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Strong edge geodetic problem in networks

Abstract: Geodesic covering problems form a widely researched topic in graph theory. One such problem is geodetic problem introduced by Harary et al. [Math. Comput. Modelling, 1993, 17, 89-95]. Here we introduce a variation of the geodetic problem and call it strong edge geodetic problem. We illustrate how this problem is evolved from social transport networks. It is shown that the strong edge geodetic problem is NP-complete. We derive lower and upper bounds for the strong edge geodetic number and demonstrate that these bounds are sharp. We produce exact solutions for trees, block graphs, silicate networks and glued binary trees without randomization.

1 Introduction

Covering problems are among the fundamental problems in graph theory, let us mention the vertex cover problem, the edge cover problem, and the clique cover problem. An important subclass of covering problems is formed by path coverings that include the edge covering problem, the geodesic covering problem, the induced path covering problem and the path covering problem. Of a particular importance are coverings with shortest paths (also known as geodesics), e.g. in the analysis of structural behavior of social networks. In particular, the optimal transport flow in social networks requires an intensive study of geodesics [2–4]. In this paper we introduce and study a related problem that we call strong edge geodetic problem. This problem is in part motivated by the following application to social transport networks.

Urban road network is modeled by a graph whose vertices are bus stops or junctions. The urban road network is patrolled and maintained by road inspectors, see Fig. 1 for an example of a network with road inspectors I1, I2, I3, and I4.
A road patrolling scheme is prepared satisfying the following conditions:

1. A road segment is a geodesic in the road network. It is patrolled by a pair of road inspectors by stationing one inspector at each end.
2. One pair of road inspectors is not assigned to more than one road segment. However, one road inspector is assigned to patrol other road segments with other inspectors.

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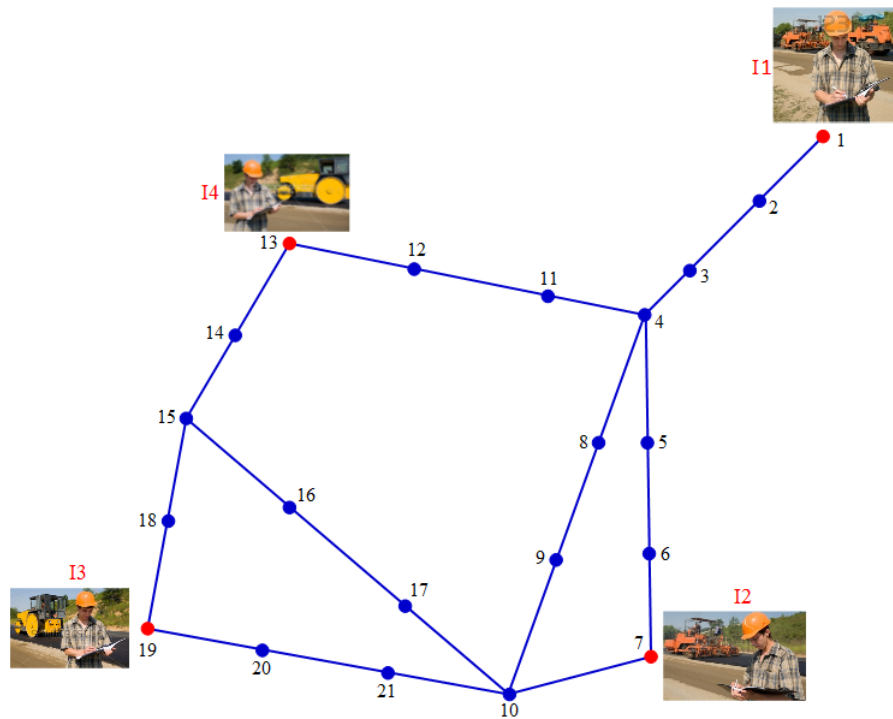**Sandi Klavžar:** Faculty of Mathematics and Physics, University of Ljubljana, Slovenia and Faculty of Natural Sciences and Mathematics, University of Maribor, Slovenia and Institute of Mathematics, Physics and Mechanics, Ljubljana, Slovenia, E-mail: sandi.klavzar@fmf.uni-lj.si
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We point out that, under the assumption that the distances in a network between bus stops/junctions are integral, we may without loss of generality assume the network to be a simple graph. Indeed, to obtain an equivalent graph from the network, each edge may be subdivided by an appropriate number of times. From this reason, we are restricting ourselves in this paper to simple graphs.

An example of a patrolling scheme for the network of Fig. 1 is given below:

I1 and I2 patrol road segment $1 - 2 - 3 - 4 - 5 - 6 - 7$;
I1 and I3 patrol road segment $1 - 2 - 3 - 4 - 8 - 9 - 10 - 21 - 20 - 19$;
I1 and I4 patrol road segment $1 - 2 - 3 - 4 - 11 - 12 - 13$;
I2 and I3 patrol road segment $7 - 10 - 21 - 20 - 19$;
I2 and I4 patrol road segment $7 - 10 - 17 - 16 - 15 - 14 - 13$; and
I3 and I4 patrol road segment $19 - 18 - 15 - 14 - 13$.

