

LOWER BOUNDS FOR DILATION, WIRELENGTH, AND EDGE CONGESTION OF EMBEDDING GRAPHS INTO HYPERCUBES

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ABSTRACT. Interconnection networks provide an effective mechanism for exchanging data between processors in a parallel computing system. One of the most efficient interconnection networks is the hypercube due to its structural regularity, potential for parallel computation of various algorithms, and the high degree of fault tolerance. Thus it becomes the first choice of topological structure of parallel processing and computing systems. In this paper, lower bounds for the dilation, wirelength, and edge congestion of an embedding of a graph into a hypercube are proved. Two of these bounds are expressed in terms of the bisection width. Applying these results, the dilation and wirelength of embedding of certain complete multipartite graphs, folded hypercubes, wheels, and specific Cartesian products are computed.

1. INTRODUCTION

A suitable interconnection network is an important part for the design of a multicomputer or multiprocessor system. Such a network is usually modeled by a symmetric graph, where the nodes represent the processing elements and the edges represent the communication channels. Desirable properties of an interconnection network include symmetry, embedding capabilities, relatively small degree, small diameter, scalability, robustness, and efficient routing [35]. One of the most efficient interconnection networks is the hypercube due to its structural regularity, potential for parallel computation of various algorithms, and the high degree of fault tolerance [36].

The hypercube has many excellent features and thus becomes the first choice of topological structure of parallel processing and computing systems. The machines based on hypercubes such as the Cosmic Cube from Caltech, the iPSC/2 from Intel, and Connection Machines have been implemented commercially [10]. Hypercubes are very popular models for parallel computation because of their symmetry and relatively small number of interprocessor connections. The hypercube embedding problem is the problem of mapping a communication graph into a hypercube multiprocessor. Hypercubes are known to simulate other structures such as grids and binary trees [8, 26].

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Graph embedding is an important technique that maps a guest graph into a host graph, usually an interconnection network [6]. The quality of an embedding can be measured by certain cost criteria. One of these criteria which is considered very often is the *dilation*. The dilation of an embedding is defined as the maximum distance between a pair of vertices of the host graph that are images of adjacent vertices of the guest graph. It is a measure for the communication time needed when simulating one network on another [23]. A closely related and important cost criterion is the *wirelength of an embedding* which is the sum of the dilations in the host graph of edges in the guest graph. The wirelength of a graph embedding arises from VLSI design, data structures and data representations, networks for parallel computer systems, biological models that deal with cloning and visual stimuli, parallel architecture, structural engineering and so on [21, 31, 41].

Another important cost criterion is the *edge congestion*. The edge congestion of an embedding is the maximum number of edges of the guest graph that are embedded on any single edge of the host graph. An embedding with a large edge congestion faces many problems, such as long communication delay, circuit switching, and the existence of different types of uncontrolled noise. In data networking, network edge congestion occurs when a link or node is carrying so much data that its quality of service deteriorates. Typical effects include packet loss or the blocking of new connections. Therefore, a minimum edge congestion is a most desirable feature in network embedding [11, 28].

Graph embeddings have been well-studied for a number of networks [2, 3, 8, 11, 12, 13, 14, 15, 22, 27, 33, 26, 29, 34, 41]. In this paper, we study the dilation, wirelength, and congestion of embeddings of graphs into hypercubes and proceed as follows. In the next section concepts needed are formally introduced. In Section 3, we consider the dilation of embedding into Q_n and in the main result characterize the graphs G with $\text{dil}(G, Q_n) \leq n - 1$: they are precisely the graphs which contain a perfect anti-matching. Then, in Section 4, we prove a general lower bound on the edge congestion and a lower bound on the wirelength of embeddings into hypercubes. Both bounds are expressed in terms of the bisection width. In the remainder of the paper these results are applied to particular graphs. The results obtained in Sections 5 to 8 are summarized in Table 1, where $K_{2^{n-p}, \dots, 2^{n-p}}$ denotes the complete 2^p -partite graph with parts of size 2^{n-p} .

TABLE 1. Summary of Sections 5 to 8

| Embedding $f : G \rightarrow Q_n$ | Dilation | Wirelength |
|--|--|---------------------------------------|
| $G = K_{2^{n-p}, \dots, 2^{n-p}}$ | n for $p = n$ $n - 1$ for $n - \log_2(n + 1) < p \leq n - 1$ (Corollary 1) | $n2^{2n-p-2}(2^p - 1)$ (Theorem 4) |
| $G = FQ_n$ | 2 [27] | $n2^n$ (Theorem 5) |
| $G = W_{2^n}$ | n (Theorem 1) | $(n + 2)2^{n-1}$ (Theorem 6) |
| $G = K_{2^{n/2}} \square K_{2^{n/2}}$ (n even) | $\leq n/2$ | $n2^{3n/2-2}$ (Theorem 8) |

2. PRELIMINARIES

In this section we give basic definitions and preliminaries related to embedding problems. Let G and H be finite graphs. Then an *embedding of G into H* is a pair (f, P_f) , where $f : V(G) \rightarrow V(H)$ is a one-to-one mapping and P_f is a mapping from $E(G)$ to the set of paths in H such that each edge $e = uv \in E(G)$ is assigned a path $P_f(e)$ in H between $f(u)$ and $f(v)$. By abuse of language we will also refer to an embedding (f, P_f) simply by f .

