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Every n-dimensional Grid is Cordial

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Abstract

Let $f : V(G) \to \{0, 1\}$ be a (binary) labelling of G. Assign 0 to all edges joining two vertices having the same label and assign 1 to the other edges. Let $v_f(0)$, $v_f(1)$, $e_f(0)$ and $e_f(1)$ be the number of vertices labelled 0, vertices labelled 1, edges labelled 0 and edges labelled 1, respectively. A graph G is *cordial* if there is a labelling f of G such that $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. In this note, it is proved that the Cartesian product of a finite number of paths is cordial. As a byproduct it is shown that $G\Box C_{4n}$ is cordial for any bipartite graph G. These generalize some known results of Cahit and Ho, Lee and Shee.

1 Introduction

Soon after Cahit [2] introduced the concept of cordial graphs they received considerable attention (see the references). A lot of results refer on cordiality of different classes of graphs. In [2] Cahit obtained several results on cordiality of trees, complete graphs, cycles, Eulerian graphs and others. Later [3] Cahit extended a result on cycles to cactus graphs with cycle blocks. Ho, Lee and Shee [4] characterized cordial unicyclic graphs and cordial generalised Petersen graphs. Benson and Lee [1] studied windmill graphs (several complete graphs identified at a point) and obtained an interesting connection with simultaneous Diophantine inequalities. Kirchherr [6] investigated

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subdivision graphs of cordial graphs and cactus graphs. Among others he gave a characterization of cordial cactus graphs.

There are also several results on cordiality dealing with different operations on graphs: the join of graphs, the Cartesian product and the lexicographic product of graphs. Kirchherr [6] investigated the join of graphs. The join of graphs is also used in [7] for a particular construction. However, the main motivation for our paper is a paper of Ho, Lee and Shee [5] where the Cartesian product of graphs and the lexicographic product of graphs are studied.

In Section 2 we prepare some auxiliary lemmas for the last section. We also prove that $G\Box C_{4n}$ is cordial for any bipartite graph G, thus extending Corollary 2 from [5]. In Section 3 we prove the main result of the paper: the Cartesian product of any number of paths is cordial. It follows in particular that hypercubes are cordial.

2 Definitions and Preliminary Lemmas

Graphs considered in this paper are undirected, finite, simple and connected.

Let $f : V(G) \to \{0, 1\}$ be a (binary) labelling of G. The labelling f induces a (binary) edge labelling in the following way: an edge joining two vertices having the same label receives 0 and an edge joining two vertices having opposite labels receives 1. Let $v_f(0)$, $v_f(1)$, $e_f(0)$ and $e_f(1)$ be the number of vertices labelled 0, vertices labelled 1, edges labelled 0 and edges labelled 1, respectively. A graph G is *cordial* if there is a labelling f of G such that $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$.

The *Cartesian product* $G\Box H$ of graphs G and H is the graph with vertex set $V(G)\times V(H)$ and $(a,x)(b,y) \in E(G\Box H)$ whenever $ab \in E(G)$ and $x = y$, or $a = b$ and $xy \in E(H)$. The Cartesian product is associative, commutative and K_1 is a unit. For $x \in V(H)$ set $G_x = G\Box\{x\}$ and for $a \in V(G)$ set $H_a = \{a\}\Box H$. We call G_x and H_a a *layer* of G and of H, respectively.

We first state an obvious remark. Since we will use it several times, we state it as a lemma.

Lemma 2.1 *For all graphs G and H,*

$$|E(G\Box H)| \;=\; |V(G)|\cdot|E(H)| + |V(H)|\cdot|E(G)|.$$

□

We also recall the following result, which can be found in [5].

Lemma 2.2 *Let G and H be cordial graphs with $|E(G)|$ and $|E(H)|$ both even. Then $G\Box H$ is cordial.* □

In the following, let $\{1, 2, \ldots, m\}$ be the vertex set of the path P_m, $m \geq 1$. For $j = 0, 1, 2, 3$ we introduce canonical labellings c^j of P_m in the following way. For each i, $1 \leq i \leq m$, let

$$c^j(i) = \begin{cases} 1; & \text{if } i \equiv 2+j \pmod 4 \text{ or } i \equiv 3+j \pmod 4, \\ 0; & \text{otherwise.} \end{cases}$$

It is easily checked, that c^0 and c^2 are cordial labellings of P_m for all $m \geq 1$ and that c^1 and c^3 are cordial for all odd m.

