

# Edge-transitive lexicographic and Cartesian products

Wilfried Imrich <sup>a</sup>      Ali Iranmanesh <sup>b</sup>      Sandi Klavžar <sup>c,d,e</sup>  
Abolghasem Soltani <sup>a</sup>

<sup>a</sup> Montanuniversität Leoben, Leoben, Austria

<sup>b</sup> Department of Mathematics, University of Tarbiat Modares, Tehran, Iran

<sup>c</sup> Faculty of Mathematics and Physics, University of Ljubljana, Slovenia

<sup>d</sup> Faculty of Natural Sciences and Mathematics, University of Maribor, Slovenia

<sup>e</sup> Institute of Mathematics, Physics and Mechanics, Ljubljana, Slovenia

## Abstract

Let  $G \circ H$  and  $G \square H$  be the lexicographic product and the Cartesian product of graphs  $G$  and  $H$ , respectively. In this note connected, edge-transitive lexicographic and Cartesian product graphs are characterized. In particular, this fixes an error from [Appl. Math. Lett. 24 (2011) 1924–1926]. If  $G$  is connected and not complete, then  $G \circ H$  is edge-transitive if and only if  $G$  is edge-transitive and  $H$  is edgeless. If  $A = G \circ H$  is edge-transitive and  $G$  is complete, then there exists a complete graph  $K$  and an edgeless graph  $D$  such that  $A = K \circ D$ . Finally, a connected Cartesian product graph is edge-transitive if and only if it is the Cartesian power of a connected, edge-, and vertex-transitive graph.

**Keywords:** edge-transitive graph; vertex-transitive graph; lexicographic product of graphs; Cartesian product of graphs

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## 1 Introduction

A graph  $G = (V(G), E(G))$  is *vertex-transitive* (resp. *edge-transitive*) if the automorphism group  $\text{Aut}(G)$  acts transitively on  $V(G)$  (resp.  $E(G)$ ). A fine source on the fundamental properties of these graphs, their applications, and related topics, is the book [4], see also the survey [10] and recent papers [3, 11, 17].

While vertex-transitivity of graph products is well understood, cf. [6], it is rather surprising that not much can be found in the literature about their edge-transitivity. Moreover, the sources are not reliable. It is the recent paper [14] that motivated us to have a closer look at the edge-transitivity of lexicographic products (and of Cartesian products). It is namely claimed in [14] that the lexicographic product of edge-transitive graphs is edge-transitive as well. In the next section we will show that

this assertion is far from being true. On the positive side, we will characterize edge-transitive lexicographic products. This will be done in two steps, first for products whose first factors are not complete (Theorem 2.2) and then for products whose first factors are complete (Theorem 2.3). In Section 3, we follow with a characterization of connected, edge-transitive Cartesian product graphs, while in the rest of this section some notation is introduced.

If  $G$  is a graph, then its vertex-connectivity is denoted by  $\kappa(G)$ . Recall that  $\kappa(G) \leq \delta(G)$ , where  $\delta(G)$  is the minimum degree in  $G$ . We will denote the edgeless graph on  $m$  vertices with  $N_m$ . The *lexicographic product*  $G \circ H$  of graphs  $G$  and  $H$  is the graph with  $V(G \circ H) = V(G) \times V(H)$ , where  $(g, h)$  is adjacent to  $(g', h')$  if either  $g = g'$  and  $hh' \in E(H)$ , or  $gg' \in E(G)$ . The *Cartesian product*  $G \square H$  of graphs  $G$  and  $H$  is the graph with the vertex set  $V(G) \times V(H)$ , vertices  $(g, h)$  and  $(g', h')$  being adjacent if  $g = g'$  and  $hh' \in E(H)$ , or  $gg' \in E(G)$  and  $h = h'$ . If  $u = (g, h) \in V(G \square H)$ , then the subgraph of  $G \square H$  induced by the vertices of the form  $(g, x)$ ,  $x \in V(H)$ , is isomorphic to  $H$ ; it is denoted with  $H^u$  and called an *H-layer* (through  $u$ ). Analogously *G-layers* are defined and the same terminology applies to lexicographic products. Finally, we say that vertices  $x$  and  $y$  of a graph  $G$  are in *relation*  $R_G$  (or simply in *relation*  $R$  if the graph  $G$  is clear from the context), if they have the same open neighborhood, that is, if  $N_G(x) = N_G(y)$  holds. It is well-known (cf. [6, Exercise 8.4]) that  $R$  is an equivalence relation on  $V(G)$ ; its equivalence classes are then called *R-classes*.

## 2 Edge-transitive lexicographic products

The lexicographic product of graphs is also known as the composition of graphs as well as graph substitution, because  $G \circ H$  can be obtained from  $G$  by substituting a copy  $H_g$  of  $H$  for every vertex  $g$  of  $G$ , and then joining all vertices of  $H_g$  with all vertices of  $H_{g'}$  if  $gg' \in E(G)$ . This graph operation was of ongoing interest in the last several decades, the references [1, 5, 9] present just a very selective list of recent papers.

