The domination game on split graphs

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Abstract

In this paper the domination game and the game domination number γ_g are investigated in the class of split graphs. It is proved that $\gamma_g(G) \le n/2$ for any isolate free n-vertex split graph G, thus strengthening the conjectured $3n/5$ general bound and supporting Rall's $\lceil n/2 \rceil$-conjecture. Split graphs of even order with $\gamma_g(G) = n/2$ are also characterized.

Key words: domination game; split graph; hamiltonian path;

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1 Introduction

If u and v are vertices of a graph $G = (V(G), E(G))$, then u *dominates* v if $u = v$ or u is adjacent to v. The domination game is played on G by *Dominator* and *Staller* who take turns choosing a vertex from G such that at least one previously undominated vertex becomes dominated. The game is over when no such move is possible. The score of the game is the number of vertices chosen by the two players. Dominator wants to minimize the score and Staller wants to maximize it. A game is called a *D-game* (resp. *S-game*) if Dominator (resp. Staller) has the first move. The *game domination number* $\gamma_g(G)$ of G is the score of a D-game played on G assuming that both players play optimally, the *Staller-start game domination number* $\gamma_g'(G)$ is the score of an optimal

S-game. This game was introduced in [3] and has been thoroughly investigated so far. A vertex u *totally dominates* v if u is adjacent to v. The total domination game is defined just as the domination game, except that everywhere "domination" is replaced with "total domination". This version of the domination game was introduced in [14].

Kinnersley, West, and Zamani in [18] posed a celebrated 3/5-*conjecture* asserting that if G is an isolate-free forest of order n or an isolate-free graph of order n, then $\gamma_g(G) \le 3n/5$. A parallel 3/4-*conjecture* for the total domination game was later posed in [15]. For a progress on these two conjectures see [5, 13, 16] and [6, 7], respectively.

To determine the game domination number can be a challenge even on simple families of graphs such as paths and cycles. The problem for the latter two families was first solved in the unpublished manuscript [17], where the result for the cycle on n vertices C_n reads as follows:

$$\gamma_g(C_n) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil - 1; & n \equiv 3 \pmod 4\,, \\ \\ \left\lceil \frac{n}{2} \right\rceil; & \textit{otherwise}\,. \end{cases}$$

The first published proof for the game domination number of paths and cycles appeared only recently in [19]. Because of these results and having in mind that paths and cycles are the simplest graphs with a hamiltonian path and a hamiltonian cycle, several years ago D. Rall proposed (here and in the rest of the paper, $n(G)$ denotes the order of G, that is, $n(G) = |V(G)|$) the following conjecture that strengthens the 3/5-conjecture for graphs containing hamiltonian paths:

Conjecture 1.1 *If a graph G contains a hamiltonian path, then* $\gamma_g(G) \le \left\lceil \frac{n(G)}{2} \right\rceil$.

Although the conjecture has been around for a while, as far as we know it has never been stated explicitly in a publication.

In this paper we consider the domination game on split graphs, a class of graphs of lasting, wide interest in graph theory, cf. [4, 10]. The class of split graphs might appear quite restrictive, however even with nested split graphs (a subclass of split graphs) one can approximate real complex graphs [21]. The paper is organized as follows. In the next section definitions and notation are given, and known results needed later are recalled. In Section 3 we prove that if G is an isolate free n-vertex split graph G, then $\gamma_g(G) \le n(G)/2$ and $\gamma_g'(G) \le \left\lfloor \frac{n(G)+1}{2} \right\rfloor$. Then, in Section 4, split graphs of even order with $\gamma_g(G) = n/2$ are characterized.

2 Preliminaries

The open neighborhood $N_G(x) = \{y : \ xy \in E(G)\}$ and the closed neighborhood $N_G[x] = N_G(x) \cup \{x\}$ will be abbreviated to $N(x)$ and $N[x]$ when G will be clear from the context. If $x \in V(G)$ and $S \subseteq V(G)$, then let $N_S(x) = N_G(x) \cap S$ and

$\deg_S(x) = |N_G(x) \cap S|$. For $m \in \mathbb{N}$ we will use the notation $[m] = \{1, \ldots, m\}$. A *chordal graph* is one in which every cycle of length 4 has a chord, that is, an edge that connects two non-consecutive vertices of the cycle. The disjoint union of two copies of a graph G is denoted with $2G$, in particular, $2K_2$ is the disjoint union of two complete graphs on two vertices.