The length of $P_f(e)$ is the *dilation* $\text{dil}_f(e)$ of the edge e (with respect to the embedding (f, P_f)). The maximum dilation over all edges of G is called *the dilation of the embedding*. The *dilation of embedding G into H* is the minimum dilation taken over all embeddings of G into H and denoted by $\text{dil}(G, H)$, cf. [3]. In other words,

$$\text{dil}(G, H) = \min_f \max_{e \in E(G)} \text{dil}_f(e),$$

where the minimum is taken over all injections $f : V(G) \rightarrow V(H)$. It is not necessary to specify P_f when we are interested in the dilation of embedding G into H , because we can take $P_f(uv)$ to be any shortest path between $f(u)$ and $f(v)$. The *diameter* of H , denoted by $\text{diam}(H)$, is the maximum distance between any two vertices of H , and this provides a trivial upper bound for $\text{dil}(G, H)$.

The wirelength of an embedding $f = (f, P_f)$ of G into H is

$$WL_f(G, H) = \sum_{e \in E(G)} \text{dil}_f(e), \quad (1)$$

and the *wirelength of embedding G into H* is

$$WL(G, H) = \min_f WL_f(G, H),$$

see [26]. We also refer to [2, 3, 30] for the search of embeddings of G into H that attain the value $WL(G, H)$.

Let $f = (f, P_f)$ be an embedding of G into H . For $e = uv \in E(H)$, let $EC_f(e)$ denote the number of edges $e' \in E(G)$ such that e is in the path $P_f(e')$ between $f(u)$ and $f(v)$ in H . Then the *edge congestion of f* is

$$EC_f(G, H) = \max_{e \in E(H)} EC_f(e)$$

and the *edge congestion of embedding G into H* is

$$EC(G, H) = \min_f EC_f(G, H),$$

where the minimum is taken over all embeddings $f = (f, P_f)$ of G into H . Note that

$$WL_f(G, H) = \sum_{e_H \in E(H)} EC_f(e_H).$$

The above problems can be specialized to several distance problems. For example, if the host graph is a path, then the dilation of embedding graph into a path is the *bandwidth problem*. Further distance problems are listed in Table 2, where P_n , C_n and K_n , are the path, the cycle, and the complete graph of order n , respectively, and $K_{a,b}$ is the complete bipartite graphs on $a + b$ vertices.

Let $n \geq 1$. The *n -cube* (n -dimensional hypercube) has vertex set $\{0, 1\}^n$, and vertices $x, y \in V(Q_n)$ are adjacent if and only if the corresponding binary strings differ exactly in one bit. Equivalently, the vertices of Q_n can be identified with integers $0, 1, \dots, 2^n - 1$, where vertices i and j are adjacent if and only if $i - j = \pm 2^p$ for some integer $p \geq 0$.

For disjoint subsets $A, B \subseteq V(G)$ we will use the notation $E_G(A, B) = \{uv \in E(G) \mid u \in A, v \in B\}$ and $E_G(A) = \{uv \mid u, v \in A\}$. If x is a vertex of a graph G , then $N_i(x)$ is the set of vertices in G at distance i from x . The maximum degree of G is denoted by $\Delta(G)$ and its order by $n(G)$.

TABLE 2. Various embedding problems

| Embedding $f : G \rightarrow H$ | Dilation | Wirelength | Congestion |
|------------------------------------|-----------------------|--------------------------|------------------------------|
| $H = P_n$ | Bandwidth [17] | MinLA [1] | Cutwidth [16] |
| $H = C_n$ | Cyclic bandwidth [24] | Cyclic wirelength [4] | Cyclic cutwidth [7] |
| $G = K_n$ | $\text{diam}(H)$ [40] | Wiener index of H [39] | Forwarding index of H [18] |
| $G = K_{a,b}$ | Makespan [37] | Average load | Maximum load |

3. GRAPHS WITH LARGE DILATION OF EMBEDDING INTO HYPERCUBES

The dilation, the wirelength, and the edge congestion problem are different in the sense that an embedding that gives the minimum dilation need not give the minimum congestion (wirelength) and vice-versa. From $\text{diam}(Q_n) = n$ we obtain immediately the upper bound $\text{dil}(G, Q_n) \leq n$ for every graph G . To characterize graphs G of order 2^n with $\text{dil}(G, Q_n) = n$, we need the following concept. If G is a graph, then the set of pairs of vertices $X = \{\{x_1, y_1\}, \dots, \{x_k, y_k\}\}$ forms an *anti-matching*, if $x_1 y_1 \notin E(G), \dots, x_k y_k \notin E(G)$. That is, X is an anti-matching if $x_1 y_1, \dots, x_k y_k$ form a matching in the complement of G . In addition, we say that X is a *perfect anti-matching* if $2|X| = n(G)$. Now we have the following characterization.

