The annihilation number does not bound the 2-domination number from the above

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Abstract

The 2-domination number $\gamma_2(G)$ of a graph G is the minimum cardinality of a set $S \subseteq V(G)$ such that every vertex from $V(G) \setminus S$ is adjacent to at least two vertices in S. The annihilation number $a(G)$ is the largest integer k such that the sum of the first k terms of the non-decreasing degree sequence of G is at most the number of its edges. It was conjectured that $\gamma_2(G) \leq a(G) + 1$ holds for every connected graph G. The conjecture was earlier confirmed, in particular, for graphs of minimum degree 3, for trees, and for block graphs. In this paper, we disprove the conjecture by proving that the 2-domination number can be arbitrarily larger than the annihilation number. On the positive side we prove the conjectured bound for a large subclass of bipartite, connected cacti, thus generalizing a result of Jakovac from [Discrete Appl. Math. 260 (2019) 178–187].

Keywords: 2-dominating set; 2-domination number; annihilation number; cactus graph

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1 Introduction

Let $d_1 \leq \cdots \leq d_n$ be the degree ordering of a graph G. The *annihilation number* $a(G)$ is the largest integer k such that $\sum_{i=1}^{k} d_i \leq m(G)$. This concept was first defined in [17], see also [11] for an earlier, closely related concept called Havel-Hakimi process. The 2-domination number $\gamma_2(G)$ of a graph G is the minimum cardinality of a set $S \subseteq V(G)$ such that every vertex from $V(G) \setminus S$ is adjacent to at least two vertices in S. Now, the following conjecture relating these two concepts was posed.

Conjecture 1.1 ([7, 9]) *If G is a connected graph with at least 2 vertices, then $\gamma_2(G) \leq a(G) + 1$.*

If $\delta(G) \geq 3$, then Caro and Roddity [5, Corollary 2] deduced from their main result that $\gamma_2(G) \leq \lfloor \frac{n(G)}{2} \rfloor$. Hence, $\gamma_2(G) \leq a(G) + 1$ holds for any graph G with $\delta(G) \geq 3$. Desormeaux, Henning, Rall, and Yeo [9] followed with a confirmation of the conjecture for trees (see also [16] for another proof of it). Moreover, they have also characterized the trees that attain the equality in the conjecture. Very recently, Jakovac [14] proved the conjecture for block graphs. In addition he proved:

Proposition 1.2 ([14]) *If G is a bipartite cactus such that every edge of G belongs to a cycle, then $\gamma_2(G) \leq a(G) + 1$.*

In Section 2 we disprove Conjecture 1.1 by a subclass of connected cactus graphs with minimum degree 1. The construction further shows that the gap between the 2-domination number and the annihilation number can be arbitrarily large. Although the conjecture is wrong, it is still interesting to find classes of graphs which satisfy the conjecture. In Section 3 we prove several lemmas needed in the subsequent section. Then, in Section 4, we show that Conjecture 1.1 holds for bipartite connected cacti which (i) contain no sun at an outer cycle and (ii) the degree of the exit vertex of any outer 4-cycle is at least 4. (A sun at a cycle is obtained from the cycle by adding a pendant vertex to each of its vertices except one.) In this way we generalize Proposition 1.2. We conclude the paper with some open problems while in the rest of this section definitions and concepts needed are given.

1.1 Preliminaries

All graphs in this paper are undirected, finite and simple. We follow [1] for graph theoretical notation and terminology not defined here.

If $G = (V(G), E(G))$ is a graph, then let $n(G) = |V(G)|$ and $m(G) = |E(G)|$. A graph G is *nontrivial* if $n(G) \geq 2$. For $v \in V(G)$, the set of its neighbors is denoted by $N_G(v)$ and called the *neighborhood* of v, and the *closed neighborhood* $N_G[v]$ of v is $N(v) \cup \{v\}$. The *degree* of a vertex $v \in V(G)$ is denoted by $d_G(v)$. For a subset $S \subseteq V(G)$, we define $\sum(S, G) = \sum_{v \in S} d_G(v)$. In the above notation we may omit the index G provided that G is clear from the context. A vertex v of degree 1 is a *leaf* while its only neighbor is called a *support vertex*. If u has at least two neighbors which are leaves, then u is referred to as

a *strong support vertex*. The *minimum* and the *maximum degree* among the vertices of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. If $X \subseteq V(G)$, then $G - X$ denotes the graph obtained from G by deleting the vertices in X and all edges incident with them. Moreover, if $u_1 u_2 \in E(G)$ and $v_1 v_2 \notin E(G)$, notations $G - u_1 u_2$ and $G + v_1 v_2$ will be used for the graphs $(V(G), E(G) - \{u_1 u_2\})$ and $(V(G), E(G) \cup \{v_1 v_2\})$, respectively. These $-$ and $+$ notations will also be used for sets of edges. A connected graph is a *cactus* if its cycles are pairwise edge-disjoint.

A vertex $v \in V(G)$ *dominates* the vertices contained in $N[v]$. A set $S \subseteq V(G)$ is a *dominating set* if each vertex of G is dominated or equivalently, if $N[S] = V(G)$, where $N[S] = \Sigma_{v \in S} N[v]$ is the *closed neighborhood* of S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. For $k \geq 1$, a *k-dominating set* of a graph G is a set $S \subseteq V(G)$ such that each vertex from $V(G) \setminus S$ is adjacent to at least k vertices in S. There always exists at least one k-dominating set in G, since $V(G)$ is clearly a k-dominating set. The *k-domination number* $\gamma_k(G)$ of G is the minimum cardinality of a k-dominating set of G. Thus, a 1-dominating set is a usual dominating set and hence $\gamma_1(G) = \gamma(G)$. The notion of the k-dominating set was introduced by Fink and Jacobson [10], a survey on it up to 2012 can be found in [4]. It has been further investigated afterwards, [6, 15] are a couple of recent papers. In this paper we focus on the 2-domination number, cf. [3].

$S \subseteq V(G)$ is an *annihilation set* of G if $\sum_{v \in S} d(v) \leq m(G)$ and is an *optimal annihilation set* if $|S| = a(G)$. Obviously, any optimal annihilation set of a connected graph of order at least three contains all leaves of G. Assuming that S is an optimal annihilation set, we denote by $d^*(G)$ the minimum vertex degree over the set $V(G) \setminus S$.

2 Counterexample to Conjecture 1.1

Let $t \geq 4$ and $k_1, \ldots, k_t \geq 1$. Then the graph $G(t; k_1, \ldots, k_t)$ is obtained in the following way. First, take a disjoint union of cycles C_{3k_i+1}, $i \in [t]$, add an additional vertex w, and connect w with an arbitrary but fixed vertex in each of the cycles. Second, in the so far constructed graph, add a pendant vertex to each of the vertices of degree 2. In Fig. 1 the graph $G(4; 1, 1, 1, 1)$ is drawn.