By condition 2., the restriction is that one pair of inspectors is assigned at most one road segment. For example, there are two road segments of equal length between inspectors I2 and I4, however these two inspectors are assigned a single road segment in the patrolling scheme. The strong edge geodetic problem is to identify a minimum number of road inspectors to patrol the urban road network.

We proceed as follows. In the next section we state definitions and notions needed in this paper, and formally introduce the strong edge geodetic problem as well as two closely related problems. Then, in Section 3, we prove that the strong edge geodetic problem is NP-complete. In the subsequent section we discuss upper bounds on the strong edge geodetic number and show that it can be bounded from above by the edge isometric path number. In Section 5, we observe that simplicial vertices are intimately related to the strong edge geodetic problem and deduce several consequences, in particular determine exactly the strong edge geodetic number of block graphs and silicate networks. In Section 6, we introduce non-geodesic edges and use them to prove another lower bound on the strong edge geodetic number. The bound is shown to be in particular exact on glued binary trees without randomization.

2 Preliminaries

Let x and y be vertices of a graph G. Then the interval $I(G; x, y)$ (or $I(x, y)$ for short if G is clear from the context) between x and y is the set of vertices u such that u lies of some shortest x, y-path. In addition, for $S \subseteq V(G)$ the geodetic closure $I(S)$ of S is

$$I(S) = \bigcup_{\{x,y\}\in\binom{S}{2}} I(x, y).$$

S is called a geodetic set if $I(S) = V(G)$. The geodetic problem, introduced by Harary et al. [1], is to find a geodetic set of G of minimum cardinality; this graph invariant is denoted with $g(G)$. Since then the problem has attracted several researchers and has been studied from different perspectives [5–10].

If x and y are vertices of a graph G, then let $I_e(G; x, y)$ denote the set of the edges e such that e lies on at least one shortest x, y-path. Again we will simply write $I_e(x, y)$ when there is no danger of confusion. For a set $S \subseteq V(G)$, the edge geodetic closure $I_e(S)$ is the set of edges defined as

$$I_e(S) = \bigcup_{\{x,y\}\in\binom{S}{2}} I_e(x, y).$$

A set S is called an edge geodetic set if $I_e(S) = E(G)$. The edge geodetic problem, introduced and studied by Santhakumaran et al. [11], is to find a minimum edge geodetic set of G. The size of such a set is denoted with $g_e(G)$. Note that $g(G) \leq g_e(G)$ holds for any graph G. As far as we know, the complexity status of the edge geodetic problem is unknown for the general case. On the other hand there are a significant number of theoretical results of the edge geodetic problem [12–14].

We now formally introduce the strong edge geodetic problem. If G is a graph, then $S \subseteq V(G)$ is called a *strong edge geodetic set* if to any pair $x, y \in S$ one can assign a shortest x, y-path P_{xy} such that

$$\bigcup_{\{x,y\}\in\binom{S}{2}} E(P_{xy}) = E(G).$$

By definition, in a strong edge geodetic set S there are $\binom{|S|}{2}$ paths P_{xy} that cover all the edges of G. The cardinality of a smallest strong edge geodetic set S will be called the *strong edge geodetic number G* and denoted by $sg_e(G)$. We will also say that the smallest strong edge geodetic set S is a $sg_e(G)$-set. The *strong edge geodetic problem* for G is to find a $sg_e(G)$-set of G. We emphasize that the strong edge geodetic problem requires, not only to determine a set $S \subseteq V(G)$, but also a list of specific geodesics, that is, precisely one geodesic between each pair of vertices from S.

The *Cartesian product* $G_1 \square \cdots \square G_k$ of graphs $G_1, \ldots, G_k$ has the vertex set $V(G_1) \times \cdots \times V(G_k)$, vertices $(g_1, \ldots, g_k)$ and $(g'_1, \ldots, g'_k)$ being adjacent if they differ in exactly one position, say in the ith, and $g_i g'_i$ is an edge of G_i [15]. A vertex v of a graph G is *simplicial* if its neighborhood induces a clique. In other words, a vertex v is simplicial if only if v lies in exactly one maximal clique. Finally, for a positive integer n we will use the notation $[n] = \{1, \ldots, n\}$.

3 Strong edge geodetic problem is NP-complete

In this section we prove:

Theorem 3.1. *The strong edge geodetic problem is NP-complete.*

If G is a graph, then a set S of its vertices is called a *shortest path union cover* if the shortest paths that start at the vertices of S cover all the edges of G. Here, we consider all the shortest paths that start at v for each $v \in S$. The *shortest path union cover problem* is to find a shortest path union cover of minimum cardinality. Boothe et al. [16] proved that the shortest path union cover problem is NP-complete. Now we show that the strong edge geodetic problem is NP-complete by a reduction from the shortest path union cover problem.