A graph $G = (V(G), E(G))$ is a *split graph*, if $V(G)$ can be partitioned into (possibly empty) sets K and I, where K is a clique and I is an independent set [12]. The pair (K, I) is called a *split partition* of G. Split graphs can be characterized in several different ways, in particular as the graphs that contain no induced subgraphs isomorphic to a graph in $\{2K_2, C_4, C_5\}$, the result proved in [12]. If G is a split graph with a split partition (K, I), then a maximal clique of G is either K or it is induced with the closed neighborhood of a vertex from I. Hence a maximum clique of G is easy to detect. Throughout the paper we may and will thus assume that if (K, I) is a split partition of a (split) graph G, then $|K| = \omega(G)$, that is, K is a largest clique of G. We will also set $k = |K|$ and $i = |I|$. We will further set $K = \{x_1, \ldots, x_k\}$ and $I = \{y_1, \ldots, y_i\}$.

The sequence of moves is a D-game will be denoted with $d_1, s_1, d_2, s_2, \ldots$, and the sequence of moves is an S-game with $s_1', d_1', s_2', d_2', \ldots$ A *partially dominated graph* is a graph together with a declaration that some vertices are already dominated, that is, they need not be dominated in the rest of the game. If $S \subseteq V(G)$, then let $G|S$ denote the partially dominated graph in which vertices from S are already dominated. If $S = \{x\}$ we will abbreviate $G|\{x\}$ to $G|x$. If $G|S$ is a partially dominated graph, then $\gamma_g(G|S)$ and $\gamma_g'(G|S)$ denote the optimal number of moves in the D-Game and the S-Game, respectively, played on $G|S$. A vertex u of a partially dominated graph $G|S$ is *saturated* if each vertex in $N[u]$ is dominated. Clearly, as soon as a vertex becomes saturated, it is not a legal move in the rest of the game.

Lemma 2.1 (Continuation Principle, [18]) *Let G be a graph with $A, B \subseteq V(G)$. If $B \subseteq A$, then $\gamma_g(G|A) \le \gamma_g(G|B)$ and $\gamma_g'(G|A) \le \gamma_g'(G|B)$.*

An utmost important consequence of the Continuation Principle is the following:

Theorem 2.2 ([3, 18]) *If G is a partially dominated graph, then $|\gamma_g(G) - \gamma_g'(G)| \le 1$.*

A graph G is said to be a *no-minus graph* if for every $A \subseteq V(G)$ we have $\gamma_g(G|A) \le \gamma_g'(G|A)$. We will need the following result due to Dorbec, Košmrlj, and Renault.

Theorem 2.3 [11, Theorem 2.7] *Connected split graphs are no-minus graphs.*

This theorem was actually proved in [11] for the so-called (connected) tri-split graphs which form a generalization of split graphs.

3 The $1/2$ upper bound

In this section we first prove the $1/2$ upper bound for the D-game and then the corresponding bound for the S-game. At the end the sharpness of both bounds is demonstrated. In the corresponding arguments we need to show that Dominator has a strategy which ensures that at most a prescribed number of moves will be played, no matter how Staller is playing. But this means that we may assume that Staller is playing optimally, because otherwise the game could only be finished faster.

Theorem 3.1 *If G is a connected split graph with $n(G) \geq 2$, then $\gamma_g(G) \leq \left\lfloor \frac{n(G)}{2} \right\rfloor$.*