Consider first the sporadic counterexamples as shown in Fig. 2. It is straightforward to verify that $m(G(4; 1, 2, 3, 4)) = 2(4 + 7 + 10 + 13) = 68$, $a(G(4; 1, 2, 3, 4)) = 3 + 6 + 9 + 12 + \lfloor \frac{38}{3} \rfloor = 42$ and $\gamma_2(G(4; 1, 2, 3, 4)) = 5 + (6 + 3) + (9 + 4) + (12 + 5) = 44$. Hence $\gamma_2(G(4; 1, 2, 3, 4)) - a(G(4; 1, 2, 3, 4)) = 44 - 42 = 2 > 1$. Similarly, in the second example we have $m(G(t; 1, \ldots, 1)) = 2(4 + \cdots + 4) = 8t$, $a(G(t; 1, \ldots, 1)) = 3t + \lfloor \frac{8t-3t}{3} \rfloor = 4t + \lfloor \frac{2t}{3} \rfloor$ and $\gamma_2(G(t; 1, \ldots, 1)) = 5t$. Therefore, $\gamma_2(G(t; 1, \ldots, 1)) - a(G(t; 1, \ldots, 1)) = 5t - (4t + \lfloor \frac{2t}{3} \rfloor) = \lceil \frac{t}{3} \rceil$.

The above examples generalize as follows.

Theorem 2.1 *Let c_0 be a given positive integer, $t \geq 4$, and $k_1, \ldots, k_t \geq 1$. If $t > 3c_0$, then*

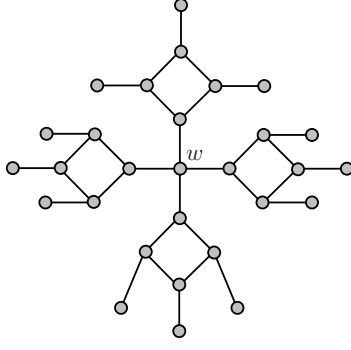$$\gamma_2(G(t; k_1, \ldots, k_t)) - a(G(t; k_1, \ldots, k_t)) > c_0 \,.$$

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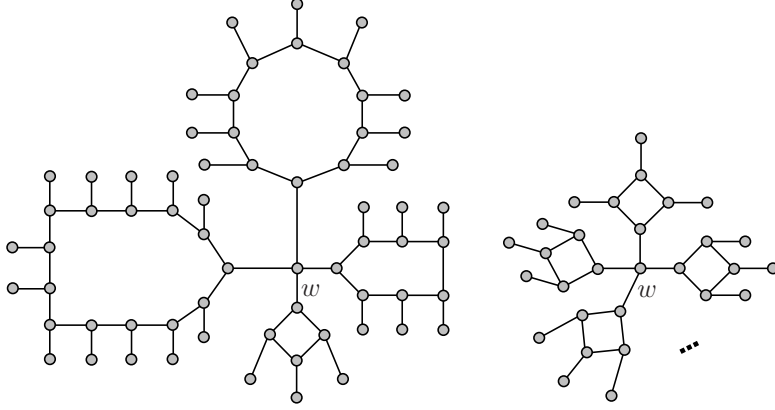Figure 1: The graph $G(4; 1, 1, 1, 1)$

Figure 2: Graphs $G(4; 1, 2, 3, 4)$ and $G(t; 1, \ldots, 1)$, $t \geq 4$

Proof. To shorten the presentation, set $G = G(t; k_1, \ldots, k_t)$ for the rest of the proof. Since to each of the constitutional cycles C_{3k_i+1} of G exactly $3k_i$ leaves are attached, as well as the edge to the vertex w of degree t, we get

$$m(G) = 2 \left(\sum_{i=1}^{t} (3k_i + 1) \right).$$

Hence, since each leaf of G is a member of every one of its optimal annihilation sets and all the other vertices of such a set are of degree 3, we get

$$a(G) = 4 \sum_{i=1}^{t} k_i + \left\lfloor \frac{2t}{3} \right\rfloor. \tag{1}$$

We now claim that $\gamma_2(G) = 4 \sum_{i=1}^{t} k_i + t$. Let X be a 2-dominating set of G with $|X| = \gamma_2(G)$.

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Then every leaf of G lies in X. Consider now a constitutional cycle $C = C_{3k_i+1}$ of G and suppose that $|X \cap V(C)| \leq k_i$. Then C contains three consecutive vertices neither of them lying in X. But then the middle of these three vertices, even if being adjacent to w, is not 2-dominated. If follows that $|X \cap V(C)| \geq k_i + 1$ for $i \in [t]$. Consequently, $\gamma_2(G) \geq 4 \sum_{i=1}^{t} k_i + t$. Since on the other hand it is easy to construct a 2-dominating set that has exactly $k_i + 1$ vertices on C, the claim is proved. Combining the claim with (1) we conclude that

$$\gamma_2(G) - a(G) = \left(4 \sum_{i=1}^{t} k_i + t \right) - \left(4 \sum_{i=1}^{t} k_i + \left\lfloor \frac{2t}{3} \right\rfloor \right) = \left\lceil \frac{t}{3} \right\rceil > c_0 .$$

$\square$

3 Some preliminary lemmas

In this section, we give some lemmas to be used in the next section. They give examples of how to obtain from a graph G a smaller graph G', such that $\gamma_2(G') \leq a(G') + 1$ implies $\gamma_2(G) \leq a(G) + 1$. First we recall [14, Lemma 4].

Lemma 3.1 *Assume that G is a graph on at least four vertices and $u \in V(G)$ a strong support vertex which is the common neighbor of pendant vertices $v_1, \ldots, v_\ell \in G$, $\ell \geq 2$. If $G' = G - \{u, v_1, \ldots, v_\ell\}$ is a connected graph, then $\gamma_2(G') \leq a(G') + 1$ implies $\gamma_2(G) \leq a(G) + 1$.*

We proceed with new lemmas for which we define a function f on a finite graph G with

$$f(G) = n(G) + 3m(G) + n_1(G) ,$$

where $n_1(G)$ denotes the number of leaves in G. Note that $f(G) \geq 7$ for every nontrivial, finite, connected graph G.

Lemma 3.2 *Let G be a connected graph with $n(G) \geq 3$ and which fulfils at least one of the following properties:*

(i) $d^*(G) \leq 2$;

(ii) G contains an induced path P_5 whose internal vertices are of degree 2;

(iii) G contains a pendant path P_4.