Given a graph $G = (V, E)$, we construct a graph $G' = (V', E')$ with the vertex set $V' = V \cup V_1 \cup V_2$ and the edge set $E' = E \cup E_1 \cup E_2 \cup E_3$. The vertex set V_1 is $\{u_i : u \in V, i \in [\deg(u)]\}$ and V_2 is $\{u'_i : u \in V, i \in [\deg(u)]\}$. Hence, for each vertex u of G, vertices $\{u_1, \ldots, u_{\deg(u)}\}$ are added to V_1 and vertices $\{u'_1, \ldots, u'_{\deg(u)}\}$ are added to V_2. The edge set E_1 is $\{uu_i : u \in V, i \in [\deg(u)]\}$. The edge set E_2 is $\{u_i v_j : u \in V, i \in [\deg(u)]$ and $v \in V, j \in [\deg(v)]\} \cup \{u_i u_j : u \in V, i, j \in [\deg(u)]\}$. The edge set E_3 is $\{u_i u'_i : u \in V, i \in [\deg(u)], i \neq j\}$. The construction is illustrated in Fig. 2. We can imagine that the graph $G' = (V', E')$ is composed of three layers where the top layer is the graph G, the middle layer is induced by the vertex set V_1, and the bottom layer corresponds to the vertex set V_2 which is an independent set.

Fig. 2. Graphs G and G'.

(a) (b)
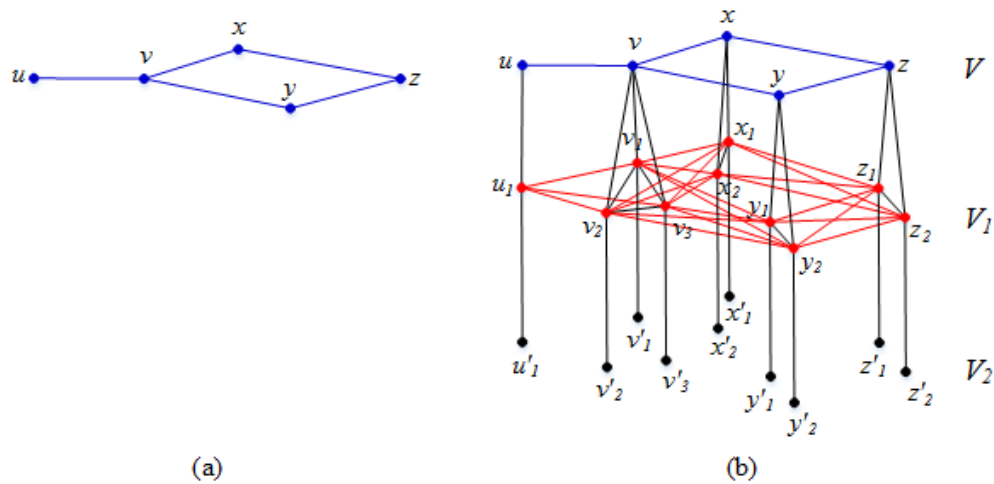
In the above construction, we thus create vertices $w'_1, \ldots, w'_{\deg(w)}$ in G' for every vertex w of G. It is important to know the reason for this construction. Here is the explanation. It is easy to verify that in the example from Fig. 2 the singleton $\{v\}$ is a shortest path union cover of G. Between the vertices v and z, there are two shortest paths vxz and vyz which cover the edges of G. See Fig. 3(a). Between one pair of vertices, more than one shortest path is allowed in the shortest path union cover problem. But in the strong edge geodetic problem between one pair of vertices v and z'_1, two shortest paths $vxzz_1 z'_1$ and $vyzz_1 z'_1$ are not allowed. See Fig. 3(b). In order to avoid this conflict, for every vertex w of G, we create $w'_1, \ldots, w'_{\deg(w)}$ in G'. The shortest paths $vxzz_1 z'_1$ and $vyzz_2 z'_2$ that cover the edges of G' do not violate the condition of the strong edge geodetic problem. See Fig. 3(c).

Fig. 3. The reason for constructing $w'_1, \ldots, w'_{\deg(w)}$ in G' for every vertex w of G.

(a) (b) (c)
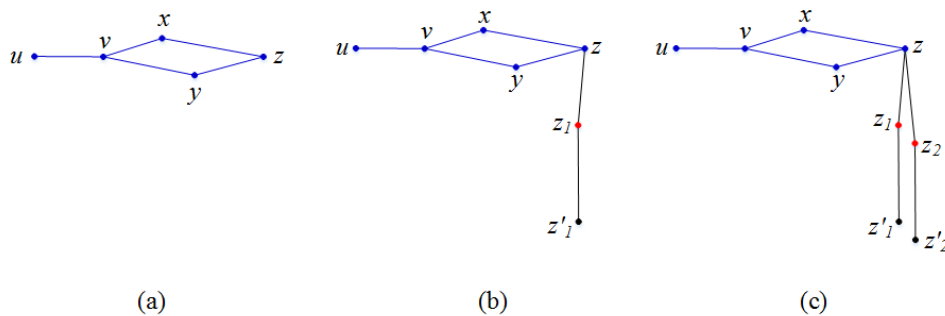
Here is a simple observation on the graph $G'(V', E')$.

