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A Convexity Lemma and Expansion Procedures for Bipartite Graphs

WILFRIED IMRICH AND SANDI KLAVŽAR[†]

A hierarchy of classes of graphs is proposed which includes hypercubes, acyclic cubical complexes, median graphs, almost-median graphs, semi-median graphs and partial cubes. Structural properties of these classes are derived and used for the characterization of these classes by expansion procedures, for a characterization of semi-median graphs by metrically defined relations on the edge set of a graph and for a characterization of median graphs by forbidden subgraphs. Moreover, a convexity lemma is proved and used to derive a simple algorithm of complexity $O(mn)$ for recognizing median graphs.

1. INTRODUCTION

Hamming graphs and related classes of graphs have been of continued interest for many years as can be seen from the list of references. As the subject unfolded, many interesting problems arose which have not been solved yet. In particular, it is still open whether the known algorithms for recognizing partial cubes or median graphs, which comprise an important subclass of partial cubes, are optimal. The best known algorithms for recognizing whether a graph G is a member of the class $\mathcal{PC}$ of partial cubes have complexity $O(mn)$, where m and n denote, respectively, the numbers of edges and vertices of G, see [1, 13]. As the recognition process involves a coloring of the edges of G, which is a special case of sorting, one might be tempted to look for an algorithm of complexity $O(m \log n)$. However, even the best known algorithm for recognizing membership in the class $\mathcal{M}$ of median graphs [12] has complexity $O(mn^{1/2})$, whereas membership in the class $\mathcal{H}$ of binary Hamming graphs, i.e., the class of hypercubes, can be tested in linear time $O(m)$, see [6, 14]. For the classes

$$\mathcal{H} \subset \mathcal{M} \subset \mathcal{PC}$$

we thus have recognition algorithms of complexities $O(m)$, $O(mn^{1/2})$ and $O(mn)$.

Most of the algorithms make use of so-called expansion procedures for generating these graphs, cf. for instance [15]. For more information on the existing algorithms we refer to [14].

To shed more light on the situation we introduce almost-median graphs $\mathcal{AM}$ and semi-median graphs $\mathcal{PM}$ in this paper. These classes are naturally defined by expansion procedures. Together with the class $\mathcal{ACC}$ of so-called acyclic cubical complexes introduced by Bandelt and Chepoi [4] we thus arrive at the following hierarchy

$$\mathcal{H} \subset \mathcal{ACC} \subset \mathcal{M} \subset \mathcal{AM} \subset \mathcal{PM} \subset \mathcal{PC}.$$

With the aim of better understanding the structure of partial cubes and median graphs and with the intention of laying the foundation for better recognition algorithms we then investigate this hierarchy.

We begin with a lemma, the so-called Convexity Lemma, which we then use for the introduction of a simple, new algorithm of complexity $O(mn)$ for median graphs.

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Then we define the new classes and prove some fundamental properties. In particular, we show that semi-median graphs can be characterized in a way very similar to Winkler's characterization of partial cubes [20] and present a new characterization of median graphs by forbidden subgraphs.

With the aid of the concepts developed here we plan a follow-up of this paper in which we wish to present an algorithm of subquadratic complexity for recognizing semi-median graphs, a new, simpler algorithm of complexity $O(mn^{1/2})$ for recognizing median graphs and efficient algorithm for recognizing acyclic cubical complexes.

All graphs considered in this paper are finite undirected graphs without loops or multiple edges. If not stated otherwise, the graphs are also connected. For a graph G and a vertex set $X \subseteq V(G)$ let $\langle X \rangle$ denote the subgraph of G induced by X.

For $u, v \in V(G)$ let $d_G(u, v)$, or simply $d(u, v)$, denote the length of a shortest path in G from u to v. A subgraph H of a graph G is an *isometric* subgraph, if $d_H(u, v) = d_G(u, v)$ for all $u, v \in V(H)$. A subgraph H of G is *convex*, if for any $u, v \in V(H)$, all shortest u–v paths belong to H. Clearly, a convex subgraph is an isometric subgraph but the converse need not be true.