Then, there exists a nontrivial connected graph G' with $f(G') < f(G)$ such that $\gamma_2(G') \leq a(G') + 1$ implies $\gamma_2(G) \leq a(G) + 1$. Moreover, if G is a connected cactus graph, then G' can be chosen to be a connected cactus graph as well.

Proof. If G is a cycle and $e \in E(C)$, then set $G' = G - e$. If G is a tree and v its pendant vertex, then set $G' = G \backslash v$. Hence in the rest of the proof we may assume that G is neither a tree nor a cycle.

(i) Assume that $d^*(G) \leq 2$. Since G is neither a tree nor a cycle, there exists a cycle C in G and a vertex $v \in V(C)$ with $d(v) \geq 3$. Let $e = vu \in E(C)$ and $G' = G - e$. Then G' is connected, $f(G') < f(G)$ and $m(G') = m(G) - 1$. The deletion of an edge does not decrease the 2-domination number, so $\gamma_2(G) \leq \gamma_2(G')$. Consider an optimal annihilation set S' of G'. Then $\sum(S', G') \leq m(G') = m(G) - 1$. If $u, v \notin S'$, then $\sum(S', G) = \sum(S', G') \leq m(G') = m(G) - 1$; if S' contains exactly one of u and v, then $\sum(S', G) = \sum(S', G') + 1 \leq m(G)$. In either case $a(G) \geq |S'| = a(G')$ follows. In the third case, $u, v \in S'$ and $\sum(S', G) = \sum(S', G') + 2 \leq m(G) + 1$. Let $V_{1,2}$ denote the set of vertices which have degree 1 or 2 in G. Then $\sum(V_{1,2}, G) \geq m(G) + 1$ because $d^*(G) \leq 2$. Since $d(v) \geq 3$, we have $\sum(V_{1,2} \cup \{v\}, G) \geq m(G) + 4$, and then there is a vertex $v^* \in V_{1,2}$ which is not contained in S'. If v is replaced with v^* in S', then we get a new annihilation set S with $\sum(S, G) \leq \sum(S', G) - 1 \leq m(G)$. This proves $a(G) \geq |S'| = a(G')$ and then $\gamma_2(G) \leq \gamma_2(G') \leq a(G') + 1 \leq a(G) + 1$.

As we have just proved the statement under the assumption (i), we can assume that $d^*(G) \geq 3$ in the sequel of the proof.

(ii) Let $vu_1u_2u_3w$ be an induced path P_5 such that $d_G(u_1) = d_G(u_2) = d_G(u_3) = 2$. Set $G' = G - \{u_1, u_2, u_3\} + vw$. Observe that $n(G') = n(G) - 3$, $m(G') = m(G) - 3$, $n_1(G') = n_1(G)$ and hence $f(G') = f(G) - 12$. Let D' be a minimum 2-dominating set of G' and define D as follows:

(a) $D = D' \cup \{u_2\}$; if $v, w \in D'$,

(b) $D = D' \cup \{u_1, u_3\}$; otherwise.

In either case, D is a 2-dominating set in G. Hence, $\gamma_2(G) \leq \gamma_2(G') + 2$. Pick an optimal annihilation set S' of G'. Since $d_G(v) = d_{G'}(v)$ and $d_G(w) = d_{G'}(w)$ we have $\sum(S', G) = \sum(S', G') \leq m(G') = m(G) - 3$. Our assumption $d^*(G) \geq 3$ implies that every vertex v with $d(v) \leq 2$ is contained in every optimal annihilation set of G. Hence, either $S' \cup \{u_1, u_2, u_3\}$ is an optimal annihilation set of G and $a(G) \geq a(G') + 2$, or there is a vertex $v^* \in S'$ with $d(v^*) \geq 3$. In the latter case, consider $S = (S' - \{v^*\}) \cup \{u_1, u_2, u_3\}$, and observe that $\sum(S, G) \leq \sum(S', G) - 3 + 3 \cdot 2 \leq m(G)$. Therefore, $a(G) \geq |S| \geq |S'| + 2 = a(G') + 2$. Then $\gamma_2(G) \leq \gamma_2(G') + 2 \leq a(G') + 1 + 2 \leq a(G) + 1$. This proves the statement under the assumption (ii).

(iii) Let $u_1u_2u_3v$ be a pendant path P_4 of G such that $d_G(u_1) = 1$ and $d_G(u_2) = d_G(u_3) = 2$. Since G is connected, $G' = G - \{u_1, u_2, u_3\}$ is also connected, and we have $m(G') = m(G) - 3$ and $f(G') < f(G)$. Let D' be a minimum 2-dominating set of G'. Then $D = D' \cup \{u_1, u_3\}$ is 2-dominating set of G. Thus, $\gamma_2(G) \leq \gamma_2(G') + 2$. Next, we choose an optimal annihilation set S' in G'. Since we have already proved (ii), we may assume that $d_G(v) \geq 3$. Consider now the following two cases. If $v \notin S'$, then $\sum(S', G) = \sum(S', G')$, and $\sum(S', G) \leq m(G') = m(G) - 3$. Hence, $S = S' \cup \{u_1, u_2\}$ satisfies $\sum(S, G) = \sum(S', G) + 3 \leq m(G)$, and $a(G) \geq |S| = |S'| + 2 = a(G') + 2$. Then $\gamma_2(G) \leq \gamma_2(G') + 2 \leq a(G') + 1 + 2 \leq a(G) + 1$. In

the second case assume $v \in S'$. Then, $\sum(S', G) = \sum(S', G') + 1 \le m(G') + 1 = m(G) - 2$. We define $S = (S' \setminus \{v\}) \cup \{u_1, u_2, u_3\}$ and observe that $\sum(S, G) = \sum(S', G) - d(v) + 5 \le m(G) - 2 - 3 + 5 \le m(G)$. Hence, S is an annihilation set in G and we may conclude $a(G) \ge |S| = |S'| + 2 = a(G') + 2$. So $\gamma_2(G) \le \gamma_2(G') + 2 \le a(G') + 1 + 2 \le a(G) + 1$.

To complete the proof note that all the above transformations result in a connected cactus graph G', if G is of the same type. $\square$

Lemma 3.3 *Let w be a vertex of a nontrivial, connected graph H and let u be a vertex of a tree T with radius at least 3, where $V(H) \cap V(T) = \emptyset$. If G is the graph obtained from the H and T by identifying w and u, then there exists a connected graph G' with $f(G') < f(G)$ such that $\gamma_2(G') \le a(G') + 1$ implies $\gamma_2(G) \le a(G) + 1$.*

Proof. By Lemma 3.2(i) we may suppose throughout the proof that $d^*(G) \ge 3$. Let $v_1 \in V(T)$ be a vertex of T at the maximum distance from u. Since T has radius at least 3, we have $d_T(u, v_1) \ge 3$. Let v_1, v_2, v_3 be the first vertices on the shortest v_1, u-path in T (and also in G). Since $d_T(u, v_1) \ge 3$ we infer that $v_i \ne u$ for $i \in [3]$. In the rest we consider T to be rooted in u, also when T being a subgraph of G.