Lemma 2.3 *Let G be a bipartite graph and $V = V_1 \cup V_2$ its bipartition. If $||V_1| - |V_2|| \leq 1$ and if either $|V_1|$ or $|V_2|$ is even then $G\Box P_{2n}$ is cordial for $n \geq 1$.*

Proof. We may assume that $|V_2|$ is even. Partition V_2 arbitrarily into two equal sized sets A and B and consider the layers G_i, $i \in V(P_{2n})$. Recall that each G_i is an isomorphic copy of G. Denote by A^i, B^i and V_1^i the partition of G_i corresponding to the above partition of G. We now define a labelling f of $G \square P_{2n}$ in the following way:

$$f(v,i) = \begin{cases} c^1(i); & (v,i) \in A^i, \\ c^3(i); & (v,i) \in B^i, \\ c^2(i); & (v,i) \in V_1^i. \end{cases}$$

Hence, all vertices in a partition A^i, B^i and V_1^i are assigned the same label which will be denoted by $f(A^i)$, $f(B^i)$ and $f(V_1^i)$, respectively.

Let $w \in V_1$. Then $v_f(0) = v_f(1)$ holds on the layer $(P_{2n})_w$. Let now $a \in A$ and $b \in B$. Then $f(a,i) = 1$ if and only if $f(b,i) = 0$. Since $|A| = |B|$ holds, it follows that $v_f(0) = v_f(1)$ on the vertex set $V(P_{2n}) \times V_2$. Consequently, the labelling is cordial with respect to the vertices.

To show that f is also cordial with respect to the edges, we consider first edges within layers G_i. Since G is bipartite there are only edges from V_1^i to A^i and B^i. By the definition of f, $f(A^{2i-1}) \neq f(V_1^{2i-1})$ and $f(A^{2i}) = f(V_1^{2i})$, $i = 1, 2, \ldots, n$. The reverse holds true for B^i and V_1^i. This implies that $e_f(0) = e_f(1)$ for the edges within any two consecutive layers G_{2i-1} and G_{2i}, $i = 1, 2, \ldots, n$. Since we are dealing with even paths we have $e_f(0) = e_f(1)$ for all edges within all layers G_i.

We now consider edges in the layers $(P_{2n})_v$, $v \in V(G)$. As c^1, c^2 and c^3 are cordial labellings on odd paths, for every path $(v,1)(v,2)\cdots(v,2n-1)$, $e_f(0) = e_f(1)$ holds. The remaining edges are edges between layers G_{2n-1} and G_{2n}. Since $f(A^{2n-1}) = f(A^{2n})$, $f(B^{2n-1}) = f(B^{2n})$ and $f(V_1^{2n-1}) \neq f(V_1^{2n})$ and the cardinality of V_1 and V_2 differ by at most one, $|e_f(0) -$

$e_f(1)| \leq 1$ for those edges. Therefore, $|e_f(0) - e_f(1)| \leq 1$ holds for all edges in the product and the proof is complete. □

Lemma 2.4 *Let G be a bipartite graph and $V = V_1 \cup V_2$ its bipartition. If $||V_1| - |V_2|| \leq 1$ then $G \square P_{4n}$ is cordial for $n \geq 1$.*

Proof. Analogously to the proof of Lemma 2.3 we denote by V_1^i and V_2^i the bipartition of layer G_i, $i = 1, 2, \ldots, 4n$. Define a labelling f by:

$$f(v,i) = \begin{cases} 0; & v \in G_i,\ i \equiv 1 \pmod 4, \\ 0; & v \in V_1^i,\ i \equiv 2 \pmod 4, \\ 1; & v \in V_2^i,\ i \equiv 2 \pmod 4, \\ 1; & v \in G_i,\ i \equiv 3 \pmod 4, \\ 1; & v \in V_1^i,\ i \equiv 0 \pmod 4, \\ 0; & v \in V_2^i,\ i \equiv 0 \pmod 4. \end{cases}$$