Proof. The proof is by induction on $n(G)$. We first check the cases when $2 \leq n(G) \leq 5$. If $n(G) = 2$, then $G = K_2$, and if $n(G) = 3$, then $G \in \{K_3, P_3\}$. For all these three (split) graphs the assertion clearly holds. From [18, Proposition 5.3] we recall that if G is a (partially dominated, isolate-free) chordal graph, then $\gamma_g(G) \leq 2n(G)/3$. As split graphs are chordal, the same conclusion holds for split graphs. Hence, if $n(G) = 4$, then $\gamma_g(G) \leq 2n(G)/3 = 8/3$, that is, $\gamma_g(G) \leq 2$. Suppose finally that $n(G) = 5$. If $k = 2$, then since G is connected, at least one of the vertices, say x_1, of K has at least two neigbors in I. Then the move $d_1 = x_1$ yields $\gamma_g(G) \leq 2$. If $k = 3$, then Dominator starts the game with $d_1 = x_1$ where x_1 is a vertex of K having at least one neighbor in I. If the game is not finished yet, then Staller must finish the game in her first move by dominating the only undominated vertex in I. Hence again $\gamma_g(G) \leq 2$. Finally, if $k \in \{4, 5\}$, then $\gamma_g(G) = 1$. This proves the basis of the induction.

Assume now that the result is true for all split graphs up to and including $n - 1$ vertices, where $n \geq 6$. We distinguish two cases.

Case 1: $\deg_I(x_r) \leq 1$, $r \in [k]$.
In this case we clearly have $|I| \leq |K|$. If $i = 0$, then $G = K_k$ and the assertion is clear. Otherwise, let Dominator start the game by playing a vertex of K with a neighbor in I. Then, in every subsequent move (either by Staller or by Dominator), exactly one new vertex (in I) will be dominated. It follows that $\gamma_g(G) = |I|$. Consequently,

$$\gamma_g(G) = |I| = \frac{|I| + |I|}{2} \leq \frac{|K| + |I|}{2} = \frac{n(G)}{2}\,.$$

Case 2: $\deg_I(x_r) \geq 2$, for some $r \in [k]$.
We may without loss of generality assume that $x_1y_1, x_1y_2 \in E(G)$. The initial strategy of Dominator is to play $d_1 = x_1$. After that Staller selects a vertex optimally which means that she plays y_s, where $s \notin [2]$, unless, of course, the game is over after the move $d_1 = x_1$. (We note that because of the Continuation Principle if $N[x] \subseteq N[w]$ and both x and w are legal moves, we may assume Staller will play x over w.) Set $Z = \{x_1, y_1, y_2, y_s\}$. Then, since Staller has played optimally (and Dominator maybe not), after the first two moves we have

$$\gamma_g(G) \leq 2 + \gamma_g\left(G \middle| \cup_{z \in Z} N[z]\right)\,.$$

Set $G' = G \setminus \{x_1, y_1, y_2, y_s\}$. After x_1 and y_s have been played, the vertices x_1, y_1, y_2, and y_s are saturated. Therefore, by the Continuation Principle,

$$\gamma_g \left(G \middle| \cup_{z \in Z} N[z]\right) \leq \gamma_g(G')\,.$$

Since $n(G') = n(G) - 4$, we can combine the above two inequalities with the induction hypothesis into

$$\gamma_g(G) \leq 2 + \gamma_g \left(G \middle| \cup_{z \in Z} N[z]\right) \leq 2 + \gamma_g(G') \leq 2 + \left\lfloor \frac{n(G) - 4}{2} \right\rfloor = \left\lfloor \frac{n(G)}{2} \right\rfloor$$

and we are done. □

The assumption of Theorem 3.1 that G is connected is essential. For instance, for the complement $\overline{K}_n$ of K_n (both of these graphs being split graphs) we have $\gamma_g(\overline{K}_n) = n$. Note also that Theorem 3.3 supports Conjecture 1.1. In this respect we mention a very interesting dichotomy that detecting hamiltonicity is difficult on $K_{1,5}$-free split graphs but polynomial on $K_{1,4}$-free split graphs [20].