Theorem 1. *Let G be a graph of order 2^n . Then $\text{dil}(G, Q_n) \leq n - 1$ if and only if G contains a perfect anti-matching.*

Proof. Let $\text{dil}(G, Q_n) \leq n - 1$ and let (f, P_f) be a corresponding embedding of G into Q_n . Consider the partition $X' = \{\{u, \bar{u}\} \mid u \in V(Q_n)\}$ of $V(Q_n)$ into 2^{n-1} pairs. Then, since $\text{dil}_f(e) \leq n - 1$ holds for any edge e of G , we must have that $f^{-1}(u)$ is not adjacent to $f^{-1}(\bar{u})$. It follows that $X = \{\{f^{-1}(u), f^{-1}(\bar{u})\} \mid u \in V(Q_n)\}$ is a perfect anti-matching.

Conversely, let X be a perfect anti-matching of G . We now define an embedding (f, P_f) of G into Q_n by mapping each pair $\{x, y\}$ from X into an antipodal pair of vertices of Q_n , that is, $\{f(x), f(y)\} = \{v, \bar{v}\}$ for some $v \in V(Q_n)$. Since each vertex of Q_n has a unique antipodal vertex, it follows that $\text{dil}_f(e) \leq n - 1$ for each edge e of G and hence $\text{dil}(G, Q_n) \leq n - 1$. \square

To determine a large class of graphs G with $\text{dil}(G, Q_n) = n - 1$ we state the following lemma that could be applied elsewhere.

Lemma 1. *If G is a graph of order 2^n , then*

$$\text{dil}(G, Q_n) \geq \max \left\{ k : \sum_{i=1}^{k-1} \binom{n}{i} < \Delta(G) \right\}.$$

Proof. Let k be a positive integer satisfying $\binom{n}{0} + \dots + \binom{n}{k-1} < \Delta(G)$, and let (f, P_f) be an embedding of G into Q_n . Let u be a vertex of G with $\deg(u) = \Delta(G)$ and let $f(u) = v \in V(Q_n)$. Setting $X = N_1(v) \cup \dots \cup N_{k-1}(v)$, the assumption on k implies that

$$|X| = |N_1(v)| + \dots + |N_{k-1}(v)| = \sum_{i=1}^{k-1} \binom{n}{i} < \Delta(G).$$

Thus, there exists at least one vertex w of G adjacent to u such that $f(w) \in V(Q_n) \setminus X$. It follows that the length of $P_f(uw)$ is at least k and consequently $\text{dil}(G, Q_n) \geq k$. \square

Combining Lemma 1 with Theorem 1 we obtain the following corollary.

Corollary 1. *If G is a graph of order 2^n with $\Delta(G) \geq 2^n - n - 1$ that contains a perfect anti-matching, then $\text{dil}(G, Q_n) = n - 1$.*

Proof. By Theorem 1, $\text{dil}(G, Q_n) \leq n - 1$ and the reverse inequality follows from Lemma 1 and

$$\sum_{i=1}^{n-2} \binom{n}{i} = 2^n - \binom{n}{0} - \binom{n}{n-1} - \binom{n}{n} = 2^n - n - 2 < \Delta(G). \quad \square$$

For illustration, consider $G = K_{4,4,4,4}$. Then $\Delta(G) = 12 \geq 2^4 - 4 - 1$. Since G contains a perfect anti-matching, $\text{dil}(G, Q_4) = 3$. In Figure 1 a corresponding embedding is shown.

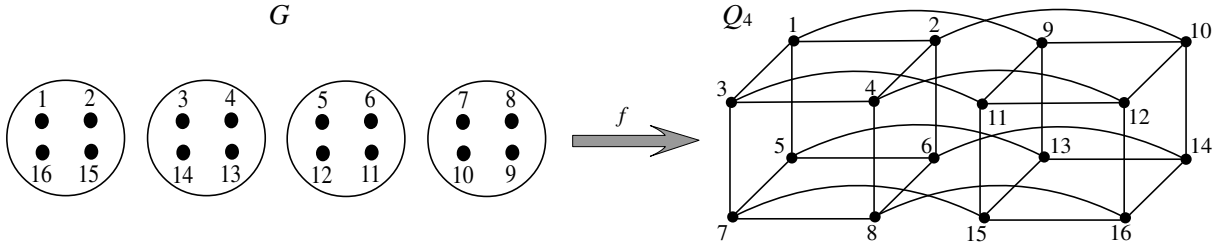


FIGURE 1. $\text{dil}(K_{4,4,4,4}, Q_4) = 3$

4. TWO LOWER BOUNDS IN TERMS OF THE BISECTION WIDTH

In this section we give a general lower bound on the edge congestion and on the wirelength of embedding into hypercubes. Both bound involve the bisection width that is defined as follows. The *bisection width* $BW(G)$ of a graph G is the minimum number of edges necessary in an edge cut so that the sizes of the two sides of the cut differ by at most one.