It is well-known that a lexicographic product  $G \circ H$  has transitive automorphism group if and only if  $G$  and  $H$  have transitive automorphism groups, see [6, Theorem 10.14]. In [14, Theorem 2.2] it is claimed that if  $G$  and  $H$  are edge-transitive graphs, then  $G \circ H$  is edge-transitive as well. To see that this need not be the case, consider the edge-transitive graphs  $K_2$  and  $P_3$  and their lexicographic product  $K_2 \circ P_3$ , which is clearly not edge-transitive. More generally, let  $G$  be an edge-transitive graph and  $H$  be an edge-transitive graph on at least two edges that is not vertex-transitive. Then, by the above,  $G \circ H$  is not vertex-transitive. Since an edge-transitive graph that is not vertex-transitive is necessarily bipartite (cf. [16, Proposition 2.2] or [4, Lemma 3.2.1]), it follows that  $G \circ H$  is not edge-transitive.

To characterize edge-transitive lexicographic products whose first factor is not complete, we will make use of the following result due to Watkins.

**Theorem 2.1** [15, Corollary 1A] *If  $G$  is a connected, edge-transitive graph, then  $\kappa(G) = \delta(G)$ .*

Our first main result now reads as follows.

**Theorem 2.2** *Let  $G$  be a connected graph that is not complete and  $H$  be a graph. Then  $G \circ H$  is edge-transitive if and only if  $G$  is edge-transitive and  $H$  is edgeless.*

**Proof.** Suppose first that  $G$  and  $H$  are as stated and that  $G \circ H$  is edge-transitive. From [6, Proposition 25.7] we know that, as  $G$  is not complete,  $\kappa(G \circ H) = \kappa(G) |V(H)|$ . Since  $\delta(G \circ H) = \delta(H) + \delta(G) |V(H)|$ , Theorem 2.1 implies that

$$\kappa(G) |V(H)| = \delta(H) + \delta(G) |V(H)|.$$

As  $\kappa(G) \leq \delta(G)$  holds, we infer that  $\delta(H) = 0$  (and that  $\kappa(G) = \delta(G)$ ). Hence  $H$  is edgeless.

It is easy to see that the  $R$ -classes of  $A = G \circ H$  are the products of the  $R$ -classes of  $G$  with  $H$ . Note that all  $R$ -classes are edgeless because we do not admit loops. The automorphisms of  $A$  preserve the  $R$ -classes, hence every automorphism of  $A$  induces one of  $A/R$ . Since  $\text{Aut}(A)$  is edge-transitive, this is also the case for  $A/R$ . Notice that  $A/R = A/R_{G \circ H}$  is the projection of  $(G/R_G) \circ H$  onto  $G/R_G$ , hence  $A/R_{G \circ H} \cong G/R_G$ . Therefore  $G/R_G$  is edge-transitive, and thus also  $G$ .

The converse is straightforward. □

The edge-transitive lexicographic products whose first factors are complete are described in the next theorem.

**Theorem 2.3** *If  $A$  is an edge-transitive lexicographic product  $G \circ H$  of a complete graph  $G$  by a graph  $H$ , then there exists a complete graph  $K$  and an edgeless graph  $D$  such that  $A = K \circ D$ .*

**Proof.** If  $G \cong K_n$  and if  $\text{Aut}(A) \neq \text{Aut}(G) \circ \text{Aut}(H)$ , then  $H$  can be uniquely decomposed into subgraphs  $B_1, \dots, B_\ell$  such that any permutation of the subgraphs  $B_i^v$  of  $A$ , where  $i$  is fixed and  $v$  runs through all vertices of  $A$ , is an automorphism of  $A$ . (The  $B_i$  are the complements of the connected components of the complement of  $H$ .) Clearly no edge within a  $B_i^v$  can be mapped into one between different  $H$ -layers, hence all  $B_i$  must be edgeless.

Suppose they do not all have the same size, say  $B_1$  and  $B_2$  have different sizes. Then any two vertices  $v \in V(B_1)$  and  $w \in V(B_2)$  have different degrees in  $H$ . For any two vertices  $x, y \in G$  the edge  $[(x, v), (y, v)]$  of  $G \circ H$  will have endpoints of degree  $d_H(v) + (n-1)|V(H)|$ , whereas the degrees of the endpoints of  $[(x, w), (y, w)]$  are  $d_H(w) + (n-1)|V(H)|$ . Since  $d_H(v) \neq d_H(w)$  these edges cannot be mapped into each other. Hence all  $B_i$  are edgeless graphs of the same size, say  $r$ . Then  $H = K_{|V(H)|/r} \circ D_r$  and  $A = K_n \circ K_{|V(H)|/r} \circ D_r = K \circ K_{|V(H)|/r}$ , where  $K = K_n \circ K_{|V(H)|/r}$ . □

Consider the following illustrative example to Theorem 2.3. It is easy to infer that the lexicographic product  $A_n = K_n \circ C_4$  is the so-called cocktail-party graph of order  $4n$ . Clearly,  $A_n$  is edge-transitive. Now,  $A_n$  can also be represented as  $A_n = K_{2n} \circ D_2$ , in accordance with Theorem 2.3.

