption, there is a homomorphism $f : H \to K_n$ and hence also a homomorphism $G[H] \to G[K_n]$. It follows $\chi(G[H]) \leqslant \chi(G[K_n])$.

Conversely, let f be an optimal coloring of $G[H]$ and let $a \in V(G)$. As $\chi(H) = n$, f restricted to H_a intersects at least n color classes. Choose n of them and in every class choose a vertex in H_a. Connect by an edge (if necessary) any two of the selected vertices. If we repeat this procedure for all vertices of G, we end up with a graph isomorphic to $G[K_n]$. It is straightforward to verify that it is properly colored. It follows that $\chi(G[K_n]) \leqslant \chi(G[H])$. □

We next give an upper bound which generalizes several previously known upper bounds. Let $<X>$ denote the subgraph of G induced by the vertices $X \subseteq V(G)$.

Theorem 3.2 (Kaschek and Klavžar [18]). *Let G and H be any graphs and let $\chi(H) = n$. Let $\{X_i\}_{i \in \{1, 2, \dots, k\}}$ be a partition of a set $X \subseteq V(G)$. Let for all i, $\chi(G - X_i) = m_i$ and let $\chi(<X>) = s$. Then,*

$$\chi(G[H]) \leqslant (m_1 + m_2 + \cdots + m_r)\left\lceil \frac{n}{k} \right\rceil$$
$$+ (m_{r+1} + m_{r+2} + \cdots + m_k + s)\left\lfloor \frac{n}{k} \right\rfloor$$
$$+ \chi(<X_1 \cup X_2 \cup \cdots \cup X_r>),$$

where $n = pk + r$, $0 \leqslant r < k$. □

Theorem 3.2 implies the next corollary from [20], which in turn includes a result on χ-critical graphs from [9].

Corollary 3.1. *If G is a χ-critical graph, then for any graph H,*

$$\chi(G[H]) \leqslant \chi(H)(\chi(G) - 1) + \left\lceil \frac{\chi(H)}{\alpha(G)} \right\rceil.$$

Proof. Let $\alpha(G) = k$ and let $X = \{x_1, x_2, \ldots, x_k\}$ be an independent set of G. Set $X_i = \{x_i\}$. Then apply Theorem 3.2. After a short calculation the result follows. $\square$

Corollary 3.2 (Geller and Stahl [9]). *If G is a χ-critical and not complete graph, and if $\chi(H) \geqslant \alpha(G)$, then*

$$\chi(G[H]) \leqslant \chi(G)\chi(H) - \left\lfloor \frac{\chi(H)}{\alpha(G)} \right\rfloor (\alpha(G) - 1).$$

Proof. Follows from Corollary 3.1 since for all $k, n \geqslant 1$ the following holds: $n - \left\lceil \frac{n}{k} \right\rceil \geqslant \left\lfloor \frac{n}{k} \right\rfloor (k-1)$. $\square$

We remark that the assumption $\chi(H) \geqslant \alpha(G)$ of the last corollary is redundant. Some more applications of Theorem 3.2 can be found in [18].

Let $L(G)$ denote the line graph of a graph G. We conclude giving upper bounds with the following result of Linial and Vazirani [22].

Theorem 3.3. *There is a constant c, such that*

$$\chi(L(G)[L(H)]) \leqslant c \cdot \max\{\chi(L(G)), \chi(L(H))\}$$

holds for all graphs G and H. $\square$

We next consider lower bounds. Vesztergombi [39] showed that if both G and H have at least one edge then $\chi(G \boxtimes H) \geqslant \max\{\chi(G), \chi(H)\} + 2$. In [17] Jha extended this lower bound to $\chi(G \boxtimes H) \geqslant \chi(G) + n$, where $n = \omega(H)$. However, in [37] Stahl introduced the *n-tuple colorings* of a graph G as an assignment of n distinct colors to each vertex of G, such that no two adjacent vertices share a color. Let $\chi_n(G)$ be the smallest number of colors needed to give G an n-tuple coloring. As $\chi_n(G) = \chi(G \boxtimes K_n)$, Stahl essentially proved:

Theorem 3.4. *If G has at least one edge, then for any H,*

$$\chi(G[H]) \geqslant \chi(G) + 2\chi(H) - 2.$$

Proof. Let $\chi(H) = n$. Due to Theorem 3.1 it is enough to prove the theorem for $H = K_n$. We may also assume $n \geqslant 2$. Let $\chi(G[K_n]) = n + s$. Since G has at least one edge, $s \geqslant n$. Let $f : V(G[K_n]) \to V(K_{n+s})$ be a coloring of $G[K_n]$. Let $V(K_n) = \{x_1, x_2, \ldots, x_n\}$.

For $a \in V(G)$ set $m_a = \min\{f(a, x_1), f(a, x_2), \ldots, f(a, x_n)\}$. Note that $m_a \leqslant s + 1$. Define a mapping $g : G \to K_{s+2-n}$ by

$$g(a) = \begin{cases} m_a, & m_a \leqslant s + 1 - n; \\ s + 2 - n, & m_a \geqslant s + 2 - n. \end{cases}$$

We claim that g is a coloring of G. Suppose that $ab \in E(G)$ and $g(a) = g(b)$. If $g(a) = g(b) \leqslant s + 1 - n$, then $m_a = m_b$, which is impossible since $ab \in E(G)$. Suppose that $g(a) = g(b) = s + 2 - n$. As $ab \in E(G)$ the vertices $\{a, b\} \times V(K_n)$ induce a complete graph K_{2n} in $G[K_n]$. Hence, these $2n$ vertices should be colored with different colors from the set $\{s+2-n, s+3-n, \ldots, s+n\}$ which contains only $2n-1$ elements. This contradiction proves the claim.

It follows from the claim that $\chi(G) \leqslant s + 2 - n$. Since $s = \chi(G[K_n]) - n$ we get $\chi(G) \leqslant \chi(G[K_n]) - 2n + 2$. $\quad\square$

Geller [8] improved this lower bound in one particular case. He proved that if G is uniquely m-colorable, $m > 2$, then $\chi(G[K_2]) \geqslant m + 3$. He also conjectured that for every uniquely m-colorable graph G, $\chi(G[K_n]) = mn$ holds. As far as we know, the conjecture is still open.

We give two more lower bounds. In the proof of the first we will repeat an elegant argument of Linial and Vazirani [22].

Theorem 3.5. *For any graphs G and H,*

$$\chi(G[H]) \geqslant \frac{(\chi(G) - 1)\chi(H)}{\ln|V(G)|}.$$

Proof. Let $\chi(G) = m$ and $\chi(H) = n$. Let $\chi(G[H]) = k$ and let f be a k-coloring of $G[H]$. Let $C_i = \{a \in V(G) | \exists x \in V(H) : f(a,x) = i\}$, $i = 1, 2, \ldots, k$.

Since C_i is an independent set, $\bigcup_{i \in S} C_i \neq V(G)$ for every index set S with $|S| = m - 1$. Furthermore, $\bigcup_{i \in S} C_i = V(G)$ for every index set S with $|S| = k - n + 1$. Indeed, for otherwise the layer H_a, $a \in (V(G) - \bigcup_{i \in S} C_i)$, would be colored using $k - (k - n + 1) = n - 1$ colors. We thus have the following two conditions:

If $|S| = m - 1$ then $\bigcap_{i \in S} \overline{C_i} \neq \emptyset$.

If $|S| = k - n + 1$ then $\bigcap_{i \in S} \overline{C_i} = \emptyset$. $\qquad\qquad\qquad (*)$

Now, in every index set S, $|S| = m - 1$, choose a canonical element from $\bigcap_{i \in S} \overline{C_i}$. We have, using the condition $(*)$:

$|V(G)| \geqslant$ number of distinct canonical elements

$$\geqslant \frac{\text{number of choices of } S}{\text{max. number of sets } S \text{ with same canonical element}}$$

$$\geqslant \binom{k}{m-1} \bigg/ \binom{k-n}{m-1}$$

$$\geqslant \left(\frac{k}{k-n}\right)^{m-1} = \left(1 - \frac{n}{k}\right)^{-(m-1)}$$

$$\geqslant e^{\frac{(m-1)n}{k}}. \qquad \square$$

The next lower bound was first proved by Stahl in [37], while in [3] a straightforward and simple proof is given. Here we give an elegant and short proof due to Zhu [45].