The first lower bound is established by the following averaging argument.

Theorem 2. *If G and H are graphs of the same order, then*

$$EC(G, H) \geq \frac{BW(G)}{BW(H)}.$$

Proof. Set $n = n(G) (= n(H))$. Let $A \cup B$ be a partition of $V(H)$ such that $|A| = \lceil n/2 \rceil$, $|B| = \lfloor n/2 \rfloor$, and $|E_H(A, B)| = BW(H)$, and let $f : G \rightarrow H$ be an embedding with $EC_f(G, H) = EC(G, H)$. Then

$$BW(G) \leq |E_G(f^{-1}(A), f^{-1}(B))| \leq \sum_{e \in E_H(A, B)} EC_f(e). \quad (2)$$

where the first inequality follows from the definition of the bisection width, and the second one from the fact that for every edge $uv \in E_G(f^{-1}(A), f^{-1}(B))$ the path $P_f(uv)$ has to use at least one edge from $E_H(A, B)$. Finally, we conclude

$$\begin{aligned} \frac{BW(G)}{BW(H)} &= \frac{BW(G)}{|E_H(A, B)|} \leq \frac{1}{|E_H(A, B)|} \sum_{e \in E_H(A, B)} EC_f(e) \leq \max_{e \in E_H(A, B)} EC_f(e) \\ &\leq \max_{e \in E(H)} EC_f(e) = EC_f(G, H) = EC(G, H). \quad \square \end{aligned}$$

Corollary 2. *If G is a graph of order 2^n , then*

$$EC(G, Q_n) \geq \frac{BW(G)}{2^{n-1}}.$$

Proof. The result follows from Theorem 2 and the fact that $BW(Q_n) = 2^{n-1}$, see [38]. \square

The lower bound for the wirelength is the following.

Theorem 3. *If G is a graph of order 2^n , then $WL(G, Q_n) \geq n \cdot BW(G)$.*

Proof. Let $E(Q_n) = S_1 \cup \dots \cup S_n$ be the partition of $E(Q_n)$ where S_i consists of the edges whose incident vertices differ in the i th coordinate. Then $Q_n \setminus S_i$ consists of two vertex-disjoint $(n-1)$ -cubes. Moreover, $|S_i| = 2^{n-1} = BW(Q_n)$, and we can now estimate as follows, where the minimum is taken over all embeddings $f : G \rightarrow Q_n$:

$$WL(G, Q_n) = \min_f \sum_{e \in E(Q_n)} EC_f(e) = \min_f \sum_{i=1}^n \sum_{e \in S_i} EC_f(e) \stackrel{(2)}{\geq} \min_f \sum_{i=1}^n BW(G) = n \cdot BW(G).$$

The first equality above holds by the second equality in (1), and the second equality follows because the sets S_i form a partition of $E(Q_n)$. \square

5. EMBEDDINGS OF COMPLETE MULTIPARTITE GRAPHS

In this section we consider the complete p -partite graph $G = K_{n_1, \dots, n_p}$ of order 2^n . Recall that G contains p independent sets containing n_i , $i \in [p]$, vertices, and all possible edges between vertices from different parts.

From Theorem 1 it follows that $\text{dil}(G, Q_n) = n$ if at least one of the p_i is odd and $\text{dil}(G, Q_n) \leq n-1$ otherwise (that is, if all the p_i are even). We will next determine the wirelengths of embedding certain complete p -partite graphs into n -cubes, and start by determining the bisection widths of balanced complete multipartite graphs.

Lemma 2. *If G is the complete t -partite graph $K_{2r, \dots, 2r}$, with $2tr$ vertices, $t \geq 2$, $r \geq 1$, then*

$$BW(G) = r^2 t(t-1).$$

Proof. Let $V(G) = A \cup B$ be a partition with $|A| = |B| = tr$ such that A and B each contains half of the vertices in every maximal independent set. Then

$$BW(G) \leq |E_G(A, B)| = |E(G)| - |E_G(A)| - |E_G(B)| = 4r^2 \binom{t}{2} - 2r^2 \binom{t}{2} = r^2 t(t-1).$$

For the reverse inequality, let $V(G) = A \cup B$ be a partition with $|A| = |B| = tr$ and $|E_G(A, B)| = BW(G)$. For $i = 1, \dots, t$, let n_i be the number of vertices of A in the i -th independent set. Then we apply the Cauchy-Schwarz inequality to obtain the bound

$$|E_G(A)| = \sum_{1 \leq i < j \leq t} n_i n_j = \frac{1}{2} \left(\left(\sum_{i=1}^t n_i \right)^2 - \sum_{i=1}^t n_i^2 \right) = \frac{1}{2} \left((tr)^2 - \sum_{i=1}^t n_i^2 \right) \leq \frac{r^2 t(t-1)}{2}.$$