If $d(v_2) \ge 3$, then v_2 is a strong support vertex by the assumption on $d(u, v_1)$. Then this lemma holds by Lemma 3.1. Thus, we may suppose that v_2 is not a strong support vertex and has degree $d(v_2) = 2$. Let v_3 have $k \ge 0$ leaf-neighbors. If $k \ge 1$, then we denote these neighbors of v_3 with $u_1, \ldots, u_k$. We also denote with $w_1, \ldots, w_r$, $r \ge 1$, the children of the vertex v_3 which are not leaves (note that we have at least one such vertex, namely v_2). Each of these children must be a support vertex since $d(u, v_1)$ is the longest distance in T. If some of the vertices $w_1, \ldots, w_r$ is a strong support vertex, we can again settle this lemma with Lemma 3.1. Thus, we may suppose that $d(w_i) = 2$ for all $i \in [r]$. Let z_i the only child of w_i, $i \in [r]$, and note that v_1 is one of these children. It is also clear that $d_G(v) = k + r + 1$ since tree T is rooted in u. The structure of T (in G) as described above is shown in Fig. 3.
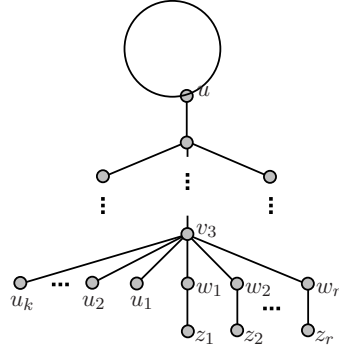

Figure 3: T rooted in vertex u with all of its descendants

Now, let $G' = G - \{u_1, \ldots, u_k, w_1, \ldots, w_r, z_1, \ldots, z_r\}$, and so $m(G') = m(G) - k - 2r$, and $d_{G'}(v_3) = 1$. Vertex v_3 must belong to any minimum 2-dominating set of G' since it is a leaf.

Hence, if D' is a minimum 2-dominating set of G', then $D = D' \cup \{u_1, \ldots, u_k, z_1, \ldots, z_r\}$ is a 2-dominating set of G, which implies $\gamma_2(G) \leq |D| = |D'| + k + r = \gamma_2(G') + k + r$.

Consider an optimal annihilation set S' of G'. If $v_3 \notin S'$, then $\Sigma(S', G) = \Sigma(S', G') \leq m(G) - k - 2r$ and let $S = S' \cup \{u_1, \ldots, u_k, z_1, \ldots, z_r\}$. Then, for $r \geq 1$, $\Sigma(S, G) = \Sigma(S', G') + d(u_1) + \cdots + d(u_k) + d(z_1) + \cdots + d(z_r) \leq m(G) - k - 2r + k + r = m(G) - r < m(G)$, and $a(G) \geq |S| = |S'| + k + r = a(G') + k + r$. On the other hand, if $v_3 \in S'$, then $\Sigma(S', G) = \Sigma(S', G') + k + r \leq m(G') + k + r = m(G) - r$ and let $S = (S' \backslash v_3) \cup \{w_1, u_1, \ldots, u_k, z_1, \ldots, z_r\}$. Since $d_G(v) = k + r + 1$, we have for $r \geq 1$, $\Sigma(S, G) = \Sigma(S', G') - d(v_3) + d(w_1) + d(u_1) + \cdots + d(u_k) + d(z_1) + \cdots + d(z_r) \leq m(G') - 1 + k + s \leq m(G) - s + 1 \leq m(G)$, and $a(G) \geq |S| = |S'| + k + r = a(G') + k + r$. Applying our inductive hypothesis to G', we can estimate as follows:

$$\gamma_2(G) \leq \gamma_2(G') + k + r \leq a(G') + k + r + 1 \leq a(G) + 1\,,$$

and we are done. $\qquad\square$

The *subdivided star* $S_s(K_{1,s+t})$, $s \geq 2$, $t \geq 0$, is the graph on $2s + t + 1$ vertices which is constructed by subdividing s edges of the star $K_{1,s+t}$ exactly once.

Lemma 3.4 *Let C be a cycle in a connected graph H and let w be a vertex of C of degree 2. If G is the graph obtained from H and $S_s(K_{1,s+t})$ by identifying w with the center of $S_s(K_{1,s+t})$, then there exists a nontrivial connected graph G' with $f(G') < f(G)$ such that $\gamma_2(G') \leq a(G') + 1$ implies $\gamma_2(G) \leq a(G) + 1$.*

Proof. Set $F = S_s(K_{1,s+t})$ and let u be the center of F. Let $uv_i'v_i$, $i \in [s]$, be the pendant paths attached to u, and let w_i, $i \in [t]$, be the leafs adjacent to u, so that $V(F) = \{u, v_1, \ldots, v_s, v_1', \ldots, v_s', w_1, \ldots, w_t\}$. If $G' = G - V(F)$, then G' is a connected graph with $m(G') = m(G) - 2s - t - 2$ and $f(G') < f(G)$. If D' is a minimum 2-dominating set of G', then $D = D' \cup \{u, w_1, \ldots, w_t, v_1, \ldots, v_s\}$ is a 2-dominating set of G. Thus, $\gamma_2(G) \leq |D| = |D'| + s + t + 1 = \gamma_2(G') + s + t + 1$. Next, let S' be an optimal annihilation set in G'. Then $\sum(S', G) \leq \sum(S', G') + 2 \leq m(G') + 2 = m(G) - 2s - t$. Consider now $S = S' \cup \{v_1', w_1, \ldots, w_t, v_1, \ldots, v_s\}$. Then $\sum(S, G) \leq \sum(S', G') + s + t + 2 \leq m(G) - 2s - t + s + t + 2 = m(G) - s + 2 \leq m(G)$. This proves $a(G) \geq |S| = |S'| + s + t + 1 = a(G') + s + t + 1$ and then $\gamma_2(G) \leq \gamma_2(G') + s + t + 1 \leq a(G') + 1 + s + t + 1 \leq a(G) + 1$. $\qquad\square$