Graphs that can be isometrically embedded into a hypercube are called *partial cubes*. In other words, a graph G is a partial cube if its vertices can be labeled by binary labels of a fixed length such that the distance between any two vertices of G is equal to the Hamming distance between the corresponding labels. A *median* of a triple of vertices u, v and w is a vertex that lies simultaneously on a shortest u–v path, a shortest u–w path, and a shortest v–w path. A graph G is then called a *median graph* if every triple of its vertices has a unique median. Finally, by Q_3^- we denote the graph obtained from the 3-cube Q_3 by removing an arbitrary vertex (i.e., the graph G_3 from Figure 1).

2. The Relation Θ and the Convexity Lemma

In this section we introduce a relation Θ, give some basic properties of it and present the Convexity Lemma. This lemma uses Θ for the characterization of convex subgraphs of bipartite graphs. At the end of the section we then use the lemma to present a simple $O(mn)$ algorithm for recognizing median graphs.

The relation Θ, which was first introduced by Djoković [9], is defined on the edge set of a graph as follows: let $e = xy$ and $f = uv$ be edges of a graph G. Then e and f are in relation Θ if

$$d_G(x, u) + d_G(y, v) \neq d_G(x, v) + d_G(y, u).$$

This relation is reflexive and symmetric. We shall denote its transitive closure by Θ^*. For further reference we list three assertions about Θ. The proofs are straightforward and can be found in [13].

LEMMA 2.1. *Let P be a shortest path in a graph G. Then no two different edges of P are in relation Θ.*

LEMMA 2.2. *Suppose Q is a path connecting the endpoints of an edge e. Then Q contains an edge f with $e\Theta f$.*

For an edge uv of a graph G let

$$W_{uv} = \{w \mid w \in V(G), d_G(w, u) < d_G(w, v)\}.$$

For later use we also introduce the sets U_{ab} as follows:

$$U_{ab} = \{u \in W_{ab} \mid u \text{ is adjacent to a vertex in } W_{ba}\}.$$

LEMMA 2.3. *Let $e = uv$ be an edge of a connected bipartite graph G and let*

$$F_{uv} = \{ f \mid f \in E(G), e\Theta f\}.$$

Then $G \setminus F_{uv}$ has exactly two connected components and they are induced by the vertex sets W_{uv} and W_{vu}. Furthermore, every shortest path P from u to a vertex x in W_{uv} is contained in W_{uv}.

The only assertion not explicitly proved in [13] is the last statement of Lemma 2.3. It follows from the fact that $vu \cup P$ is a shortest path from v to x. Thus, by Lemma 2.1, P cannot contain an edge in relation Θ to vu, i.e., an edge of F_{uv}.

We wish to remark that the relation Θ was recently rediscovered by Barthélemy and Constantin [5]. They proved several properties of Θ in the case of median graphs, obviously unaware of the previous work of Djoković [9], Graham and Winkler [11], and others. The following theorem of Winkler [20] will be used several times later.

THEOREM 2.4 ([20]). *Let G be a connected graph. Then G is a partial cube if and only if G is bipartite and $\Theta^* = \Theta$.*

Before stating the Convexity Lemma, we need one more observation.

LEMMA 2.5. *Let G be bipartite and $e\Theta f$ for two edges $e = uv$ and $f = xy$ of G. Then the notation can be chosen such that*

$$d(u, x) = d(v, y) = d(u, y) - 1 = d(v, x) - 1.$$

PROOF. Clearly $d(u, x) \neq d(u, y)$, for otherwise the bipartite graph G would contain a circuit of odd length (containing u, x and y). As x and y are connected by an edge these distances can differ by at most one. Let the notation be chosen such that $d(u, y) = d(u, x) + 1$. By the same argument as before we have $d(v, x) \neq d(v, y)$. Then

$$d(v, y) \leq d(v, u) + d(u, x) = 1 + d(u, x) = 1 + d(u, y) - 1,$$

and we infer $d(u, y) = d(v, y) + 1$. □

We are now ready for the Convexity Lemma. First a definition: let H be a subgraph of a graph G. Then the *boundary* ∂H of H in G is the set of all edges xy of G with $x \in H$ and $y \notin H$.