Combining Theorem 3.1 with Theorem 2.2 we get that if G is a connected split graph with $n(G) \geq 2$, then

$$\gamma_g{}'(G) \leq \gamma_g(G) + 1 \leq \left\lfloor \frac{n(G)}{2} \right\rfloor + 1 = \left\lfloor \frac{n(G) + 2}{2} \right\rfloor\,. \tag{1}$$

To slightly improve this bound, we first show the following:

Lemma 3.2 *Let G be a connected split graph. If there exists a vertex $x_r \in K$ with $\deg_I(x_r) = 0$, then x_r is an optimal first move of Staller in S-game.*

Proof. Suppose that $s_1' = x_r$. Then Dominator has an optimal reply in K, say $d_1' = x_s$, $s \neq r$. Indeed, the Continuation Principle implies that if $d_1' = y_t \in I$, then any neighbor of y_t is at least as good for Dominator as y_t. After the moves $s_1' = x_r$ and $d_1' = x_s$ are played, the set of vertices dominated is $X = K \cup N_G(x_s)$. Hence if Staller had played some other vertex, Dominator can still play x_s, unless Staller played x_s. In any case, if Y is the set of vertices dominated after such two moves, then $X \subseteq Y$. By the Continuation Principle it follows that $s_1' = x_r$ is an optimal move. □

Now we can improve (1) as follows:

Theorem 3.3 *If G is a connected split graph with $n(G) \geq 2$, then $\gamma_g'(G) \leq \left\lfloor \frac{n(G)+1}{2} \right\rfloor$.*

Proof. The assertion is clearly true for K_2, hence we may assume in the rest that $n(G) \geq 3$. By Lemma 3.2 and the Continuation Principle, Staller's first move s_1' is either a vertex of I, or a vertex from K with no neighbour in I. Let $G' = G \setminus s_1'$. Clearly,

G' is a connected split graph with $n(G') = n(G) - 1 \geq 2$, hence from Theorem 3.1 we get $\gamma_g(G') \leq \lfloor (n(G)-1)/2 \rfloor$. Therefore, applying the Continuation Principle again, we have

$$\gamma_g'(G) = 1 + \gamma_g(G|N[s_1']) \leq 1 + \gamma_g(G') \leq 1 + \left\lfloor \frac{n(G)-1}{2} \right\rfloor = \left\lfloor \frac{n(G)+1}{2} \right\rfloor$$

as claimed. □

In view of Theorem 3.1 we say that G is a $1/2$-*split graph* if $\gamma_g(G) = \lfloor n(G)/2 \rfloor$. To conclude the section we present two families of 1/2-split graphs.

Let G_k, $k \geq 2$, be the split graph with the split partition (K, I), where $K = \{x_1, \ldots, x_k\}$ and $I = \{y_1, \ldots, y_k\}$ (that is, $i = k$), and where $x_r y_r$, $r \in [k]$, are the only edges between K and I. Then it is straightforward to see that $\gamma_g(G_k) = {\gamma_g}'(G_k) = k$, that is, G_k is a 1/2-split graph and the bounds of Theorems 3.1 and 3.3 cannot be improved in general.

The above graphs G_k are of even order, hence the bounds of Theorems 3.1 and 3.3 are the same. Let next H_k, $k \geq 2$, be a split graph obtained from G_k by adding one more vertex y_{k+1} to I and the edge $x_k y_{k+1}$. Then $\deg_I(x_k) = 2$. From Dominator's first move $d_1 = x_k$ in D-game and Staller's first move $s_1' = y_{k+1}$ in S-game we respectively infer that $\gamma_g(H_k) = k$ and ${\gamma_g}'(H_k) = k + 1$. These values again achieve the upper bounds in the respective theorems.

4 1/2-split graphs of even order

We now characterize the 1/2-split graphs that have even order. In the following two lemmas we first exclude split graphs that are not such.

Lemma 4.1 *Let G be a connected split graph of even order and suppose that at least one of the following conditions is fulfilled:*

(i) $i < k$*;*

(ii) $i > 2k$*;*

(iii) there exists a vertex $x_r \in K$ *with* $\deg_I(x_r) = 0$*;*

(iv) there exists a vertex $x_r \in K$ *with* $\deg_I(x_r) \geq 3$*;*

(v) there exist $x_r, x_s \in K$ *with* $\deg_I(x_s) = 2$ *and* $N_I(x_r) \subseteq N_I(x_s)$*.*

Then G is not a 1/2-split graph.

Proof. In view of Theorem 3.3 we need to show that if one of the conditions (i)-(v) holds, then $\gamma_g(G) < \left\lfloor \frac{n(G)}{2} \right\rfloor$.