Observation 3.2. *The vertex set V_2 is a subset of any strong edge geodetic set Y of G'.*

Proof. Suppose a vertex u'_i of V_2 is not in Y. Then the edge $u_i u'_i$ is not covered by any shortest path generated by the vertices of Y. □

Property 3.3. *For each strong edge geodetic set Y of G', there exists a strong edge geodetic set Y' of G' such that $Y' \subseteq Y$ and $Y' = X \cup V_2$ where X is a subset of V.*

Proof. Suppose that Y contains a member u_i of V_1. By Observation 3.2, its corresponding vertex u'_i of V_2 is in Y. Then the vertex u_i can be replaced by u'_i which is already in Y. Set $Y' = Y \setminus V_1$. A shortest path $u_i P$ is a subpath of the shortest path $u'_i u_i P$. Hence Y' is also a strong edge geodetic set of G' and $Y' = X \cup V_2$, where X is a subset of V. □

Property 3.4. *X is a shortest path union cover of G if and only if $X \cup V_2$ is a strong edge geodetic set of G'.*

Proof. Assuming that X is a shortest path union cover of G, we will prove that $X \cup V_2$ is a strong edge geodetic set of G'. First we cover an edge $u_i v_j$ of E_2 by the shortest path $u'_i u_i v_j v'_j$ where u'_i and v'_j are in $X \cup V_2$. Note that this also holds true in the case when $u = v$. The described paths in addition cover all the edges of E_3 too. Since X is a shortest path union cover of G, the edges of E are covered by the shortest paths uPv where $u \in X$, $v \in V$ and P is a shortest path in G. Thus, the edges of E and E_1 are covered by the shortest paths $uPvv_i v'_i$ where $uPvv_i v'_i$ is a shortest path generated by $X \cup V_2$. Thus $X \cup V_2$ is a strong edge geodetic set of G'.

Conversely, suppose that $X \cup V_2$ is a strong edge geodetic set of G'. Then, for each edge e of E, there exists a shortest path P_{xy} where $x, y \in X \cup V_2$ such that P_{xy} covers e. By the structure of graph G', for any shortest path P_{uv}, if both u and v are not in V, then P_{uv} will not cover any edge of E. Thus either x or y is in V. Say $x \in V$. Thus a sub path of P_{xy} starting at x which is also a shortest path indeed covers e. Hence, X is a shortest path union cover of G. □

By Property 3.3 and 3.4, we have thus proved that a minimum shortest path union cover of G can be determined by finding a strong edge geodetic set of G'. Since G' can clearly be constructed from G in polynomial time, the argument is complete.

4 Upper bounds of $sg_e(G)$

In this section an upper bound on $sg_e(G)$ is given and another possible upper bound discussed. The upper bound given is in terms of (edge) isometric path covers, where an *isometric path* has the same meaning as a geodesic (alias a shortest path). Since the term isometric path cover is well-established, we use this terminology here.

Let $G = (V, E)$ be a graph. A set S of isometric paths of G is said to be an *isometric path cover* of G if every vertex $v \in V$ belongs to at least one path from S. The cardinality of a minimum isometric path cover is called the *isometric path number* of G and denoted by $ip(G)$ [17, 18]. We now introduce the edge version of the isometric path cover in the natural way. A set S of isometric paths of a graph $G = (V, E)$ is an *edge isometric path cover* of G if every edge $e \in E$ belongs to at least one path from S. The cardinality of a minimum edge isometric path cover is called the *edge isometric path number* and denoted by $ip_e(G)$. With this concept we have the following bounds.

Theorem 4.1. *If G is a connected graph, then*

$$\frac{1 + \sqrt{(8 \times ip_e(G) + 1)}}{2} \leq sg_e(G) \leq 2 \times ip_e(G).$$

Proof. A strong edge geodetic cover S generates $\binom{|S|}{2}$ number of geodesics. Thus $\binom{|S|}{2} \geq ip_e(G)$ which in turn implies that $|S| \geq \frac{1+\sqrt{(8 \times ip_e(G)+1)}}{2}$. Since this inequality is true for every strong edge geodetic cover, we conclude that $sg_e(G) \geq \frac{1+\sqrt{(8 \times ip_e(G)+1)}}{2}$.