LEMMA 2.6 (CONVEXITY LEMMA). *An induced connected subgraph H of a bipartite graph G is convex if and only if no edge of ∂H is in relation Θ to an edge in H.*

PROOF. Suppose H is convex and there are edges $uv = e \in H$ and $xy = f \in \partial H$ with $e\Theta f$, where $x \in H$ and $y \notin H$. By Lemma 2.5 f is on a shortest path from v to x, in contradiction to the convexity of H.

For the converse, let a, b be two vertices in the induced connected subgraph H of G and suppose no edge of the boundary ∂H is in relation Θ to an edge in H. Let P be a shortest path in G from a to b and Q be a path in H from a to b. If P is not in H, it must contain an edge in ∂H. Let this edge be e. By Lemma 2.1 it is not in relation Θ with any other edge of P. However, by Lemma 2.2 it must be in relation Θ to an edge in $P \cup Q \setminus e$ and thus in relation Θ to an edge in $Q \subseteq H$, in contradiction to our assumptions. □

To obtain a simple algorithm for recognizing median graphs we are going to combine the Convexity Lemma with the following characterization of median graphs due to Bandelt [2]. We also refer to Bandelt, Mulder and Wilkeit [3] where a generalization to quasi-median graphs is given.

THEOREM 2.7 ([2]). *A graph G is a median graph if and only if G is bipartite and for every edge ab of G, the sets U_{ab} and U_{ba} are convex.*

We will use Theorem 2.7 also in Section 3.

The crucial step of recognizing median graphs by the above approach is thus checking if, for every edge ab of a given graph, the subgraphs $\langle U_{ab}\rangle$ and $\langle U_{ba}\rangle$ are convex. Moreover, since $\Theta = \Theta^*$ for median graphs, we need not compute the two sets for each edge, because these sets are equal for all edges of a given matching F_{ab}.

In our algorithm we first verify whether a given graph G is bipartite and whether $\Theta = \Theta^*$ holds. By Theorem 2.4 this is equivalent to testing whether G is a partial cube. In [13] a simple $O(n^2 \log n)$ algorithm is given which recognizes such graphs and, furthermore, the algorithm also yields an embedding into a hypercube if it exists, cf. also [1]. In other words, we obtain the subgraphs $\langle U_{ab}\rangle$ and $\langle U_{ba}\rangle$ and colors for any edge of G. Here two edges are of the same color if they are in relation Θ.

Having all this in mind we can reformulate the convexity lemma for median graphs as follows. The subgraph $\langle U_{ab}\rangle$ is convex if and only if no color from $\langle U_{ab}\rangle$ appears on the edges of ∂H.

Our algorithm for recognizing median graphs consists of only two steps:

(1) Check if G is a partial cube.
(2) For each color obtained in Step 1 check if no color from $\langle U_{ab}\rangle$ and $\langle U_{ba}\rangle$ appears on the edges of $\partial\langle U_{ab}\rangle$ and $\partial\langle U_{ba}\rangle$, respectively.

THEOREM 2.8. *The above algorithm recognizes median graphs and can be implemented to run in $O(mn) = O(n^2 \log n)$ steps.*

PROOF. The correctness of the algorithm follows from the above discussion.

Concerning the time complexity we first recall [10, 13] that for partial cubes (and hence also for median graphs) $m = O(n \log n)$.