(i) Suppose $i < k$. Let Dominator start the game by playing a vertex $x_r \in K$ with at least one neighbor in I. After this move the vertices left undominated are $X = I \backslash N_I(x_r)$. Clearly, $|X| \le i-1$. Since in the rest of the game at least one new vertex is dominated on each move, $\gamma_g(G) \le 1 + (i-1) = i < (k+i)/2 = n(G)/2 = \lfloor n(G)/2 \rfloor$.

(ii) Assume $i > 2k$. Then there exists a vertex $x_r \in K$ with $\deg_I(x_r) \ge 3$. Let Dominator start a D-game with $d_1 = x_r$, and let Staller reply with an optimal move. After these two moves the graph G' obtained from G by removing all saturated vertices is again a connected partially dominated split graph with at most $n(G) - 5$ vertices. Indeed, G' does not contain $d_1 = x_r$, the neighbors of x_r in I (at least three of them), and s_1. Therefore,

$$\gamma_g(G) \le 2 + \gamma_g(G') \le 2 + (n(G) - 5)/2 = (n(G) - 1)/2 < n(G)/2 = \lfloor n(G)/2 \rfloor \ ,$$

where the second inequality holds by Theorem 3.1.

(iii) Suppose that there exists a vertex $x_r \in K$ with $\deg_I(x_r) = 0$. Because of (i) we can assume that $k \le i$. Therefore, since $\deg_I(x_r) = 0$, there exists a vertex $x_s \in K$ with $\deg_I(x_s) \ge 2$. Let Dominator start the game by playing $d_1 = x_s$. Then, after the first move of Staller, the graph G' obtained from G by removing all saturated vertices is a connected partially dominated split graph with at most $n(G) - 5$ vertices because it does not contain $d_1 = x_s$, the neighbors of x_s in I (at least two of them), the first move of Staller s_1, and x_r. The conclusion now follows by the same argument as in (ii).

(iv) If there exists a vertex $x_r \in K$ with $\deg_I(x_r) \ge 3$, then after Dominator plays x_r and Staller an arbitrary (optimal) move, we again have a connected partially dominated split graph with at most $n(G) - 5$ vertices after removing all saturated vertices.

(v) Let Dominator start the game by playing $d_1 = x_s$. Then x_r, x_s, and the two neighbors of x_s in I have no role in the continuation of the game. So again, after the first move of Staller, removing all saturated vertices from G we have a partially dominated connected split graph of order at most $n(G) - 5$. □

Lemma 4.2 *If G is a connected split graph of even order and there exists a vertex in K which is not adjacent to a leaf in I, then $\gamma_g(G) < \lfloor n(G)/2 \rfloor$.*

Proof. Let $x_1 \in K$ be a vertex that is not adjacent to a leaf in I. If $\deg_I(x_1) \ge 3$, then we are done by Lemma 4.1(iv).

Suppose next that $\deg_I(x_1) = 1$. Let y_1 be the vertex of I adjacent to x_1. Since y_1 is not a leaf, we may assume that $x_2 \in K$ is another neighbor of y_1. If $\deg_I(x_2) \ge 2$, then we are done by Lemma 4.1(iv) and (v). Suppose therefore that $\deg_I(x_2) = 1$. Then $N[x_1] = N[x_2]$, hence by [1, Proposition 1.4] we have $\gamma_g(G) = \gamma_g(G|x_1) = \gamma_g(G - x_1)$. Therefore, having in mind Theorem 3.1 and the fact that n is even,

$$\gamma_g(G) = \gamma_g(G - x_1) \le \lfloor (n(G) - 1)/2 \rfloor < \lfloor n(G)/2 \rfloor \ .$$