Theorem 3.6. *Let G be a nonbipartite graph. Then for any graph H,*

$$\chi(G[H]) \geqslant 2\chi(H) + \left\lceil \frac{\chi(H)}{k} \right\rceil,$$

where $2k + 1$ is the length of a shortest odd cycle in G.

Proof. Let $\chi(H) = n$. Clearly, it is enough to prove the theorem for $G = C_{2k+1}$.

Suppose that $\chi(C_{2k+1}[H]) = m$, and let $S_1, S_2, \ldots, S_m$ be the coloring classes of an m coloring of $C_{2k+1}[H]$.

Since $\alpha(C_{2k+1}) = k$ we have

$$|\{v \in C_{2k+1} | S_i \cap H_v \neq \emptyset\}| \leqslant k$$

for $i = 1, 2, \ldots, m$. On the other, because $\chi(H) = n$ the inequality

$$|\{i | S_i \cap H_v \neq \emptyset\}| \geqslant n$$

holds for every vertex $v \in C_{2k+1}$. Therefore $n(2k + 1) \leqslant mk$ which immediately implies the result. $\square$

In the rest of the section we collect several exact results. The vertices of Kneser graph $KG_{n,k}$ are the n–subsets of the set $\{1, 2, \ldots, 2n+k\}$ and two vertices are adjacent if and only if the corresponding subsets are disjoint. The result involving Kneser graphs is included to be used in the last section.

Theorem 3.7. (i) ([39,15]) *For $k \geqslant 2$ and $n \geqslant 2$, $\chi(C_{2k+1} \boxtimes C_{2n+1}) = 5$.*

(ii) ([3]) *For $n \geqslant k \geqslant 2$, $\chi(C_{2k+1}[C_{2n+1}]) = 8, k = 2, and it is 7 for $k \geqslant 3$.*

(iii) ([20]) *For $k \geqslant 2$, $\chi(C_5 \boxtimes C_5 \boxtimes C_{2k+1}) = 10 + \left\lceil \frac{5}{k} \right\rceil$.*

(iv) ([37]) *For $k \geqslant 2$ and $n \geqslant 1$, $\chi(C_{2k+1} \boxtimes K_n) = 2n + \left\lceil \frac{n}{k} \right\rceil$.*

(v) ([21]) *For $k \geqslant 0$ and $n \geqslant 1$, $\chi(KG_{n,k} \boxtimes K_n) = 2n + k$.*

Proof. (i) As $\chi(C_{2k+1} \boxtimes C_{2n+1}) \geqslant \chi(C_{2k+1} \boxtimes K_2) = \chi(C_{2k+1}[K_2])$, the lower bound follows from Theorem 3.4. To construct a 5-coloring we first color $C_5 \boxtimes C_5$ and then extend this coloring. See [39] for details, cf. also [15].

(ii) The lower bound follows from Theorem 3.6. Furthermore, it is not difficult to construct a coloring with desired number of colors. See [3] for details.

(iii) Note first that $\alpha(C_{2k+1}) = k$ and $\alpha(C_5 \boxtimes C_5) = 5$. If we apply the following result from [35, p. 142]

$$\alpha(C_5 \boxtimes C_5 \boxtimes C_{2k+1}) = \alpha(C_5 \boxtimes C_5)\alpha(C_{2k+1}),$$

we obtain

$$\chi(C_5 \boxtimes C_5 \boxtimes C_{2k+1}) \geqslant \frac{5 \times 5 \times (2k + 1)}{5 \times k} = 10 + \frac{5}{k}.$$

On the other hand, writing $\chi(C_5 \boxtimes C_5 \boxtimes C_{2k+1}) = \chi(C_{2k+1} \boxtimes (C_5 \boxtimes C_5))$ and using Corollary 3.1 we get

$$\chi(C_{2k+1} \boxtimes (C_5 \boxtimes C_5)) \leqslant 15 - 5 + \lceil \tfrac{5}{k} \rceil.$$

(iv) The lower bound follows from Theorem 3.6. It is also not difficult to construct a coloring with desired number of colors. In fact, one can use a similar approach as in the proof of Theorem 3.6.