For Step 1 of the algorithm we use a simple algorithm from [13] which runs in $O(n^2 \log n)$ time. For the second step we know from [13], cf. also [11], that there are at most $n - 1$ different colors obtained in the first step. Therefore we have to check at most $2n - 2$ sets for their convexity. By the convexity lemma it is trivial to do it by checking every edge, i.e., in $O(m)$ time. Thus the overall complexity of Step 2 is $O(mn) = O(n^2 \log n)$. □

3. EXPANSION PROCEDURES, SEMI-MEDIAN AND ALMOST-MEDIAN GRAPHS

In this section we investigate two new classes of graphs, semi-median graphs and almost-median graphs. In particular we study their role in the so-called expansion procedures and from this point of view we contribute to the 'masterplan' for studying expansion procedures proposed by Mulder [19]. In his setting expansions studied here are connected (resp. isometric, resp. convex) Cartesian expansions. A Winkler-like characterization of semi-median graphs is also given and it is shown that a graph is a median graph if and only if it is an almost-median graph with no convex Q_3^- as a subgraph.

Let G be a graph and let V_1 and V_2 be vertex subsets of G such that $V_1 \cap V_2 \neq \emptyset$ and $V_1 \cup V_2 = V(G)$. Assume in addition that $\langle V_1\rangle$ and $\langle V_2\rangle$ are isometric subgraphs of G and that there is no edge between a vertex of $V_1 \setminus V_2$ and a vertex of $V_2 \setminus V_1$. An *expansion* of a graph G (with respect to V_1 and V_2) is a graph H, obtained from G in the following way.

(i) Replace each vertex $v \in V_1 \cap V_2$ by vertices v_1 and v_2 and join then by an edge.
(ii) Join v_1 and v_2 to all neighbors of v in $V_1 \setminus V_2$ and $V_2 \setminus V_1$, respectively.
(iii) If $v, u \in V_1 \cap V_2$ are adjacent in G, then join v_1 to u_1 and v_2 to u_2.

A *contraction* is just the inverse operation of the expansion, i.e., G is a contraction of H. An *expansion procedure* is just a sequence of expansions.

Observe that in the expansion as defined, there is **no** condition on the intersection $S = V_1 \cap V_2$ (except that it is not empty). We call an expansion a *connected expansion* if S is connected, an *isometric expansion* if S is isometric, and a *convex expansion* if S is convex.

The point of departure for the above expansions is the convex expansion theorem due to Mulder [17, 18]. He proved that a graph is a median graph if and only if it can be obtained from the one vertex graph by a sequence of convex expansions. Later Chepoi [7], see also [8], proved the analogous result for partial cubes. They can be obtained by a sequence of expansions from the one vertex graph. For further reference we state these two results as a theorem.

THEOREM 3.1.

(i) ([17]) A graph G is a median graph if and only if G can be obtained from the one vertex graph by a sequence of convex expansions.
(ii) ([7]) A graph G is a partial cube if and only if G can be obtained from the one vertex graph by a sequence of expansions.

Concerning the theorem of Chepoi we wish to add that he used the term isometric expansion for what we call expansion here. Note also that the acyclic cubical complexes are precisely the graphs obtained by the convex expansion procedure from the one vertex graph provided that the intersection $V_1 \cap V_2$ in every expansion step is a hypercube.

In the sequel we shall consider the classes of graphs obtained from the one vertex graph by connected resp. by isometric expansions. Recall that

$$U_{ab} = \{u \in W_{ab} \mid u \text{ is adjacent to a vertex in } W_{ba}\}.$$

We call a bipartite graph G a *semi-median* graph, if $\Theta = \Theta^*$ and for any edge ab of G the subgraph $\langle U_{ab} \rangle$ is connected. Similarly, we say that a bipartite graph G is an *almost-median* graph, if $\Theta = \Theta^*$ and for any edge ab of G the subgraph $\langle U_{ab} \rangle$ is isometric. The characterizations of these new classes of graphs will be facilitated by the following lemma.