The remaining case to consider is that $\deg_I(x_1) = 2$. Let $y_1, y_2 \in I$ be the neighbors of x_1 in I. Recall that by our assumption y_1 and y_2 are not pendant vertices. If y_1 and y_2 have a common neighbor x_r in K, $r \neq 1$, then in view of Lemma 4.1(iv) we may assume that $\deg_I(x_r) = 2$, but then $N_I(x_1) \subseteq N_I(x_r)$ and we are done by Lemma 4.1(v). It follows that there exist vertices $x_2, x_3 \in K$ such that x_2 is adjacent to y_2 and x_3 is adjacent to y_1. Using Lemma 4.1(v) again we see that $\deg_I(x_2) = \deg_I(x_3) = 2$. Let y_3 and y_4 be the other neighbors in I of x_3 and x_2, respectively. Let $Z = \{x_1, x_2, x_3, y_1, y_2, y_3, y_4\}$ and let G_1 and G_2 be the subgraphs of G induced by Z and $V(G)\backslash Z$ respectively. Clearly, G_1 is a connected split graph. The same holds for G_2 unless it is the empty graph. It can be easily verified that $\gamma_g(G_1) = \gamma_g'(G_1) = 3$. Hence by Theorem 2.3 they are no-minus graphs with $\gamma_g(G_1) = \gamma_g'(G_1)$ and hence by [11, Theorem 2.11] we have $\gamma_g(G_1 \cup G_2) = \gamma_g(G_1) + \gamma_g(G_2)$. Moreover, by Theorem 3.1 and because n is even we have

$$\gamma_g(G_2) \leq \lfloor (n(G) - 7)/2 \rfloor = (n(G) - 8)/2$$

and consequently

$$\gamma_g(G_1 \cup G_2) \leq 3 + (n(G) - 8)/2 = (n(G) - 2)/2 < \lfloor n(G)/2 \rfloor \,.$$

The argument will be complete by proving that $\gamma_g(G) \leq \gamma_g(G_1 \cup G_2)$. For this sake we proceed by the imagination strategy as follows. Consider a real D-game played on G and at the same time Dominator imagines a D-game played on $G_1 \cup G_2$. Dominator plays optimally in the game on $G_1 \cup G_2$ and copies his moves from there to the real game on G. On the other hand, Staller plays optimally in the real game on G (this is the only game being played by Staller), and Dominator copies each move of Staller to the imagined game. Since a D-game is played in both games, Dominator will first play a vertex of K in the real game which is played on G. Hence every move of Staller will be a vertex from I, thus newly dominating only this vertex. It follows that every move of Staller in the real game is a legal move in the imagined game. On the other hand, a legal move of Dominator in the imagined game may not be legal in the real game. If this happens, Dominator cannot copy this move to the real game; instead, he selects an arbitrary legal move in the real game (if there is such a move available, otherwise the game is over). Under this strategy, the set of vertices dominated in the imagined game is always a subset of the set of vertices dominated in the real game. Hence, if s is the number of moves played in the real game and t the number of moves in the imagined game, then $s \leq t$. Moreover, since Dominator may not play optimally on G (but Staller does), we have $\gamma_g(G) \leq s$. Similarly, as Dominator plays optimally on $G_1 \cup G_2$, we infer that $\gamma_g(G_1 \cup G_2) \geq t$. Therefore, $\gamma_g(G) \leq s \leq t \leq \gamma_g(G_1 \cup G_2)$ which completes the argument. $\square$

Theorem 4.3 *A connected split graph of even order is a* $1/2$*-split graph if and only if every vertex in* K *is adjacent to at least one leaf in* I *and* $\deg_I(x_i) \in [2]$ *for* $i \in [k]$.

Proof. Suppose that $\gamma_g(G) = \lfloor n(G)/2 \rfloor$. Then by Lemma 4.2 every vertex of K is adjacent to at least one leaf in I and by Lemma 4.1(iii) and (iv), $\deg_I(x_i) \in [2]$ for every vertex $x_i \in K$.

Conversely, suppose that G is a connected split graph of even order in which every vertex in K is adjacent to at least one leaf in I and $\deg_I(x_i) \in [2]$ for $i \in [k]$. By Theorem 3.1 we need only to prove that Staller has a strategy that guarantees that a D-game will last at least $\lfloor n(G)/2 \rfloor$ moves. After each move we consider that the resulting graph is a partially dominated graph without saturated vertices. The corresponding Strategy of Staller is the following.

First, In Phase 1, she selects vertices which are not pendent vertices in I. After this is no longer possible for Staller, Phase 1 is over and Phase 2 begins. At that time the vertices from I that are not yet dominated are pendent vertices. In Phase 2 Staller selects pendent vertices which are neighbors of degree-2 vertices from K as long as this is possible. Phase 3 starts when the only not yet dominated vertices from I are those that are adjacent to vertices of K with exactly one neighbor in I.