The inequality $sg_e(G) \leq 2 \times ip_e(G)$ follows from the fact that an edge isometric path cover of cardinality $ip_e(G)$ contains at most $2 \times ip_e(G)$ end-vertices. $\square$

Remark 4.2. *The upper bound of $sg_e(G) \leq 2 \times ip_e(G)$ is sharp. For instance, for the star graphs $K_{1,2r}$, it is easy to verify that $sg_e(K_{1,2r}) = 2r = 2 \times ip_e(K_{1,2r})$. Additional sharpness examples can be constructed using tree-like graphs.*

To conclude the section we note that the bound $sg_e(G) \leq |V(G)| - \text{diam}(G) + 1$ which one would be tempted to conjecture does not hold. For this sake consider the graph G obtained from the 6-cycle on the vertices $v_1, \ldots, v_6$ with natural adjacencies, by adding the edges v_1v_3 and v_4v_6. Then $\text{diam}(G) = 3$ yet it can be verified that $sg_e(G) = 5$.

5 Lower bound using simplicial vertices

In view of Theorem 3.1 it is natural to derive upper and lower bounds for the strong edge geodetic number. In this section we use simplicial vertices for this purpose and derive some related consequences.

Clearly, every simplicial vertex u belongs to every geodetic set, to every edge geodetic set, as well as to every strong edge geodetic set because u cannot be an inner vertex of a geodesic. We have already observed that $g(G) \leq g_e(G)$ holds for any graph G. In addition, we also have $g_e(G) \leq sg_e(G)$. In the next result we collect these observations for the latter use.

Lemma 5.1. *Let X be the set of simplicial vertices of a graph G. Then*

$$|X| \leq g(G) \leq g_e(G) \leq sg_e(G).$$

Using Lemma 5.1, we compute strong edge geodetic number for certain graphs. First we proceed with block graphs and trees. Recall that a graph G is a *block graph* if every block of G is a clique.

Proposition 5.2. *The set of simplicial vertices of a block graph G on at least two vertices is a $sg_e(G)$-set of G.*

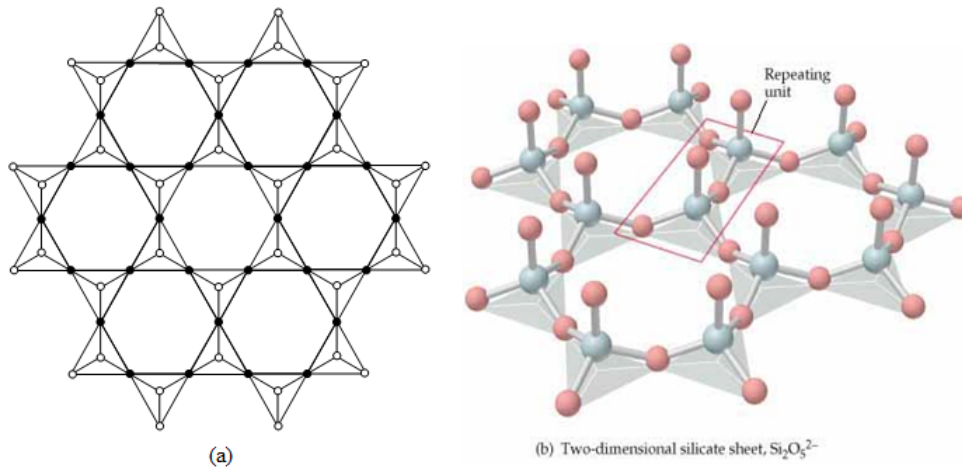*Proof.* Let X be the set of simplicial vertices of a block graph G. By Lemma 5.1, $sg_e(G) \geq |X|$.

It remains to prove that X is a strong edge geodetic set of G. We proceed by induction on the number b of blocks of G. If $b = 1$ then G is a complete graph on at least two vertices, and the assertion clearly holds. Suppose now that $b \geq 2$. Let Q be a pendant block of G and let $V(Q) = \{x_1, \ldots, x_k\}$, $k \geq 2$. We may without loss of generality assume that x_k is the cut vertex of Q. Then $x_i, i \in [k-1]$, are simplicial vertices of G. The graph G' induced by the vertices $(V(G) \setminus V(Q)) \cup \{x_k\}$ is a block graph (on at least two vertices) with one block less than G, hence by the induction hypothesis the set of its simplicial vertices forms a $sg_e(G)$-set of G'. Let Y' be the set of geodesics that correspond to a $sg_e(G')$-set. Suppose first that x_k is a simplicial vertex of G'. Then the set of paths $\{Px_i : P \in Y', P = x_k P', i \in [k-1]\} \cup \{P : P \in Y', x_k \text{ is not end vertex of } P\} \cup \{x_i x_j : i, j \in [k-1], i \neq j\}$ forms a required strong edge geodetic set of G. Similarly, if x_k is not a simplicial vertex of G', then we replace every shortest path from Y' that is of the form $Px_k R$ with shortest paths $Px_k x_i$ and $x_i x_k R$ for all $i \in [k-1]$ to find a required strong edge geodetic set of G also in this case. $\square$

Proposition 5.2 immediately implies:

Corollary 5.3. *The set of leaves of a tree T is a unique $sg_e(G)$-set of T.*

Next we determine the strong edge geodetic number of hexagonal silicate networks which are well-known chemical networks. A hexagonal silicate network [19] is shown in Fig. 4.