LEMMA 3.2. *Let G be a triangle free graph with transitive Θ and let $ab \in E(G)$. Then F_{ab} is a matching and induces an isomorphism between $\langle U_{ab} \rangle$ and $\langle U_{ba} \rangle$.*

PROOF. As G is a triangle free graph with transitive Θ, we see easily that G must be bipartite.

Let $e = uw$ and $f = uv$ be distinct edges of F_{ab}. As Θ is transitive we have $e\Theta f$. But G is bipartite, so this is impossible. Hence F_{ab} is a matching.

To show that F_{ab} induces an isomorphism, let $uu', vv' \in F_{ab}$ and assume that $uv \in E(G)$. As Θ is transitive we have $uu'\Theta vv'$ and therefore $u'v' \in E(G)$. □

THEOREM 3.3. *A graph G is a semi-median graph if and only if G can be obtained from the one vertex graph by the connected expansion procedure.*

PROOF. Let G be a semi-median graph. Then G is a partial cube and by Chepoi's expansion theorem (Theorem 3.1 (ii)) it can be obtained from the one vertex graph by the expansion procedure. We claim that each step of the expansion procedure is connected and prove the claim by the induction on the number of expansion steps. Let G be a graph obtained by an expansion from a semi-median graph G' with the corresponding vertex subsets V_1 and V_2. Let F_{ab} be the new Θ-class of G. By Lemma 3.2 we find that $\langle V_1 \cap V_2\rangle$ is isomorphic to $\langle U_{ab}\rangle$ in G. Moreover, since G is a semi-median graph, $\langle U_{ab}\rangle$ is connected, hence G was obtained from G' by a connected expansion.

Conversely, assume that G can be obtained from the one vertex graph by connected expansions. We need to show that if a graph G is obtained from a semi-median graph G' (with the corresponding vertex subsets V_1 and V_2) by a connected expansion, then G is semi-median as well. By Theorem 3.1 (ii) we know that G is a partial cube. We must thus show that $\langle U_{uv}\rangle$ is connected for any edge uv of G. Let F_{ab} be the new Θ-class of G. If $uv \in F_{ab}$ then U_{uv} is clearly connected hence we may assume that $uv \notin F_{ab}$. Let F'_{uv} be the Θ-class of G' which naturally corresponds to the class F_{uv} of G. Then F_{uv} is obtained from F'_{uv} in the following way. If an edge $u'v'$ of F'_{uv} belongs to $V_1 \cap V_2$, then add to F_{uv} the two corresponding expanded edges. All the other edges of F'_{uv} are also edges of F_{uv}. It follows that $\langle U_{uv}\rangle$ is connected, because the corresponding subgraph of G' is connected. $\square$

Mulder (personal communication) pointed out that an isometric expansion of an almost-median graph need not be an almost-median graph. Consider, for instance, the graph G_3 from Figure 1. It is almost-median and its outer 6-cycle is isometric. Thus we can expand G_3 such that $V_1 = V(G_3)$ and V_2 is the vertex set of the 6-cycle. The graph we obtain is the graph G_2 from the same figure, which is not an almost-median graph. However, one may follow the lines of the first part of the proof of Theorem 3.3 (replacing 'connected' by 'isometric') to obtain the following result.

PROPOSITION 3.4. *Let G be an almost-median graph. Then G can be obtained from the one vertex graph by the isometric expansion procedure.*

In Figure 1 the graph G_1 is a partial cube which is not a semi-median graph. G_2 is a semi-median graph but not an almost-median graph, while G_3 is almost-median but not median. G_4 is a median graph.