4   A class of cacti for which Conjecture 1.1 holds

If H_1 and H_2 are subgraphs of a graph G, then the distance $d_G(H_1, H_2)$ between H_1 and H_2 is defined as $\min\{d_G(u, v) : u \in V(H_1), v \in V(H_2)\}$, where $d_G(u, v)$ is the standard distance between vertices u and v. Let C and C' be cycles of a cactus graph G. If $x \in V(C)$ and $x' \in V(C')$ are such that $d_G(x, x') = d_G(C, C')$, then we say that x and x' are *exit vertices* of cycles C and C', respectively. A cycle of G is said to be an *outer cycle* if it has at most one exit vertex. In the case that G is unicyclic, then we also declare its cycle to be outer. Hence,

if a cactus graph is not a tree, then it contains at least one outer cycle. We say that there is a *sun* at an outer cycle of a cactus if at all of its vertices, but at the exit vertex, there is exactly one pendant vertex attached. In Fig. 4 a cactus that contains two suns is drawn.
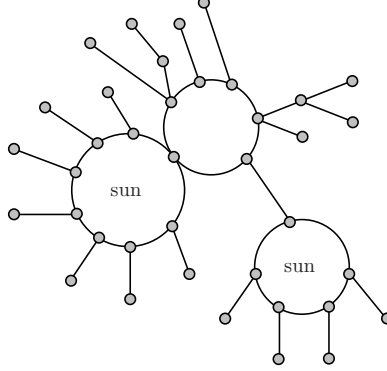

Figure 4: A cactus with two suns

With the above terminology in hands the main result of this section reads as follows.

Theorem 4.1 *Let G be a connected, bipartite cactus. If G contains no sun at an outer cycle, and the exit vertex of every outer 4-cycle is of degree at least 4, then $\gamma_2(G) \leq a(G) + 1$.*

Proof. We proceed by induction on the value of the function f defined in the previous section. For $f(G) = 7$ we have $G \cong K_2$, and $\gamma_2(K_2) = 2 = a(K_2) + 1$. For the inductive hypothesis, let $f(G) \geq 8$ and assume that for every nontrivial graph G' with $f(G') < f(G)$, we have $\gamma_2(G') \leq a(G') + 1$, where G' and G are connected, bipartite cactus graphs. If G is a tree, then the result follows. Also, if G is a cycle, then the statement is true. Thus, we may suppose that G contains at least one cycle as a proper subgraph. We denote with C_ℓ, where $\ell \geq 4$ is an even number, an outer cycle of G, and with x the exit vertex of C_ℓ.

In the rest of the proof we will consider subgraphs G' formed from G by removing a set of vertices or edges and adding edges in such a way that $f(G') < f(G)$ will hold and such that G' will fulfil the assumptions of the theorem. Also, throughout the proof, D' will denote a minimum 2-domination set of G', and S' an optimal annihilation set in G'. We are going to construct a 2-domination set $D = D' \cup D''$ and an annihilation set $S = S' \cup S''$ in G that will satisfy $|D''| = |S''| = s$, where s is a positive integer. Applying our inductive hypothesis to G', we will estimate that $\gamma_2(G') \leq a(G') + 1$ and consecutively $\gamma_2(G) \leq \gamma_2(G') + s \leq a(G') + s + 1 \leq a(G) + 1$. In this way the theorem will be proved.

Case 1: All vertices from $V(C_\ell) \setminus \{x\}$ have degree 2.
Let $C_\ell = x, v_1, \ldots, v_{\ell-1}, x$. Set $G' = G - \{v_2, \ldots, v_{\ell-2}\}$. Then $m(G') = m(G) - (\ell - 2)$. Since $d_{G'}(v_1) = d_{G'}(v_{\ell-1}) = 1$, both vertices v_1 and $v_{\ell-1}$ belong to D'. Set $D = D' \cup \{v_3, v_5, \ldots, v_{\ell-3}\}$. Then D is a 2-dominating set of G and hence $\gamma_2(G) \leq |D| = |D'| + \frac{\ell-4}{2} = \gamma_2(G') + \frac{\ell-4}{2}$. Since $d_{G'}(v_1) = d_{G'}(v_{\ell-1}) = 1$, then v_1 and v_ℓ are also both in S'. Set

$S = S' \cup \{v_3, v_5, \ldots, v_{\ell-3}\}$, Then $\sum(S, G) \leq \sum(S', G') + 2 + 2\frac{\ell-4}{2} \leq m(G') + 2 + 2\frac{\ell-4}{2} \leq m(G) - \ell + 2 + (\ell - 4) < m(G)$. It follows that $a(G) \geq a(G') + \frac{\ell-4}{2}$. So $\gamma_2(G) \leq \gamma_2(G') + 1 \leq a(G') + 1 + \frac{\ell-4}{2} \leq a(G) + 1$.

Case 2: $V(C_\ell) \setminus \{x\}$ contains a vertex of degree at least 3.

Since $V(C_\ell) \setminus \{x\}$ contains some vertices of degree at least 3, and C_ℓ is an outer cycle, there are trees attached to these vertices. We root each of these trees in the vertex of the tree that lies in $V(C_\ell)$. Amongst these trees select a tree T such that T has the largest height among the trees, where the height of T is $\max\{d(u, v) : u = V(C_\ell) \cap V(T), v \in V(T)\}$. Denote the height ot T with h, and let $u = V(T) \cap V(C_\ell)$.

Subcase 2.1: $h \geq 3$.

Since $h \geq 3$, there exists a leaf $v \in V(T)$ such that $d(u, v) = h \geq 3$. By Lemma 3.3 and our inductive hypothesis, the theorem holds.

Subcase 2.2: $h = 2$.

We consider Cases (a), (b), (c), (d), and (e) which are schematically presented in Fig. 2. All the other cases can be proved with the help of Lemma 3.1 and Lemma 3.2(ii).

Case (a): In this case, we have a subdivided star $S_s(K_{1,s+t})$, ($s \geq 2$ and $t \geq 0$), attached to the vertex u in C_ℓ. By Lemma 3.4 and our inductive hypothesis for $G' = G - V(S_s(K_{1,s+t}))$, the result holds.

In the following cases we will only consider subdivided stars with $s = 1$ and $t \geq 0$, that is, the subdivided star $S_1(K_{1,1+t})$. Set $V(S_1(K_{1,1+t})) = \{u, v_1, v_2, w_1, \ldots, w_t\}$, where u is the vertex of degree $t + 1$, $w_1, \ldots, w_t$ are leaves adjacent to u, and uv_1v_2 is the pendant path of length 2.