(a)

(b) Two-dimensional silicate sheet, $Si_2O_5^{2-}$

Corollary 5.4. *The simplicial vertices of the hexagonal silicate network G form a $sg_e(G)$-set of G.*

Proof. The simplicial vertices of G are marked by white bullets in Fig. 4. It is easy to verify that the set of simplicial vertices forms a strong edge geodetic set of G. The result follows from Lemma 5.1. □

Corollary 5.4 considers only silicate networks of hexagonal type. This result can be extended to any type of silicate sheets. The verification is left to the reader.

6 Lower bound using convex components

For a different kind of a lower bound we introduce the following concept. We say that edges e and f of a graph G form a *geodesic pair* if they belong to some shortest path of G. Otherwise, e and f form a *non-geodesic pair*. Since (non)-geodesic pairs play a key role in Theorem 6.2, we next characterize such edges in the following result that might be of independent interest.

Proposition 6.1. *Let $e = uv$ and $f = xy$ be edges of a connected graph G. Then e and f are geodesic if and only if $|\{d(u,x), d(u,y), d(v,x), d(v,y)\}| = 3$.*

Proof. Suppose that e and f are geodesic edges and let P be a shortest path containing these two edges. We may without loss of generality assume that $d(v,x) = d(u,y) + 2$ and set $d(v,x) = k$. Then $d(u,x) = d(v,y) = k+1$, so that

$$\{d(u,x), d(u,y), d(v,x), d(v,y)\} = \{k, k+1, k+2\}.$$

Conversely, suppose that $|\{d(u,x), d(u,y), d(v,x), d(v,y)\}| = 3$. We may without loss of generality assume that $k = d(u,x)$ is the minimal among the four distances. If also $d(v,x) = k$, then $d(u,y) \leq k+1$ and $d(v,y) \leq k+1$ which is not possible. Therefore, $d(v,x) = k+1$. Since $d(u,y) \leq k+1$ it follows that $d(v,y) = k+2$. It now readily follows that e and f are geodesic edges. □

We recall that a subgraph H of a graph G is *convex*, if for any vertices $x, y \in V(H)$, every shortest x, y-path in G lies completely in H.

Theorem 6.2. *Let F be a set of pairwise non-geodesic edges of a graph G. If $G - F$ consists of $t \geq 2$ convex components, then*

$$sg_e(G) \geq \left\lceil t \times \sqrt[t]{\frac{2|F|}{t(t-1)}} \right\rceil .$$

Proof. Let $G[X_1], \ldots, G[X_t]$ be the convex components of $G - F$. Let U be an arbitrary strong edge geodetic set of G and set $U_i = U \cap X_i$, $i \in [t]$. Clearly, $U = \cup_{i=1}^t U_i$ and $U_i \cap U_j = \emptyset$ for $i \neq j$. As $G[X_i]$ is convex, no shortest path between two vertices of U_i contains an edge from F. Since in addition a shortest path between a vertex from U_i and a vertex from U_j, $i \neq j$, contains at most one edge from F, if follows that

$$\sum_{i \neq j} |U_i||U_j| \geq |F|. \tag{1}$$

Using the fact that for any non-negative real numbers their arithmetic mean is at least as large as their geometric mean, we get

$$|U| = \sum_{k=1}^t |U_k| \geq t \times \sqrt[t]{\prod_{k=1}^t |U_k|}. \tag{2}$$

Since $|U_k| \geq 1$ for $k \in [t]$, the following inequality is straightforward:

$$\prod_{k=1}^t |U_k| \geq |U_i||U_j|, \ i, j \in [t]. \tag{3}$$

Applying Inequality (3) for all $\binom{t}{2}$ pairs $\{i, j\}$ we get

$$\binom{t}{2} \prod_{k=1}^t |U_k| \geq \sum_{i \neq j} |U_i||U_j|. \tag{4}$$

Using (2), (4), and (1) in that order, we can now estimate as follows:

$$|U| \geq t \times \sqrt[t]{\prod_{k=1}^t |U_k|}$$

$$\geq t \times \sqrt[t]{\frac{1}{\binom{t}{2}} \sum_{i \neq j} |U_i||U_j|}$$

$$\geq t \times \sqrt[t]{\frac{1}{\binom{t}{2}} |F|}.$$

Since U is an arbitrary strong edge geodetic set we conclude that

$$sg_e(G) \geq t \times \sqrt[t]{\frac{2|F|}{t(t-1)}}.$$

Since $sg_e(G)$ is an integral, the result follows. □

To show that the bound of Theorem 6.2 is sharp, let us say that a graph G is *r-good* if it contains vertices u and v such that $I(u, v)$ contains r shortest u, v-paths which cover all the edges of G. For example, uniform theta graphs [20] are r-good graphs. To specify the vertices u and v we will denote such a graph with G_u^v. Clearly, an r-good graph G_u^v is necessarily bipartite and u and v are diametrical vertices of G.