It follows from Theorem 3.1 that a graph is a median graph if and only if $\Theta = \Theta^*$ and if the sets U_{ab} are convex. We have seen in Theorem 2.7 that the condition $\Theta = \Theta^*$ can be replaced in the characterization of median graphs by bipartiteness. On the other hand, the condition $\Theta = \Theta^*$ cannot be relaxed to bipartiteness in the characterization of semi-median graphs and almost-median graphs. Consider, for instance, the graph $K_{2,3}$ in which the sets U_{ab} are connected and isometric, yet $K_{2,3}$ is not even a partial cube. Note that the last fact immediately follows from Theorem 2.4.

We now introduce a relation δ defined on the edge set of a bipartite graph. We say an edge e is in relation δ to an edge f if e and f are opposite edges of a square in G or if $e = f$. Clearly δ is reflexive and symmetric. Moreover, it is contained in Θ. Thus, the transitive closure δ^* is contained in Θ^*.

THEOREM 3.5. *A bipartite graph is a semi-median graph if and only if $\Theta = \delta^*$.*

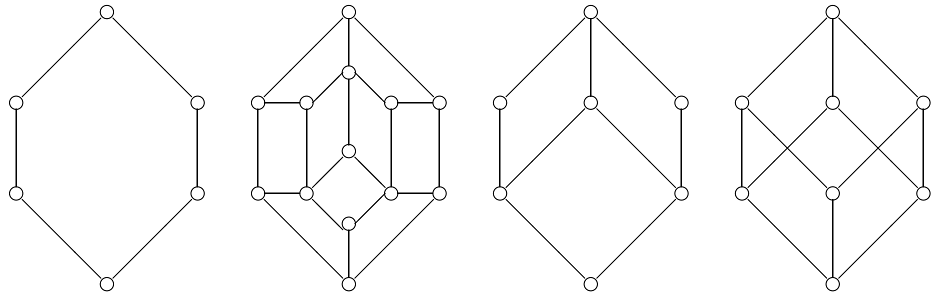

FIGURE 1. The introduced hierarchy of classes of graphs is strict.

PROOF. Suppose G is a semi-median graph. It follows from Lemma 2.3 that only an edge uv, with $u \in U_{ab}$ and $v \in U_{ba}$, can be in relation Θ to ab. By the connectedness of U_{ab} every edge uv in relation Θ to ab clearly also is in relation δ^* to ab. Thus, $\Theta \subseteq \delta^*$. As $\delta^* \subseteq \Theta^* = \Theta$ we deduce that $\Theta = \delta^*$.

On the other hand, if $\Theta = \delta^*$ we infer that Θ is transitive and thus we can embed G isometrically into a hypercube. It remains to show that $\langle U_{ab}\rangle$ and $\langle U_{ba}\rangle$ are connected. Let the edge uv, where $u \in U_{ab}$ and $v \in U_{ba}$, be in relation Θ to ab. As $\Theta = \delta^*$ there are paths $u = u_0u_1 \dots u_k = a$ and $v = v_0v_1 \dots v_k = b$ such that all $u_iv_i, 0 \le i \le k$ are in $E(G)$. But then $u = u_0u_1 \dots u_k = a$ is in $\langle U_{ab}\rangle$ and $v = v_0v_1 \dots v_k = b$ in $\langle U_{ba}\rangle$. As uv was arbitrarily chosen, this means that both $\langle U_{ab}\rangle$ and $\langle U_{ba}\rangle$ are connected. □

Using Theorem 3.5 we can generalize several results from [5] from median to semi-median graphs. For instance, Proposition 1.7 of [5] reads (for semi-median graphs) as follows.

COROLLARY 3.6. *Let G be a semi-median graph. For every cycle C of G and any edge e of C there is another edge e' of C such that $e\delta^*e'$.*

PROOF. Apply Lemma 2.2 and Theorem 3.5. □

Many characterizations of median graphs are known, see [16] for a recent survey. In our next theorem we add another characterization via almost-median graphs.

THEOREM 3.7. *A graph is a median graph if and only if it is almost-median and contains no convex Q_3^- as a subgraph.*

PROOF. A median graph is clearly almost-median and contains no convex Q_3^-.