Consider the number of saturated vertices during this game. Since $\deg_I(x_i) \in [2]$, $i \in [k]$, after each move of Dominator in Phases 1 and 2 the number of newly saturated vertices is at most three. By the strategy of Staller, after each of her moves in these two phases the number of saturated vertices increases by exactly one. Suppose that Phase 2 is finished with the k^{th} move of Staller. Then the number of saturated vertices is at most $3k + k = 4k$. If there are l vertices in Phase 3 yet to be dominated, then the game is finished by the next l moves. After each such move, no matter whether it was done either by Dominator or Staller, two newly saturated vertices are created and therefore $n(G) \leq 4k + 2l$. The described strategy of Staller may not be optimal, hence

$$\gamma_g(G) \geq 2k + l = \frac{2(2k+l)}{2} \geq \frac{n(G)}{2} = \left\lfloor \frac{n(G)}{2} \right\rfloor .$$

Suppose next that Phase 2 is finished with the k^{th} move of Dominator. In this case the number of saturated vertices at this stage of the game at most $3k + k - 1 = 4k - 1$. Let again l be the number of vertices yet to be dominated in Phase 3. Then the number of not yet saturated vertices is exactly $2l$. Since G is of even order, the number of vertices already saturated is at most $4k - 2$. Hence $n(G) \leq 4k - 2 + 2l$ and therefore

$$\gamma_g(G) \geq (2k-1) + l = \frac{2(2k-1+l)}{2} = \frac{4k-2+2l}{2} \geq \frac{n(G)}{2} = \left\lfloor \frac{n(G)}{2} \right\rfloor ,$$

and we are done. $\square$

5 Concluding remarks

In [3] it was proved that the game domination number of a graph G is bounded by the domination number $\gamma(G)$ of G as follows:

$$\gamma(G) \le \gamma_g(G) \le 2\gamma(G) - 1\,.$$

Consequently, to prove Conjecture 1.1, it suffices to consider "only" graphs G with the property $\gamma(G) > (n(G)+2)/4$. Moreover, since for a graph G with a hamiltonian path we clearly have $\gamma(G) \le \lceil n(G)/3 \rceil$, it suffices to concentrate just on graphs G with the domination number roughly between $n(G)/4$ and $n(G)/3$.

In Section 4 we have characterized 1/2-split graphs of even order. It would likewise be of interest to characterize 1/2-split graphs of odd order. It seems possible to proceed along the similar lines as in Section 4, however the consideration turned out to be more lengthy and technical.

Split graphs have different important generalizations. Chordal graphs form one of them. Since trees are chordal graphs and there exist infinite families of the so-called 3/5-trees (see [2, 16]), Theorem 3.1 does not extend to chordal graphs. Another important generalization of split graphs are $2K_2$-free graphs, that is, graphs that do not contain two independent edges as an induced subgraph, cf. [8, 9]. Now, C_5 belongs to this class and $\gamma_g(C_5) = 3$, hence Theorem 3.1 also does not extend to $2K_2$-free graphs. Let us therefore ask whether there is some natural superclass of split graphs to which Theorem 3.1 extends. Actually we know of one such class (tri-split graphs), see below. But this extension is rather straightforward, hence let us rephrase the question as follows:

Problem 5.1 *Is there a natural superclass of split graphs to which Theorem 3.1 extends "non-trivially"?*

At the end of Section 2 we have mentioned tri-split graphs that were inroduced in [11]. They are defined as follows. A graph G is a *tri-split graph* if $V(G)$ can be partitioned into three disjoint sets $A \ne \emptyset$, B, and C with the following properties. The set A induces a clique, B induces an independent set, and C and arbitrary graph. Each vertex from A is adjacent to each vertex from C (that is, there is a join between A and C), and no vertex of B is adjacent to a vertex in C. So the only neighbors of the vertices from C are in A. Now, if a D-game is played of a tri-split graph G, then the first move of dominator will be on A, and after this move all vertices in C and in A are dominated. This means that every vertex of C is saturated and the game continues as it would be played on the split graph induced by $A \cup B$. But then Theorem 3.1 extends to tri-split graphs.

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