Case (b): In this case there are subdivided stars $S_1(K_{1,1+t_1})$ and $S_1(K_{1,1+t_2})$ with adjacent respective roots u and u' on C_ℓ. Let $V(S_1(K_{1,1+t_1})) = \{u, v_1, v_2, w_1, \ldots, w_{t_1}\}$ and $V(S_1(K_{1,1+t_2})) = \{u', v_1', v_2', w_1', \ldots, w_{t_2}'\}$. Set

$$G' = G - [V(S_1(K_{1,1+t_1})) \cup (V(S_1'(K_{1,1+t_2})) - \{u'\})] .$$

Then $m(G') = m(G) - (t_1 + t_2 + 6)$. Since $d_{G'}(u') = 1$, we have $u' \in D'$. Set further $D = D' \cup \{u, v_2, v_2', w_1, \ldots, w_{t_1}, w_1', \ldots, w_{t_2}'\}$. Since D is a 2-domination set of G we get $\gamma_2(G) \leq |D| = |D'| + t_1 + 2 + t_2 + 1 = \gamma_2(G') + t_1 + t_2 + 3$. Let y be the neighbor of u on C_ℓ different from u'. (Note that y may be x.). We now consider four subcases with respect to whether y and u' belong to S'.

If $y \notin S'$ and $u' \notin S'$, we have $\sum(S', G) = \sum(S', G') \leq m(G') = m(G) - (t_1 + t_2 + 6)$. Let $S = S' \cup \{w_1, \ldots, w_{t_1}, w_1', \ldots, w_{t_2}', v_1, v_2, v_2'\}$. Then $\sum(S, G) = \sum(S', G) + d(w_1) + \cdots + d(w_{t_1}) + d(w_1') + \cdots + d(w_{t_2}') + d(v_1) + d(v_2) + d(v_2') \leq m(G) - (t_1 + t_2 + 6) + t_1 + t_2 + 2 + 1 + 1 = m(G) - 2 < m(G)$, and we have $a(G) \geq |S'| = |S'| + t_1 + t_2 + 3 = a(G') + t_1 + t_2 + 3$.

If $y \in S'$ and $u' \notin S'$, we have $\sum(S', G) = \sum(S', G') + 1 \leq m(G') + 1 = m(G) - (t_1 + t_2 + 5)$. Let $S = S' \cup \{w_1, \ldots, w_{t_1}, w_1', \ldots, w_{t_2}', v_1, v_2, v_2'\}$. Then $\sum(S, G) = \sum(S', G) + d(w_1) + \cdots + d(w_{t_1}) + d(w_1') + \cdots + d(w_{t_2}') + d(v_1) + d(v_2) + d(v_2') \leq m(G) - (t_1 + t_2 + 5) + t_1 + t_2 + 2 + 1 + 1 = m(G) - 1 < m(G)$, and we have $a(G) \geq |S'| = |S'| + t_1 + t_2 + 3 = a(G') + t_1 + t_2 + 3$.

If $y \notin S'$ and $u' \in S'$, we have $\sum(S', G) = \sum(S', G') + 2 + t_2 \leq m(G') + 2 + t_2 = m(G) - t_1 - 4$. Let $S = (S' - \{u'\}) \cup \{w_1, \ldots, w_{t_1}, w_1', \ldots, w_{t_2}', v_1, v_2, v_2', v_1'\}$. Then $\sum(S, G) =$
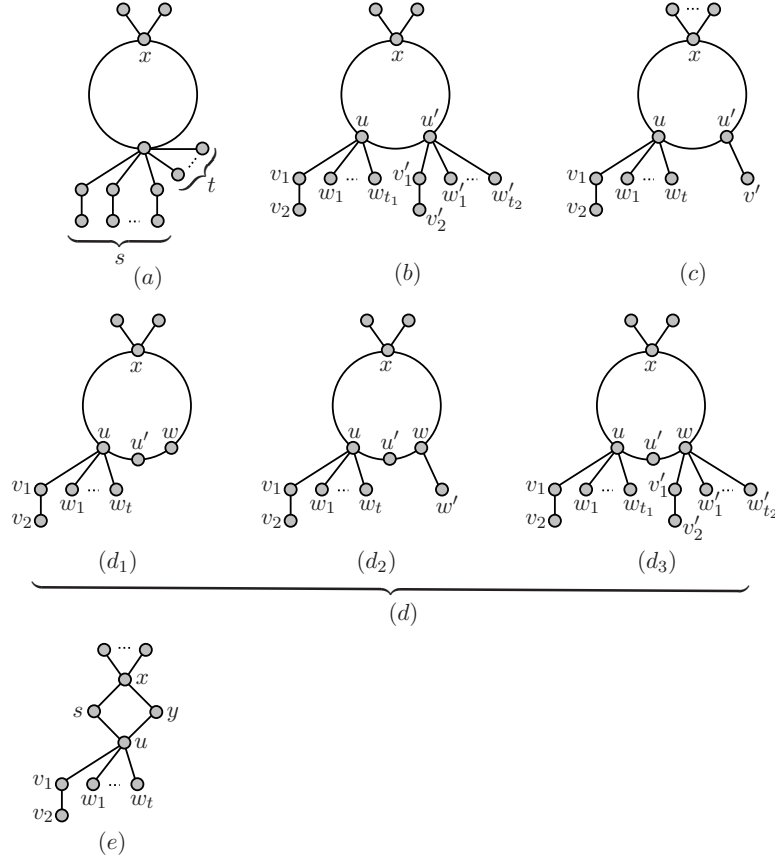
Figure 5: The subcases (a), (b), (c), (d), and (e) for the case $h = 2$ from the proof

$\sum(S', G) - d(u') + d(w_1) + \cdots + d(w_{t_1}) + d(w'_1) + \cdots + d(w'_{t_2}) + d(v_1) + d(v_2) + d(v'_2) + d(v'_1) \leq m(G) - t_1 - 4 - t_2 - 3 + t_1 + t_2 + 2 + 1 + 1 + 2 = m(G) - 1 < m(G)$, and we have $a(G) \geq |S| = |S'| + t_1 + t_2 + 3 = a(G') + t_1 + t_2 + 3$.

If $y \in S'$ and $u' \in S'$, we have $\sum(S', G) = \sum(S', G') + 3 + t_2 \leq m(G') + 3 + t_2 = m(G) - t_1 - 3$. Let $S = (S' - \{u'\}) \cup \{w_1, \ldots, w_{t_1}, w'_1, \ldots, w'_{t_2}, v_1, v_2, v'_2, v'_1\}$. Then $\sum(S, G) = \sum(S', G) - d(u') + d(w_1) + \cdots + d(w_{t_1}) + d(w'_1) + \cdots + d(w'_{t_2}) + d(v_1) + d(v_2) + d(v'_2) + d(v'_1) \leq m(G) - t_1 - 3 - t_2 - 3 + t_1 + t_2 + 2 + 1 + 1 + 2 = m(G)$, and we have $a(G) \geq |S'| = |S'| + t_1 + t_2 + 3 = a(G') + t_1 + t_2 + 3$.