For the same reason, $|E_G(B)| \leq r^2 t(t-1)/2$, and consequently,

$$BW(G) = |E_G(A, B)| = |E(G)| - (|E_G(A)| + |E_G(B)|) \geq 4r^2 \binom{t}{2} - r^2 t(t-1) = r^2 t(t-1). \quad \square$$

We now want to determine the wirelength for embedding the complete 2^p -partite graph $G = K_{2^{n-p}, \dots, 2^{n-p}}$, $1 \leq p < n$ into Q_n .

Theorem 4. *If G is the complete 2^p -partite graph $K_{2^{n-p}, \dots, 2^{n-p}}$, where $1 \leq p < n$, then*

$$WL(G, Q_n) = n \cdot 2^{2n-p-2} (2^p - 1).$$

Proof. Combining Lemma 2 for $t = 2^p$ and $r = 2^{n-p-1}$ with Theorem 3, we obtain

$$WL(G, Q_n) \geq n \cdot BW(G) = n \cdot 2^{2n-p-2} (2^p - 1).$$

In order to prove the reverse inequality we construct an embedding $f : G \rightarrow Q_n$ in the following way. Let $V(G) = V_1 \cup \dots \cup V_{2^p}$ be the partition of $V(G)$ into 2^p independent sets of size 2^{n-p} . Label the vertices of $V(G)$ with the numbers $0, 1, 2, \dots, 2^n - 1$, in such a way that

$$V_i = \{j2^{p+1} + i - 1 : j = 0, 1, 2, \dots, 2^{n-p-1} - 1\} \cup \{j2^{p+1} - i : j = 1, 2, 3, \dots, 2^{n-p-1}\}.$$

Recall that $V(Q_n) = \{0, 1\}^n$ and define $f : V(G) \rightarrow V(Q_n)$ by mapping $x \in V(G)$ to its binary representation. That is, $x = x_1 \cdot 2^{n-1} + x_2 \cdot 2^{n-2} + \dots + x_n \cdot 2^0$ is mapped to $f(x) = x_1 x_2 \dots x_n$. We will now show that for this embedding, $WL_f(G, Q_n) = n \cdot 2^{2n-p-2} (2^p - 1)$, which concludes the proof. For $x, y \in \{0, 1\}^n$, let $d(x, y)$ denote their Hamming distance, that is, $d(x, y) = |x_1 - y_1| + \dots + |x_n - y_n|$, which is equal to the distance between x and y in Q_n . For every $i \in \{1, 2, \dots, 2^p\}$,

$$\sum_{\{x, y\} \in \binom{V_i}{2}} d(f(x), f(y)) = n 2^{2(n-1-p)},$$

and consequently,

$$\begin{aligned} WL_f(G, Q_n) &= \sum_{\{x, y\} \in \binom{V(Q_n)}{2}} d(x, y) - \sum_{i=1}^{2^p} \sum_{\{x, y\} \in \binom{V_i}{2}} d(f(x), f(y)) \\ &= n 2^{2n-2} - 2^p n 2^{2(n-1-p)} = n 2^{2n-2-p} (2^p - 1). \quad \square \end{aligned}$$

6. EMBEDDINGS OF FOLDED HYPERCUBES

The n -dimensional *folded hypercube* FQ_n is the graph obtained from Q_n by adding, for every vertex $x = x_1 \dots x_n$, the edge between x and its antipodal vertex $\bar{x} = \bar{x}_1 \dots \bar{x}_n$, where $\bar{x}_i = 1 - x_i$, cf. [41]. Folded hypercubes are of wide current interest in reliability and fault tolerance of interconnection networks, see [9, 21, 25, 42]. In this section we add to the known results on folded hypercubes their wirelength.

Theorem 5. *For all $n \geq 2$, $WL(FQ_n, Q_n) = n 2^n$.*

Proof. In [32] it was proved that $BW(FQ_n) = 2^n$, and with Theorem 3 this implies $WL(FQ_n, Q_n) \geq n 2^n$. To prove the reverse inequality, let $\text{id} : FQ_n \rightarrow Q_n$ be the identity embedding, that is, $\text{id}(v) = v$, for every $v \in V(FQ_n)$ (and where the second v is considered as a vertex of Q_n). For this embedding, the $n 2^{n-1}$ edges of FQ_n that are also the edges of Q_n contribute $n 2^{n-1}$ to the wirelength. and each of the 2^{n-1} diagonal edges contributes n to the wirelength. We conclude

$$WL_{\text{id}}(FQ_n, Q_n) = n 2^{n-1} + 2^{n-1} n = n \cdot 2^n. \quad \square$$

7. EMBEDDINGS OF WHEELS

Let $n \geq 4$. The *wheel* W_n of order n is the graph obtained from a cycle C_{n-1} by adding a new vertex x and joining x to all the vertices of the cycle. Vertex x is called the *hub* of the wheel and the edges incident with x are *spokes* of the wheel. Since the hub vertex is adjacent to all other vertices, the wheel does not contain an anti-matching, and by Theorem 1, $\text{dil}(W_{2^n}, Q_n) = n$. We next determine the wirelength of embedding W_{2^n} into Q_n .