Clearly, f is a cordial labelling with respect to vertices. The edges in the layers G_{2i-1}, $i = 1, 2, \ldots, 2n$, are all labelled 0. Since G is bipartite, the edges in the layers G_{2i}, $i = 1, 2, \ldots, 2n$, are all labelled 1. Therefore, $e_f(0) = e_f(1)$ holds for all edges within the layers G_i, $i = 1, 2, \ldots, 4n$. Next, we consider the edges between layers G_i and G_{i+1}, $i = 1, 2, \ldots, 4n-1$. If i is odd, then there are $|V_1|$ edges labelled 0 and $|V_2|$ edges labelled 1. For even i, the situation is complementary. Therefore, $e_f(0) = e_f(1)$ for all edges between consecutive layers G_i, $1 \leq i \leq 4n-1$. Since $|V_1|$ and $|V_2|$ differ by at most 1, the number of edges from layer G_{4n-1} to layer G_{4n} labelled 0 and 1 differs by at most 1. Labelling f is therefore also cordial with respect to the edges. □

A small modification of the proof of Lemma 2.4 gives us the following result.

Theorem 2.5 *If G is a bipartite graph then $G \Box C_{4n}$ is cordial for $n \geq 1$.*

Proof. Define a labelling f as the labelling in the proof of Lemma 2.4. As in the proof of Lemma 2.4, f is cordial with respect to the vertices, and $e_f(0) = e_f(1)$ holds for the edges within layers G_i, $1 \leq i \leq 4n$. If we treat the layer G_1 as the layer G_{4m+1} then by the same arguments as above, $e_f(0) = e_f(1)$ holds for all edges between layers G_i, $1 \leq i \leq 4m+1$. □

Corollary 2.6 ([5]) *$P_m \Box C_{4n}$ is cordial for $n \geq 1$.*

In fact, Corollary 2.6 was proved in [5] only for the special case of odd m.

3 Products of Paths

In [2] Cahit observed that all ladders (i.e. the graphs $P_2 \Box P_n$, $n \geq 2$) are cordial. Ho, Lee and Shee [5] showed that the graphs $P_n \Box P_n$, $n \geq 2$, are cordial. These results are special cases of the main result of our paper:

Theorem 3.1 *The Cartesian product of a finite number of paths is cordial.*

Note that Theorem 3.1 in particular implies that all hypercubes are cordial. In the rest of this section we prove the theorem.

Lemma 3.2 *$P_{4n+2} \Box P_{4m+2}$ is cordial for $n, m \geq 0$.*

Proof. We define a labelling f for $P_{4n+2} \Box P_{4m+2}$ in the following way. For $1 \leq i \leq 4n+1$ and $1 \leq j \leq 4m+1$,

$$f(i,j) = \begin{cases} c^0(j); & i \equiv 0 \pmod 4 \text{ or } i \equiv 1 \pmod 4, \\ c^2(j); & i \equiv 2 \pmod 4 \text{ or } i \equiv 3 \pmod 4. \end{cases}$$

The remaining column and row are labelled as follows:

$$f(4n+2,j) = c^3(j); \quad 1 \le j \le 4m+2,$$
$$f(i,4m+2) = c^1(i); \quad 1 \le i \le 4n+1.$$

Observe that layers $(P_{4m+2})_i$ and $(P_{4m+2})_{i+2}$, $i = 1, 2, \ldots, 4n-2$, are complementarily labelled. It follows that $v_f(0) = v_f(1)$ holds on the layers $(P_{4m+2})_i$, $1 \le i \le 4n$. Furthermore, it can be easily verified that $v_f(0) = v_f(1)$ is true also on the layers $(P_{4m+2})_{4n+1}$ and $(P_{4m+2})_{4n+2}$. It follows that f is cordial on vertices.