Proposition 6.3. *Let $G_{u_i}^{v_i}$, $i \in [n]$, be $(n-1)$-good graphs, $n \geq 2$. Let $V(K_n) = [n]$ and let X be the graph obtained from the disjoint union of K_n and $G_{u_i}^{v_i}$, $i \in [n]$, by connecting u_i with i for $i \in [n]$. Then $sg_e(X) = n$.*

Proof. Clearly, a strong edge geodetic set of X must contain at least one vertex from each of the subgraphs $G_{u_i}^{v_i}$, hence $sg_e(X) \geq n$.

To prove the other inequality it suffices to show that $\{v_i : i \in [n]\}$ is a $sg_e(G)$-set of X. Let P_j^i, $j \in [n] \setminus i$, be the shortest v_i, u_i-paths in $G_{u_i}^{v_i}$ that cover all the edges. (Such paths exist because $G_{u_i}^{v_i}$ is an $(n-1)$-good graph.) For any $i \leq j$ let P_{ij} be the path in X that is a concatenation of the paths P_j^i and P_i^j, and the edges iu_i, ij, and ju_j. It is straightforward to verify that each P_{ij} is a shortest v_i, v_j-path in G and that the paths P_{ij}, $i, j \in [n]$, $i \neq j$, cover all the edges of X. Hence $sg_e(X) \leq n$. $\square$

Consider the graph X of Proposition 6.3 and let $F = E(K_n)$. Then F is a set of pairwise non-geodesic edges and $X - F$ consists of n convex components. Since $|F| = \binom{n}{2}$, it follows that the bound of Theorem 6.2 is sharp for X.

An important special case of Theorem 6.2 is the following. A set F of edges of a connected graph G is a *convex edge-cut* if $G - F$ consists of two convex components. Note that the edges of a convex edge-cut are pairwise non-geodesic for otherwise at least one of the components of $G - F$ would not be convex. Hence Theorem 6.2 immediately implies:

Corollary 6.4. *If F is a convex edge-cut of a graph G, then $sg_e(G) \geq \left\lceil 2\sqrt{|F|} \right\rceil$.*

We next give an application of Corollary 6.4. Glued binary trees were introduced by physicists as a tool to design quantum algorithms [21] and quantum circuits [22]. It plays a significant role in Quantum Information Theory [23]. It is also used to study the transmission properties of continuous time quantum walks in quantum physics [24, 25]. Lockhart et al. [26] designed glued tree algorithm using glued binary trees.

An r-level complete binary tree $T(r)$ has 2^r leaves. An r-level glued binary tree $GT(r)$ is formed by connecting the leaves of two r-level complete binary trees $T_1(r)$ and $T_2(r)$. Fig. 5 displays two 3-level glued binary trees $GT(3)$. In general, the vertex set of $GT(r)$ is the union of the vertex sets of complete binary trees $T_1(r)$ and $T_2(r)$. Let L_1 and L_2 denote the vertex sets of leaves of $T_1(r)$ and $T_2(r)$ respectively. Notice that each set L_1 and L_2 is an independent set. The sets L_1 and L_2 induce a bipartite graph in $GT(r)$. Feder [23] classifies glued binary trees into those without randomization and those with randomization. If the edges of the bipartite subgraph induced by the sets L_1 and L_2 are in some fixed order, then they are called the *glued binary trees without randomization*, while if the sets L_1 and L_2 induce an arbitrary bipartite graph, then they are called the *glued binary trees with randomization*. Fig. 5(a) shows a 3-level glued binary tree without randomization. Fig. 5(b) displays a 3-level glued binary tree with randomization.

Fig. 5. (a) Glued binary trees without randomization $GT_c(r)$ and (b) glued binary trees with randomization.

(a)   (b)
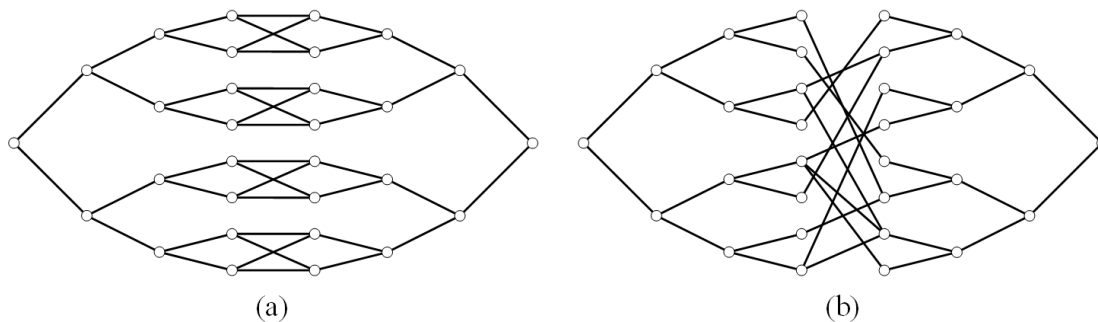
Using Corollary 6.4, we can solve the strong edge geodetic problem for certain classes of glued binary trees without randomization. We define two graphs $GT_p(r)$ and $GT_c(r)$ which are glued binary trees without randomization. The graph $GT_p(r)$ is obtained from $T_1(r)$ and $T_2(r)$ by adding straight edges between the corresponding leaves, see Fig. 6(a) for $GT_p(4)$. The graph $GT_c(r)$ is obtained from $GT_p(r)$ by adding additional cross edges between the leaves as shown in Fig. 5(a) for the case $GT_c(3)$.