Case (c): In this case there exists a subdivided star $S_1(K_{1,1+t})$ whose vertex u on C_ℓ has a neighbor u' on C_ℓ with an attended pendant vertex v'.

If $\ell \geq 6$, then let $G' = G - (V(S_1(K_{1,1+t})) \cup \{u', v'\})$. Then $m(G') = m(G) - (t + 6)$. Setting $D = D' \cup \{u, v_2, v', w_1, \ldots, w_t\}$ we have $\gamma_2(G) \leq |D| = |D'| + t + 3 = \gamma_2(G') + (t + 3)$. Independently of whether the neighbors of u and u' in G' are inside S' or not, we have $\Sigma(S', G) \leq \Sigma(S', G') + 2 \leq m(G') + 2 = m(G) - (t + 4)$. Let $S = S' \cup \{v_1, v_2, v', w_1, \ldots, w_t\}$. Then $\Sigma(S, G) = \Sigma(S', G) + (1 + 1 + 2 + t) \leq m(G) - (t + 4) + (t + 4) = m(G)$, and we have $a(G) \geq |S| = |S'| + (t + 3) = a(G') + (t + 3)$.

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Suppose now that $\ell = 4$ and let $C_\ell = x, u, u', y$. If $d(y) \geq 3$, then we can proceed as in the above case $\ell \geq 6$. Suppose next that $d(y) = 2$. Setting $G' = G - (V(C_\ell) \cup V(S_1(K_{1,1+t})))$ we have $m(G') = m(G) - (t + 7)$. Let $D = D' \cup \{u, v_2, y, v', w_1, \ldots, w_t\}$, and hence $\gamma_2(G) \leq |D| = |D'| + t + 4 = \gamma_2(G') + (t + 4)$. If $x \notin S'$, then set $S = S' \cup \{v_1, v_2, y, v', w_1, \ldots, w_t\}$. Then $\Sigma(S, G) = \Sigma(S', G) + (2 + 3 + 1 + t) \leq m(G) - (t + 7) + (t + 6) < m(G)$. If $x \in S'$, then set $S = (S' \backslash x) \cup \{u'\} \cup \{v_1, v_2, y, v', w_1, \ldots, w_t\}$. Then $\Sigma(S, G) = \Sigma(S', G) + 2 - d(x) + 3 + (2 + 3 + 1 + t) \leq m(G) - (t + 7) - 1 + (t + 6) = m(G)$. So we have $a(G) \geq |S| = |S'| + (t + 4) = a(G') + (t + 4)$.

Case (d): In this case we have a subdivided star $S_1(K_{1,1+t})$ such that its vertex u on C_ℓ, has a neighbor u' on C_ℓ of degree 2. We consider thres subcases.

Case (d1): In this subcase u' has another neighbor $w \in V(C_\ell)$ such that $d(w) = 2$. If $\ell \geq 6$, select $G' = G - (V(S_1(K_{1,1+t})) \cup \{u'\})$. Then $m(G') = m(G) - (t + 5)$. Since $d_{G'}(w) = 1$, we much have $w \in D'$. Let $D = D' \cup \{u, v_2, w_1, \ldots, w_t\}$. Since D is a 2-dominating set of G, we get $\gamma_2(G) \leq |D| = |D'| + (t + 2) = \gamma_t(G') + (t + 2)$. Independently of whether the neighbors of u and u' in G' are inside S' or not, we have $\Sigma(S', G) \leq \Sigma(S', G') + 2 \leq m(G') + 2 = m(G) - (t + 3)$. Let $S = S' \cup \{v_1, v_2, w_1, \ldots, w_t\}$. Then $\Sigma(S, G) \leq \Sigma(S', G) + (t + 3) \leq m(G) - (t + 3) + (t + 3) \leq m(G)$, and we have $a(G) \geq |S| = |S'| + (t + 2) = a(G') + (t + 2)$.

Consider now the case that $\ell = 4$. Setting $G' = G - (V(S_1(K_{1,1+t})) \cup \{u', w\})$ we have $m(G') = m(G) - (t + 6)$. Let $D = D' \cup \{u, v_2, w, w_1, \ldots, w_t\}$, and hence $\gamma_2(G) \leq |D| = |D'| + t + 3 = \gamma_2(G') + (t + 3)$. If $x \notin S'$, then set $S = S' \cup \{v_1, v_2, w, w_1, \ldots, w_t\}$. Then $\Sigma(S, G) = \Sigma(S', G) + (2 + 2 + 1 + t) \leq m(G) - (t + 6) + (t + 5) < m(G)$. If $x \in S'$, then set $S = (S' \backslash x) \cup \{u'\} \cup \{v_1, v_2, w, w_1, \ldots, w_t\}$. Then $\Sigma(S, G) \leq \Sigma(S', G) + 2 - d(x) + 2 + (2 + 2 + 1 + t) \leq m(G) - (t + 6) + (t + 5) < m(G)$. So we have $a(G) \geq |S| = |S'| + (t + 3) = a(G') + (t + 3)$.

Case (d2): Suppose new that the other neighbor w of u' has a pendant vertex w'. If $\ell \geq 6$, then let $G' = G - (V(S_1(K_{1,1+t})) \cup \{u', w'\})$. Then $m(G') = m(G) - (t + 6)$. Since $d_{G'}(w) = 1$, we have $w \in D'$. Set $D = D' \cup \{u, v_2, w', w_1, \ldots, w_t\}$. Then D is a 2-dominating set of G and therefore $\gamma_2(G) \leq |D| = |D'| + (t + 3) = \gamma_t(G') + (t + 3)$. If $w \notin S'$, then we have $\Sigma(S', G) \leq \Sigma(S', G') + 1 \leq m(G) - (t + 5)$. Set next $S = S' \cup \{v_1, v_2, w', w_1, \ldots, w_t\}$. Then $\Sigma(S, G) = \Sigma(S', G) + (1 + 2 + 1 + t) \leq m(G) - (t + 5) + (t + 4) < m(G)$, and we have $a(G) \geq |S| = |S'| + (t + 3) = a(G') + (t + 3)$. If $w \in S'$, we have $\Sigma(S', G) \leq \Sigma(S', G') + 3 \leq m(G') + 3 = m(G) - 3$. Setting $S = (S' - \{w\}) \cup \{u', v_1, v_2, w', w_1, \ldots, w_t\}$, we have $\Sigma(S, G) = \Sigma(S', G) - 3 + (2 + 2 + 1 + 1 + t) \leq m(G) - (t + 3) + (t + 3) = m(G)$, and we have $a(G) \geq |S| = |S'| + (t + 3) = a(G') + (t + 3)$.