Theorem 6. *If $n \geq 2$, then $WL(W_{2^n}, Q_n) = (n+2)2^{n-1}$.*

Proof. For every embedding the spokes contribute

$$\sum_{k=1}^n k \binom{n}{k} = n2^{n-1}$$

to the wirelength. Since Q_n is bipartite, the $2^n - 1$ cycle edges contribute at least 2^n . This gives a lower bound of $n2^{n-1} + 2^n = (n+2)2^{n-1}$, and using a Gray code this bound can be achieved. \square

8. EMBEDDINGS OF SOME CARTESIAN PRODUCTS

Recall that the *Cartesian product* $G \square H$ of graphs G and H is the graph with the vertex set $V(G) \times V(H)$, vertices (g, h) and (g', h') being adjacent if either $g = g'$ and $hh' \in E(H)$, or $h = h'$ and $gg' \in E(G)$. For more information on this product see the book [20]. We will use the following result on the edge-isoperimetric problem for Cartesian products of complete graphs (see [5] for a survey of related results).

Theorem 7. ([19]) *Let $G = K_{p_1} \square K_{p_2} \square \cdots \square K_{p_t}$ with $p_1 \leq p_2 \leq \cdots \leq p_t$, and let m be an integer with $1 \leq m \leq p_1 p_2 \cdots p_t$. If H is a subgraph of G with $|V(H)| = m$, then $|E(H)| \leq |E(H^*)|$, where H^* is the subgraph induced by the first m vertices in lexicographic order, where we take*

$$V(G) = \{(a_1, a_2, \dots, a_t) : 1 \leq a_i \leq p_i \text{ for } i = 1, 2, \dots, t\},$$

and the lexicographic order is defined by $(a_1, a_2, \dots, a_t) < (b_1, b_2, \dots, b_t)$ if there exists an index i with $a_i < b_i$ and $a_j = b_j$ for all $j < i$.

As a corollary, we obtain the bisection widths for products of complete graphs.

Corollary 3. *If $G = K_{2p_1} \square K_{p_2} \square K_{p_3} \square \cdots \square K_{p_t}$ with $2p_1 \leq p_2 \leq p_3 \leq \cdots \leq p_t$, then $BW(G) = p_1^2 p_2 \cdots p_t$.*

Proof. We have $|E(G)| = p_1 p_2 \cdots p_t (2p_1 + p_2 + p_3 + \cdots + p_t - t)$, and for any $A \subseteq V(G)$ with $|A| = p_1 p_2 \cdots p_t$, Theorem 7 implies

$$|E_G(A)| \leq \frac{1}{2} p_1 p_2 \cdots p_t (p_1 + p_2 + \cdots + p_t - t)$$

with equality if A is the set of vertices (a_1, \dots, a_t) with $a_1 \leq p_1$ or the set of vertices (a_1, \dots, a_t) with $a_1 > p_1$. This implies that for every partition $V(G) = A \cup B$ with $|A| = |B| = p_1 p_2 \cdots p_t$,

$$|E_G(A, B)| = |E(G)| - |E_G(A)| - |E_G(B)|$$

$$\geq p_1 p_2 \cdots p_t (2p_1 + p_2 + p_3 + \cdots + p_t - t) - p_1 p_2 \cdots p_t (p_1 + p_2 + \cdots + p_t - t) = p_1^2 p_2 \cdots p_t$$

with equality if A is the set of vertices (a_1, \dots, a_t) with $a_1 \leq p_1$. \square

Theorem 8. *Let n be a positive even integer, and let $G = K_{2^{n/2}} \square K_{2^{n/2}}$. Then*

$$WL(G, Q_n) = n2^{3n/2-2}.$$

Proof. From Corollary 3 with $t = 2$, $p_1 = 2^{n/2-1}$ and $p_2 = 2^{n/2}$, we obtain $BW(G) = 2^{3n/2-2}$, and together with Theorem 3 this implies the lower bound. To prove the upper bound we embed G into Q_n , by mapping a vertex $(a, b) \in V(G) = \{0, 1, \dots, 2^{n/2} - 1\} \times \{0, 1, \dots, 2^{n/2} - 1\}$ with