For the converse suppose that G is an almost-median graph which contains no convex Q_3^- as a subgraph. By Theorem 2.7 it is enough to show that the sets U_{ab} are convex. Suppose on the contrary that some set U_{ab} is not convex and let u and v be two vertices of U_{ab} such that there is a u–v shortest path, say P, which is not completely in U_{ab}. We may assume that P is as short as possible. As U_{ab} is isometric, it follows by the minimality that no internal vertex of P lies in U_{ab}. Let Q be a shortest u–v path which lies in U_{ab}. Clearly, $|Q| = |P|$. Let u' and v' be the vertices of U_{ba} with $uu', vv' \in F_{ab}$. Let w be the vertex of Q adjacent to v and let w' be the corresponding vertex of U_{ba}. Let x be the vertex of P adjacent to u and recall that $x \notin U_{ab}$.

CASE 1. $|P| = 2$.
Let $H = \langle\{x, u, w, v, u', w', v'\}\rangle$. We wish to show that H is a convex Q_3^-. Note that w and x are adjacent to both u and v in this case.

We first consider the pairs of vertices in H of distance 2. Observe that w and x are the only common neighbors of v and u because G is a $K_{2,3}$-free graph. For the same reason x and w have no other common neighbor than u and v. As F_{ab} is a matching, we also quickly see that the only paths of length 2 between the pairs of vertices $\{x, u'\}$, $\{x, v'\}$, $\{u, w'\}$, $\{v, w'\}$, $\{w, u'\}$ and $\{w, v'\}$ are those induced by H. Assume that there is a path v'–w''–u', where $w'' \neq w'$. If $w'' \notin U_{ba}$ then by the transitivity of Θ we find that $vx\Theta v'w''$, which is impossible, and if $w'' \in U_{ba}$ then we find a $K_{2,3}$ in W_{ab}.

It remains to consider the pairs of vertices in H of distance 3 (in H). By symmetry it is enough to consider the pairs $\{x, w'\}$ and $\{u, v'\}$. Because F_{ab} is a matching, it follows that $d_G(x, w') = d_G(u, v') = 3$. Suppose that there is a path x–x'–x''–w', where $x' \in U_{ab}$, $x' \neq v, u$ and $x'' \in U_{ba}$, $x'' \neq v', u'$. Then $x'x'' \in F_{ab}$ and as Θ is transitive x' is adjacent to w. But then the vertices x, w and u, v, x' induce a $K_{2,3}$. Analogously, a path u–x'–x''–v', where $x' \in U_{ab}$, $x' \neq v, w$ and and $x'' \in U_{ba}$, $x'' \neq u', w'$ would give us another $K_{2,3}$. We have thus found a convex Q_3^- in G, which settles the first case.

CASE 2. $|P| = k \geq 3$.
Clearly $d(x, v) = d(u, w) = k-1$. Because of the minimality of $d(u, v)$ we have $d(x, w) \geq k-1$ and as G is bipartite we conclude that $d(x, w) = k$. It follows that the edges ux and wv are in relation Θ. Set $Q = u - q_1 - q_2 - \cdots - q_{k-2} - w - v$ and consider the set U_{ux}. By the minimality, the u–w subpath of Q belongs to U_{ux}. By Lemma 3.2 there is a path $x - r_1 - r_2 - \cdots - r_{k-2} - v$ in U_{xu}, where r_i is adjacent to q_i, for $i = 1, 2, \ldots, k-2$. As $x \in W_{ab} \setminus U_{ab}$ we have $r_1 \in W_{ab}$. Suppose $r_1 \notin U_{ab}$. Then $q_1 - r_1 - r_2 - \cdots - r_{k-2} - v$ is a shortest q_1–v path (of length $k-1$) which is not in U_{ab}, a contradiction with the minimality of $d(u, v)$. So we must have $r_1 \in U_{ab}$. But then $d(u, r_1) = 2$, yet u–x–r_1 is a path not in U_{ab}, the final contradiction. □