Let now $\ell = 4$. Setting $G' = G - (V(S_1(K_{1,1+t}) \cup \{u', w, w'\})$ we have $m(G') = m(G) - (t + 7)$. Let $D = D' \cup \{u, v_2, w, w', w_1, \ldots, w_t\}$, and hence $\gamma_2(G) \leq |D| = |D'| + t + 4 = \gamma_2(G') + (t + 4)$. If $x \notin S'$, then set $S = S' \cup \{u, v_2, u', w', w_1, \ldots, w_t\}$ and $\Sigma(S, G) = \Sigma(S', G) + (2 + 2 + 1 + 1 + t) \leq m(G) - (t + 7) + (t + 6) < m(G)$. If $x \in S'$, then set $S = (S' \backslash x) \cup \{w\} \cup \{u, v_2, u', w', w_1, \ldots, w_t\}$ and therefore $\Sigma(S, G) = \Sigma(S', G) + 2 - d(x) + 3 + (2 + 2 + 1 + 1 + t) \leq m(G) - (t + 7) + 1 + (t + 6) = m(G)$. So we have $a(G) \geq |S| = |S'| + (t + 4) = a(G') + (t + 4)$.

Case (d3): Suppose now that at the other neighbor w of u' we have another subdivided star $S_1(K_{1,1+t_2})$. Set $G' = G - (V(S_1(K_{1,1+t_1}) \cup V(S_1(K_{1,1+t_2}) \cup \{u'\})$ and hence $m(G') = m(G) - (t_1 + t_2 + 8)$. Setting $D = D' \cup \{u, w, v_2, v'_2, w_1, \ldots, w_{t_1}, w'_1, \ldots, w'_{t_2}\}$ we get $\gamma_2(G) \leq |D| = |D'| + (t_1 + t_2 + 4) = \gamma_2(G') + (t_1 + t_2 + 4)$. Independently of whether the neighbors of

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u and w in G' are inside S' or not, we have $\Sigma(S', G) \leq \Sigma(S', G') + 2 \leq m(G) - (t_1 + t_2 + 6)$. Set further $S = S' \cup \{v_1, v_2, v'_1, v'_2, w_1, \ldots, w_{t_1}, w'_1, \ldots, w'_{t_2}\}$, so that $\Sigma(S, G) = \Sigma(S', G) + (1 + 2 + 1 + 2 + t_1 + t_2) \leq m(G) - (t_1 + t_2 + 6) + (t_1 + t_2 + 6) = m(G)$, and $a(G) \geq |S| = |S'| + (t_1 + t_2 + 4) = a(G') + (t_1 + t_2 + 4)$.

Case (e): Let $\ell = 4$ and let $C_4 = x, a, u, y, x$, where u is attended with a subdivided star $S(K_{1,1,t})$. Setting $G' = G - V(S_1(K_{1,1+t}))$ we have $m(G') = m(G) - (t + 4)$. Since $d'_G(s) = d'_G(y) = 1$, the vertices s and y belong to D'. Let $D = \{D' \setminus \{a, y\}\} \cup \{x, u, v_2, w_1, \ldots, w_t\}$. Then $\gamma_2(G) \leq |D| = |D'| + (t + 1) = \gamma_2(G') + (t + 1)$. Set further $S = S' \cup \{v_2, w_1, \ldots, w_t\}$. Then $\Sigma(S, G) = \Sigma(S', G) + 2 + (1 + t) \leq m(G) - (t + 4) + (t + 3) < m(G)$, and $a(G) \geq |S| = |S'| + (t + 1) = a(G') + (t + 1)$.

Subcase 2.3: $h = 1$.

We need to consider only one case which is shown in Fig. 6, because, as we have already seen in Subcase 2.2, all the other cases for $h = 1$ either have a sun at an outer cycle or can be proved with the help of Lemma 3.1.
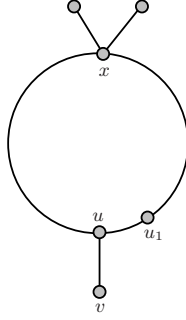

Figure 6: The situation to be considered in the subcase $h = 1$

Assume that $d(u) = 3$, and at least one of its neighbors in $V(C_\ell) \setminus \{x\}$, is of degree 2, denote it with u_1. Denote the child of u with v. Setting $G' = G - \{u, v\}$ we have $m(G') = m(G) - 3$. Then $u_1 \in D'$, since it is leaf in G'. Setting $D = D' \cup \{v\}$ we get $\gamma_2(G) \leq |D| = |D'| + 1 = \gamma_2(G') + 1$. Independently of whether the neighbors of u in G' are inside S' or not, we have $\sum(S', G) \leq \sum(S', G') + 2 \leq m(G') + 2 = m(G) - 1$. Let $S = S' \cup \{v\}$. Then $\sum(S, G) = \sum(S', G) + d(v) \leq m(G) - 1 + 1 = m(G)$, and we have $a(G) \geq |S'| = |S'| + 1 = a(G') + 1$. $\qquad \square$

5 Concluding remarks

Based on the results of this paper, the following problem is very natural.

Problem 5.1 Characterize the cactus graphs for which Conjecture 1.1 holds true.

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Note that the class of cacti in question does not contain bipartite cacti as a subclass since some of the counterexamples from Section 2 are bipartite. More generally, it would be interesting to know the answer to the following:

Problem 5.2 Characterize the graphs for which Conjecture 1.1 holds true.

As we already mentioned, in [9] trees were characterized for which the equality in Conjecture 1.1 holds. Hence we pose:

Problem 5.3 Characterize the cactus graphs for which the equality in Conjecture 1.1 holds. More generally, characterize the graphs with the same property.

Let $\gamma_t(G)$ denote the total domination number of a graph G. (For an extensive information on γ_t see the book [13].) In [7, 9] a parallel conjecture to Conjecture 1.1 was posed for the total domination number, that is, it was conjectured that

$$\gamma_t(G) \leq a(G) + 1 \tag{2}$$

holds for every nontrivial connected graph G. This conjecture holds for graphs of minimum degree at least 3, and has been verified for trees [8] and for cactus graphs and block graphs [2]. The counterexamples to Conjecture 1.1 presented in this paper are far from being counterexamples for (2) since their total domination number is significantly smaller and, after all, the counterexamples to Conjecture 1.1 are cactus graphs for which (2) holds. Hence we are inclined to believe that (2) holds true.

Acknowledgments

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