$$a = \sum_{i=1}^{n/2} a_i 2^{n/2-i}, \quad b = \sum_{i=1}^{n/2} b_i 2^{n/2-i}$$

to the binary string $a_1a_2\dots a_{n/2}b_1b_2\dots b_{n/2} \in \{0,1\}^n = V(Q_n)$. For $i \in \{1, \dots, n\}$, let $V(Q_n) = A_i \cup B_i$ be the partition with $A_i = \{x \in V(Q_n) : x_i = 0\}$ and $B_i = \{x \in V(Q_n) : x_i = 1\}$. Then

$$E(Q_n) = \bigcup_{i=1}^n E_{Q_n}(A_i, B_i)$$

and since each of the sets $f^{-1}(A_i)$ and $f^{-1}(B_i)$ induces a subgraph of G isomorphic to $K_{2^{n/2-1}} \square K_{2^{n/2}}$, we obtain

$$\begin{aligned} WL_f(G, Q_n) &= \sum_{i=1}^n \sum_{e \in E_{Q_n}(A_i, B_i)} EC_f(e) = \sum_{i=1}^n |E_G(f^{-1}(A_i), f^{-1}(B_i))| \\ &= \sum_{i=1}^n BW(G) = n \cdot BW(G) = n2^{3n/2-2}. \quad \square \end{aligned}$$

CONCLUDING REMARKS

In this paper, we have obtained lower bounds for dilation, wirelength, and edge congestion of an embedding of graphs into n -cubes. In particular, we found lower bounds for congestion and wirelength in terms the bisection width of the guest graph. This technique allows the computation of the exact dilation and wirelength for a range of networks. An open problem remains to determine the edge congestion of these networks. It also opens up the study of embedding parameters which remains unsolved for several good candidates of guest architectures.

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REFERENCES

1. Donald L. Adolphson and Te Chiang Hu, *Optimal linear ordering*, SIAM Journal on Applied Mathematics **25** (1973), no. 3, 403–423.
2. Sergei L. Bezrukov, Joseph D. Chavez, Larry H. Harper, Marcus Röttger, and Ulf-Peter Schroeder, *The congestion of n -cube layout on a rectangular grid*, Discrete Mathematics **213** (2000), no. 1-3, 13–19.
3. Sergei L. Bezrukov, Joseph D. Chavez, Larry H. Harper, Markus Röttger, and Ulf-Peter Schroeder, *Embedding of hypercubes into grids*, Mathematical Foundations of Computer Science 1998, Lecture Notes in Computer Science, vol. 1450, Springer Berlin Heidelberg, 1998, pp. 693–701.
4. Sergei L. Bezrukov and Ulf-Peter Schroeder, *The cyclic wirelength of trees*, Discrete Applied Mathematics **87** (1998), no. 1-3, 275–277.
5. Sergei S. Bezrukov, *Edge isoperimetric problems on graphs*, Graph theory and combinatorial biology (László Lovász, ed.), Bolyai Society mathematical studies, vol. 7, János Bolyai Mathematical Society, 1999, pp. 157–197.
6. Vipin Chaudhary and Jake K. Aggarwal, *A generalized scheme for mapping parallel algorithms*, IEEE Transactions on Parallel and Distributed Systems **4** (1993), no. 3, 328–346.
7. Joseph D. Chavez and Rolland Trapp, *The cyclic cutwidth of trees*, Discrete Applied Mathematics **87** (1998), no. 1-3, 25–32.
8. Wai-Kai Chen and Matthias F. Stallmann, *On embedding binary trees into hypercubes*, Journal of Parallel and Distributed Computing **24** (1995), no. 2, 132–138.
9. Qi Cheng, Pingshan Li, and Min Xu, *Conditional (edge)-fault-tolerant strong Menger (edge) connectivity of folded hypercubes*, Theoretical Computer Science **728** (2018), 1–8.
10. Sheshayya A. Choudum and V. Sunitha, *Augmented cubes*, Networks **40** (2002), no. 2, 71–84.
11. Tomáš Dvořák, *Dense sets and embedding binary trees into hypercubes*, Discrete Applied Mathematics **155** (2007), no. 4, 506–514.
12. Jianxi Fan and Xiaohua Jia, *Embedding meshes into crossed cubes*, Information Sciences **177** (2007), no. 15, 3151–3160.