(v) Lovász proves in [23] that $\chi(KG_{n,k}) = k+2$, thus settling a conjecture of Kneser. Combining this result with Theorem 3.4 we get $\chi(KG_{n,k} \boxtimes K_n) \geqslant k + 2 + 2n - 2 = 2n + k$. Since Vesztergombi observed in [40] that $\chi(KG_{n,k} \boxtimes K_n) \leqslant 2n + k$, the result follows. □

We conclude the section with the following problem. Determine the chromatic number of the strong product of several odd cycles. In particular, for $k, n, m \geqslant 2$ determine $\chi(C_{2k+1} \boxtimes C_{2n+1} \boxtimes C_{2m+1})$.

4. Applications

We will give three applications of product colorings, one for each product. Greenwell and Lovász [10] proved the following application of the direct product:

Theorem 4.1. *For all $n \geqslant 3$ there is a uniquely n-colorable graph without odd cycles shorter than a given number s.*

Proof. Let G be a graph with $\chi(G) > n$, and let G be without odd cycles shorter than a given number s (it is well-known that such graphs exist for all n and s). Then neither $G \times K_n$ does contain odd cycles shorter than s. It is also not hard to see that $\chi(G \times K_n) = n$. Furthermore, the graph $G \times K_n$ is uniquely colorable (for brief proofs of the last two facts see [24], problems 9.7(c) and 9.7(d)). This completes the proof. □

Results about the chromatic number of strong products turned out to be important in understanding retracts of strong products [16,19,21]. It is shown in [16] that every retract R of $G \boxtimes H$ is of the form $R = G' \boxtimes H'$, where G' is a (isometric) subgraph of G and H' is a (isometric) subgraph of H. Furthermore, if G and H are triangle-free, then G' and H' are retracts of G and H, respectively. However, using Theorem 3.7(v) we have:

Theorem 4.2. *For every $n \geqslant 2$ there is an infinite sequence of pairs of graphs G and G' such that G' is not a retract of G while $G' \boxtimes K_n$ is a retract of $G \boxtimes K_n$.*

Proof. Let $k \geqslant 2$ and let $H_{n,k}$ be a graph which we get from a copy of the graph $KG_{n,k}$ and a copy of the complete graph K_{k+1} by joining a vertex x of $KG_{n,k}$ with a vertex y of K_{k+1}. Using Theorem 3.7(v) it is easy to see that $\chi(H_{n,k}) = n(k+1)$. It follows that we have a retraction from $H_{n,k} \boxtimes K_n$ onto the subgraph $K_{k+1} \boxtimes K_n$ (take any color preserving map). But since $\chi(H_{n,k}) = k+2$ and the chromatic number is preserved by a retraction, there is no retraction $V(H_{n,k}) \rightarrow V(K_{k+1})$. $\square$

Linial and Vazirani [22] applied colorings of the lexicographic product. They used Theorem 3.5 together with the trivial upper bound to study approximation algorithms for computing the chromatic number of a graph. However, the idea of using products of a given graph with certain fixed graphs goes back to Garey and Johnson [7].

Some other graph products have also been treated with respect to their chromatic numbers. The *Cartesian sum* of graphs was studied by Yang [42], Borowiecki [1], Hell and Roberts [15], Puš [30], Čižek and Klavžar [3] and the *alternative negation* by Borowiecki [1] and Schäffer and Subramanian [34]. Although we didn't consider the Cartesian sum it turned out that it is natural to study in parallel the chromatic number of the strong product, the lexicographic product and the Cartesian sum of graphs (cf. [15,3]).

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