Theorem 6.5. $sg_e(GT_p(r)) = \left\lceil 2\sqrt{2^r} \right\rceil$.

Proof. The set of straight edges (dotted edges in Fig. 6(b)) between the leaves forms a convex edge-cut of cardinality 2^r. By Corollary 6.4, $sg_e(GT_p(r)) \geq \left\lceil 2\sqrt{2^r} \right\rceil$.

Fig. 6. Glued binary tree $GT_p(4)$.

(a)   (b)
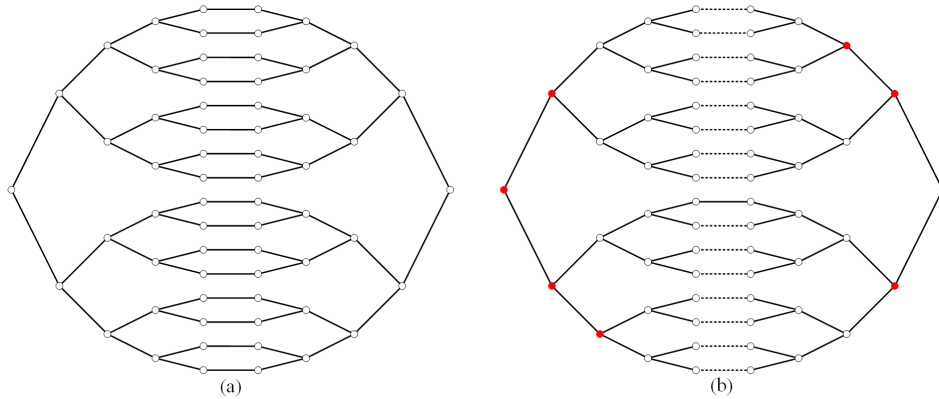
Let $k = \left\lceil 2\sqrt{2^r} \right\rceil$. The vertex set of $GT_p(r)$ is the union of the vertex sets of complete binary trees $T_1(r)$ and $T_2(r)$. Starting from the root of $T_1(r)$, select $k/2$ vertices of $T_1(r)$ in a breadth first search (BFS) traversal order. Name this set as S_1. In the same way, build a set S_2 by selecting another $k/2$ vertices from $T_2(r)$. Set $S = S_1 \cup S_2$. The red bullets in Fig. 6(b) represent this set S. It is a simple exercise to verify that the set S of vertices forms a strong edge geodetic set. Thus $sg_e(GT_p(r)) = \left\lceil 2\sqrt{2^r} \right\rceil$.   □

Using an approach from the proof of Theorem 6.5 we can also deduce the following result.

Theorem 6.6. $sg_e(GT_c(r)) = \left\lceil 2\sqrt{2 \times 2^r} \right\rceil$.

Further research is to identify the classes of glued binary trees without randomization for which the strong edge geodetic problem can be solved. The strong edge geodetic problem for glued binary trees with randomization is left as an open problem.

7 Further research

The vertex version of the strong edge geodetic problem is the strong geodetic problem defined analogously as follows. A set $S \subseteq V(G)$ is a *strong geodetic set* if for any pair $x, y \in S$ there exists a shortest x, y-path P_{xy} such that

$$\bigcup_{x,y \in \binom{S}{2}} V(P_{xy}) = V(G).$$

The *strong geodetic problem* is to find a smallest strong geodetic set of G. To our knowledge, there is no literature yet on the strong geodetic problem. We believe it is also NP-complete, similar to the edge version of the problem. As already mentioned in Section 4, Fitzpatrick et al. [17] have studied the isometric path problem. It would be interesting to investigate relationships between the strong geodetic problem and the isometric path problem. Fitzpatrick et al. [17] have derived the isometric path number for two dimensional grids. Further research would be to design techniques to derive the strong geodetic number for grid-like architectures and similar architectures.

8 Conclusion

By modeling the urban road network problem as a graph combinatorial problem we proved that the urban road network problem is NP-complete. Naming this problem as the *strong edge geodetic problem*, we have studied the properties and characteristics of the strong edge geodetic problem from the perspectives of graph theory. We have derived some sharp lower bounds for strong edge geodetic number. Using these lower bounds, we have computed the strong edge geodetic number of certain classes of graphs such as trees, block graphs, silicate networks and some glued binary trees without randomization. The complexity of the problem is unknown for other graphs such as grid-like architectures, intersection graphs, Cayley graphs, chordal graphs and some of their subclasses (interval graphs, split graphs, k-tree, . . .), bipartite graphs and planar graphs.

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