The edge set $E(P_{4n+2} \square P_{4m+2})$ will be briefly denoted by E. In order to prove $e_f(0) = e_f(1)$, we are going to partition E into six parts E_1, E_2, …, E_6, and show the desired equality for each part separately.

1. $E_1 = \{ (i,j)(k,l) \in E \mid i, k \le 4n+1, \ j, l \le 4m+1 \}$.
   Since c^0 and c^2 are both cordial labellings on a path, on the path $(i,1)(i,2)\cdots(i,4m+1)$, $i \in \{1, 2, \ldots, 4n+1\}$, $e_f(0) = e_f(1)$ holds. An analogous argument holds for the path $(1,j)(2,j)\cdots(4n+1,j)$, $j \in \{1, 2, \ldots, 4m+1\}$. Therefore, $e_f(0) = e_f(1)$ holds for the edges in E_1.

2. $E_2 = \{ (4n+1,j)(4n+2,j) \mid 1 \le j \le 4m \}$.

3. $E_3 = \{ (i,4m+1)(i,4m+2) \mid 1 \le i \le 4n \}$

4. $E_4 = \{ (4n+2,j)(4n+2,j+1) \mid 1 \le j \le 4m \}$

5. $E_5 = \{ (i,4m+2)(i+1,4m+2) \mid 1 \le i \le 4n \}$
   All the edge sets E_2, …, E_5 contain an even number of edges with an alternating labelling. Therefore, $e_f(0) = e_f(1)$ holds for edges in all these sets.

6. Let E_6 consist of the remaining four edges: $(4n+1, 4m+1)(4n+1, 4m+2)$, $(4n+2, 4m+1)(4n+2, 4m+2)$, $(4n+1, 4m+1)(4n+2, 4m+1)$ and $(4n+1, 4m+2)(4n+2, 4m+2)$. The first two edges are labelled 0 and the last two are labelled 1.

Since $E_1, \ldots, E_6$ is a partition of E, f is also cordial with respect to the edges. □

Lemma 3.3 *The Cartesian product of a finite number of odd paths is cordial.*

Proof. The lemma is clearly true for $n = 1$. Assume now that the lemma holds for n, $n \geq 1$. Let G be a product of $n+1$ odd paths. Then G can be written as $(\prod_{i=1}^{n} P^i)\Box P^{n+1}$, where P^i, $i = 1, 2, \ldots, n+1$, are odd paths. By Lemma 2.1 both factors of G have an even number of edges. Furthermore, $\prod_{i=1}^{n} P^i$ is cordial by induction hypothesis. Then by Lemma 2.2, the result follows. □

Lemma 3.4 *The Cartesian product of a finite number of even paths is cordial.*

Proof. The statement is clearly true for one path. Consider next the product $P_n \Box P_m$ of two paths. Assume first that $n = 4k$ for some $k \geq 1$. Since P_m satisfies the conditions of Lemma 2.4, the product is cordial in this case. Otherwise the product is cordial by Lemma 3.2.

To prove the statement for a product of ≥ 3 paths, we use Lemma 2.3 and induction. □

Finally, we are ready to prove Theorem 3.1. Let G be a product of n odd and m even paths, $m, n > 0$. G can be represented as $G = H_1 \Box H_2$, where $H_1 = \prod_{i=1}^{n} P^i$ and $H_2 = \prod_{j=1}^{m} R^j$. Here P^i, $i = 1, 2, \ldots, n$, are odd paths and R^i, $i = 1, 2, \ldots, m$, are even paths. By Lemma 3.3 and Lemma 3.4 both H_1 and H_2 are cordial. If $m = 1$, then by Lemma 2.3, G is cordial.

Otherwise, if $m > 1$ and even, then by Lemma 2.1 the number of edges in H_2 is even and both H_1 and H_2 satisfy the condition of Lemma 2.2, therefore G is cordial. Finally, if $m \geq 3$ and odd, we can write G in the following way:

$$G = (H_1 \Box \prod_{i=1}^{m-1} R^i) \Box R^m.$$

Then $H_1 \Box \prod_{i=1}^{m-1} R^i$ is cordial by the argument above and satisfies the condition of Lemma 2.3. This completes the proof of Theorem 3.1.

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