13. Jianxi Fan, Xiaohua Jia, and Xiaola Lin, *Complete path embeddings in crossed cubes*, Information Sciences **176** (2006), no. 22, 3332–3346.
14. Jywe-Fei Fang and Kuan-Chou Lai, *Embedding the incomplete hypercube in books*, Information Processing Letters **96** (2005), no. 1, 1–6.
15. Yuejuan Han, Jianxi Fan, Shukui Zhang, Jiwen Yang, and Peide Qian, *Embedding meshes into locally twisted cubes*, Information Sciences **180** (2010), no. 19, 3794–3805.
16. L. H. Harper, *Optimal assignments of numbers to vertices*, Journal of the Society for Industrial and Applied Mathematics **12** (1964), no. 1, 131–135.
17. Larry H. Harper, *Optimal numberings and isoperimetric problems on graphs*, Journal of Combinatorial Theory **1** (1966), no. 3, 385–393.
18. Marie-Claude Heydemann, J. C. Meyer, and Dominique Sotteau, *On forwarding indices of networks*, Discrete Applied Mathematics **23** (1989), no. 2, 103–123.
19. John H. Lindsey II, *Assignment of numbers to vertices*, The American Mathematical Monthly **71** (1964), no. 5, 508.
20. Wilfried Imrich, Sandi Klavžar, and Douglas F. Rall, *Topics in graph theory: graphs and their cartesian product*, CRC Press, 2008.
21. Che-Nan Kuo and Yu-Huei Cheng, *Cycles embedding in folded hypercubes with conditionally faulty vertices*, Discrete Applied Mathematics **220** (2017), 55–59.
22. Yung-Ling Lai and Kenneth Williams, *A survey of solved problems and applications on bandwidth, edgesum, and profile of graphs*, Journal of Graph Theory **31** (1999), no. 2, 75–94.
23. F. Thompson Leighton, *Introduction to parallel algorithms and architectures: arrays: trees: hypercubes*, Morgan Kaufmann Publishers, 1992.
24. Joseph Y.-T. Leung, Oliver Vornberger, and James D. Witthoff, *On some variants of the bandwidth minimization problem*, SIAM Journal on Computing **13** (1984), no. 3, 650–667.
25. Aixia Liu, Shiyang Wang, Jun Yuan, and Jing Li, *On g -extra conditional diagnosability of hypercubes and folded hypercubes*, Theoretical Computer Science **704** (2017), 62–73.
26. Paul Manuel, Indra Rajasingh, Bharati Rajan, and Helda Mercy, *Exact wirelength of hypercubes on a grid*, Discrete Applied Mathematics **157** (2009), no. 7, 1486–1495.
27. Paul Manuel, Indra Rajasingh, and R. Sundara Rajan, *Embedding variants of hypercubes with dilation 2*, Journal of Interconnection Networks **13** (2012), 1250004.
28. Akira Matsubayashi, *Separator-based graph embedding into multidimensional grids with small edge-congestion*, Discrete Applied Mathematics **185** (2015), 119–137.
29. Mirka Miller, R. Sundara Rajan, Natarajan Parthiban, and Indra Rajasingh, *Minimum linear arrangement of incomplete hypercubes*, The Computer Journal **58** (2014), no. 2, 331–337.
30. Jaroslav Opatrný and Dominique Sotteau, *Embeddings of complete binary trees into grids and extended grids with total vertex-congestion 1*, Discrete Applied Mathematics **98** (2000), no. 3, 237–254.
31. R. Sundara Rajan, T. M. Rajalaxmi, Jia-Bao Liu, and Guruswamy Sethuraman, *Wirelength of embedding complete multipartite graphs into certain graphs*, Discrete Applied Mathematics (2018), in press, doi:[10.1016/j.dam.2018.05.034](https://doi.org/10.1016/j.dam.2018.05.034).
32. Indra Rajasingh and Micheal Arockiaraj, *Linear wirelength of folded hypercubes*, Mathematics in Computer Science **5** (2011), no. 1, 101–111.
33. Indra Rajasingh, Bharati Rajan, and R. Sundara Rajan, *Embedding of hypercubes into necklace, windmill and snake graphs*, Information Processing Letters **112** (2012), no. 12, 509–515.
34. ———, *Embedding of special classes of circulant networks, hypercubes and generalized Petersen graphs*, International Journal of Computer Mathematics **89** (2012), no. 15, 1970–1978.
35. Parameswaran Ramanathan and Kang G. Shin, *Reliable broadcast in hypercube multicomputers*, IEEE Transactions on Computers **37** (1988), no. 12, 1654–1657.
36. Yousef Saad and Martin H. Schultz, *Topological properties of hypercubes*, IEEE Transactions on Computers **37** (1988), no. 7, 867–872.
37. Jadranka Skorin-Kapov and Asoo J. Vakharia, *Scheduling a flow-line manufacturing cell: a tabu search approach*, International Journal of Production Research **31** (1993), no. 7, 1721–1734.
38. Ladislav Stacho and Imrich Vrt'ko, *Bisection width of transposition graphs*, Discrete Applied Mathematics **84** (1998), no. 1-3, 221–235.
39. Harry Wiener, *Structural determination of paraffin boiling points*, Journal of the American Chemical Society **69** (1947), no. 1, 17–20.
40. Angela Y. Wu, *Embedding of tree networks into hypercubes*, Journal of Parallel and Distributed Computing **2** (1985), no. 3, 238–249.

41. Junming Xu, *Topological structure and analysis of interconnection networks*, Network theory and applications, vol. 7, Springer Science & Business Media, 2013.
42. Weihua Yang, Shuli Zhao, and Shurong Zhang, *Strong Menger connectivity with conditional faults of folded hypercubes*, Information Processing Letters **125** (2017), 30–34.