A closely related result should be mentioned here. Recall that a graph is *super-connected* if every minimum vertex cut isolates a vertex. Then Meng [12] proved that a connected, vertex- and edge-transitive graph  $G$  is not super-connected if and only if  $G$  is isomorphic to  $C_n \circ N_m$ , ( $n \geq 6, m \geq 1$ ), or to  $L(Q_3) \circ N_m$  ( $m \geq 1$ ), where  $L(Q_3)$  is the line graph of the 3-cube.

### 3 Edge-transitive Cartesian products

It is well-known that a Cartesian product of connected graphs has transitive automorphism group if and only if every factor has transitive automorphism group, see [6, Proposition 6.16]. On the other hand, vertex- and edge-transitivity of the factors does not imply in general that their Cartesian product is edge-transitive. A simple example is  $K_3 \square K_2$ , and, more generally,  $K_n \square K_m$ , where  $n, m \geq 2$  and  $n \neq m$ . Indeed, since  $K_n$  and  $K_m$  are relatively prime, no automorphism of  $K_n \square K_m$  maps an edge of a  $K_n$ -layer onto an edge of a  $K_m$ -layer.

The main result of this section characterizes edge-transitive connected Cartesian products. In its proof we will use the description of the automorphisms of the Cartesian product of connected prime graphs from [7, 13], see [6, Theorem 6.10]. It asserts that the automorphism group of a connected, Cartesian product graph is generated by the automorphisms and the transpositions of the prime factors.

**Theorem 3.1** *A connected Cartesian product graph is edge-transitive if and only if it is the power of a connected, edge- and vertex-transitive graph.*

**Proof.** Suppose  $H_1 \square H_2 \square \dots \square H_k$  is the prime factorization of an edge-transitive graph  $G$ . Let  $e$  be an edge in an  $H_1$ -layer of  $G$  and  $f$  be an edge in an  $H_i$ -layer,  $1 \leq i \leq k$ . By edge-transitivity there is an automorphism  $\varphi$  that maps  $e$  into  $f$ . By the above description of the automorphisms of  $G$ ,  $\varphi$  maps the  $H_1$ -layer containing  $e$  into the  $H_i$ -layer containing  $f$ . Hence all factors are isomorphic. If  $f$  is in the same  $H_1$ -layer as  $e$ , then  $\varphi$  maps this  $H_1$ -layer, say  $H_1^v$ , into itself. Thus  $\varphi|_{H_1^v}$  is an automorphism of  $H_1^v$ . Since  $f$  was arbitrarily chosen, the action of the restriction of  $\varphi|_{H_1^v}$  on  $H_1^v$  is edge-transitive. As  $H_1 \cong H_1^v$  it is clear that  $G$  is the power of an edge-transitive graph  $H \cong H_1$ . If  $H$  is not vertex-transitive, then it is easily seen that  $G$  has at least three vertex orbits under the action of  $\text{Aut}(G)$ . However, an edge-transitive graph can have only one or two.

To complete the proof we have to show that every connected graph  $G$  is edge-transitive if it is a product of the form  $H_1 \square H_2 \square \dots \square H_k$ , where the factors are isomorphic copies of an edge- and vertex-transitive graph  $H$ .

For, given edges  $e \in H_i^v$  and  $f \in H_j^w$ , we first apply  $\pi_{i,j}$ , the automorphism that interchanges the  $i$ -th with the  $j$ -th coordinate of every vertex  $v \in V(G)$ .

Hence  $\pi_{i,j}(e) \in H_j^{\pi_{i,j}(v)}$ . Then we chose automorphisms  $\alpha_i \in \text{Aut}(H_i)$  for every  $i \in \{1, \dots, k\}$  such that  $\alpha_i(\pi_{i,j}(v)_i) = w_i$ . Setting  $\alpha = (\alpha_1, \dots, \alpha_k)$ , we thus have  $w = (\alpha\pi_{i,j})(v)$ , and  $\alpha\pi_{i,j}(e) \in H_j^w$ . Since  $H_j$  is edge-transitive, there clearly exists an automorphism  $\varphi$  of  $G$  that maps  $\alpha\pi_{i,j}(e)$  into  $f$ . Hence  $f = \varphi\alpha\pi_{i,j}(e)$ .  $\square$

There are graphs that are vertex- and edge-transitive, but, given any edge  $e$ , there is no automorphism that interchanges the endpoints of  $e$ . Such graphs are called *half-transitive*, cf. [2]. By Theorem 3.1 any connected half-transitive graph  $G$  is either prime or the Cartesian power of a prime, vertex- and edge-transitive graph  $H$ . It is easy to see that  $H$  must be also be half-transitive, and that any Cartesian power of a half-transitive graph is also half-transitive. We thus have the following corollary:

**Corollary 3.2** *A connected Cartesian product graph  $G$  is half-transitive if and only if it is the power of a connected, half-transitive graph.*

We conclude with the remark that so-called weak Cartesian products, that is, connected components of Cartesian products with infinitely many factors, can be vertex transitive, even if no factor is vertex transitive, see [8].

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