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REFERENCES

1. F. Aurenhammer and J. Hagauer, Recognizing binary Hamming graphs in $O(n^2 \log n)$ time, in: *Proceedings of the 16th International Workshop on Graph Theoretical Concepts in Computer Science, Lecture Notes in Computer Science*, **484**, Springer-Verlag, New York, 1991, pp. 90–98. Also: *Math. Systems Theory*, **28** (1995), 387–395.
2. H.-J. Bandelt, Characterizing median graphs, manuscript, 1982.
3. H.-J. Bandelt, H.M. Mulder, and E. Wilkeit, Quasi-median graphs and algebras, *J. Graph Theory*, **18** (1994), 681–703.
4. H.-J. Bandelt and V. Chepoi, Graphs of acyclic cubical complexes, *Eur. J. Combin.*, **17** (1996), 113–120.
5. J.-P. Barthélemy and J. Constantin, Median graphs, parallelism and posets, *Discrete Math.*, **111** (1993), 49–63.
6. K.V.S. Bhat, On the complexity of testing a graph for n-cube, *Inform. Process. Lett.*, **11** (1980), 16–19.
7. V. Chepoi, d-Convexity and isometric subgraphs of Hamming graphs, *Cybernetics*, **1** (1988), 6–9.

8. V. Chepoi, Separation of two convex sets in convexity structures, *J. Geom.*, **50** (1994), 30–51.
9. D. Djoković, Distance preserving subgraphs of hypercubes, *J. Combin. Theory Ser. B*, **14** (1973), 263–267.
10. R.L. Graham, On primitive graphs and optimal vertex assignments, *Ann. NY Acad. Sci.*, **175** (1970), 170–186.
11. R.L. Graham and P.M. Winkler, On isometric embeddings of graphs, *Trans. Am. Math. Soc.*, **288** (1985), 527–536.
12. J. Hagauer, W. Imrich, and S. Klavžar, Recognizing median graphs in subquadratic time, *Theoret. Comput. Sci.*, in press.
13. W. Imrich and S. Klavžar, A simple $O(mn)$ algorithm for recognizing Hamming graphs, *Bull. Inst. Comb. Appl.*, **9** (1993), 45–56.
14. W. Imrich and S. Klavžar, On the complexity of recognizing Hamming graphs and related classes of graphs, *Eur. J. Combin.*, **17** (1996), 209–221.
15. P.K. Jha and G. Slutzki, Convex-expansions algorithms for recognizing and isometric embedding of median graphs, *Ars Combin.*, **34** (1992), 75–92.
16. S. Klavžar and H.M. Mulder, Median graphs: characterizations, location theory and related structures, *J. Combin. Math. Combin. Comput.*, in press.
17. H.M. Mulder, The structure of median graphs, *Discrete Math.*, **24** (1978), 197–204.
18. H.M. Mulder, *The Interval Function of a Graph, Mathematical Centre Tracts*, **132**, Mathematisch Centrum, Amsterdam, 1980.
19. H.M. Mulder, The expansion procedure for graphs, in: *Contemporary Methods in Graph Theory*, R. Bodendiek (ed.), B.I.-Wissenschaftsverlag, Mannheim/Wien/Zürich, 1990, 459–477.
20. P. Winkler, Isometric embeddings in products of complete graphs, *Discrete Appl. Math.*, **7** (1984), 221–225.

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W. IMRICH
Institute of Applied Mathematics,
Montanuniversität Leoben,
A-8700 Leoben,
Austria
E-mail: imrich@unileoben.ac.at

S. KLAVŽAR
Department of Mathematics,
PEF, University of Maribor,
Koroška cesta 160,
2000 Maribor,
Slovenia
E-mail: sandi.klavzar@uni-lj.si