The general position number of integer lattices

Sandi Klavžar [a,b,c] Gregor Rus [c,d]

[a] Faculty of Mathematics and Physics, University of Ljubljana, Slovenia

[b] Faculty of Natural Sciences and Mathematics, University of Maribor, Slovenia

[c] Institute of Mathematics, Physics and Mechanics, Ljubljana, Slovenia

[d] Faculty of Organizational Sciences, University of Maribor, Slovenia

Abstract

Given a connected graph G, the general position number $\mathrm{gp}(G)$ of G is the cardinality of a largest set S of vertices such that no three pairwise distinct vertices from S lie on a common geodesic. The n-dimensional grid graph P_∞^n is the Cartesian product of n copies of the two-way infinite path P_∞. It is proved that if $n \in \mathbb{N}$, then $\mathrm{gp}(P_\infty^n) = 2^{2^{n-1}}$. The result was earlier known only for $n \in \{1, 2\}$ and partially for $n = 3$.

E-mails: sandi.klavzar@fmf.uni-lj.si, gregor.rus4@um.si

Key words: general position problem; Cartesian product of graphs; integer lattice; Erdős-Szekeres theorem

AMS Subj. Class.: 05C12, 05C76, 11B75

1 Introduction and preliminaries

Let G be a connected graph. A set $S \subseteq V(G)$ is a *general position set* if $d_G(u,v) \neq d_G(u,w) + d_G(w,v)$ holds for every $\{u,v,w\} \in \binom{S}{3}$, where $d_G(x,y)$ denotes the shortest-path distance between x and y in G. The *general position number* $\mathrm{gp}(G)$ of G is the cardinality of a largest general position set in G. This concept and terminology was introduced in [10], in part motivated by the century old Dudeney's No-three-in-line problem [3]. A couple of years earlier and in different terminology, the problem was also considered in [13]. Moreover, in the special case of hypercubes, the general position problem has been studied back in 1995 by Körner [9]. Following these seminal papers, the general position problem has been studied from different perspectives in several subsequent papers [1, 5, 7, 8, 11, 12].

The *Cartesian product* $G \,\square\, H$ of graphs G and H is the graph with the vertex set $V(G) \times V(H)$, vertices (g,h) and (g',h') being adjacent if either $g = g'$ and $hh' \in E(H)$, or $h = h'$ and $gg' \in E(G)$. The Cartesian product $G_1 \,\square\, \cdots \,\square\, G_n$, where each factor G_i is isomorphic to G, will be shortly denoted by G^n. If P_∞ denotes the two-way infinite path, then one of the main results from [11] asserts that ${\rm gp}(P_\infty^2) = 4$. In the same paper it was also proved that $10 \le {\rm gp}(P_\infty^3) \le 16$. The lower bound 10 was improved to 14 in [8]. In this note we round these investigations by proving the following result.

Theorem 1 *If $n \in \mathbb{N}$, then ${\rm gp}(P_\infty^n) = 2^{2^{n-1}}$.*

In the rest of the section we list some further notation and and preliminary results. In the next section we prove the theorem. In the concluding section we give a couple of consequences of the theorem and pose an open problem.

For a positive integer k we will use the notation $[k] = \{1,\ldots,k\}$. Throughout we will set $V(P_\infty) = \mathbb{Z}$, where $u, v \in V(P_\infty)$ are adjacent if and only if $|u - v| = 1$. With this convention we have $V(P_\infty^n) = \mathbb{Z}^n$. If $u \in V(P_\infty^n)$, then for the coordinates of u we will use the notation $u = (u_1,\ldots,u_n)$. If a vertex from $V(P_\infty^n)$ will be indexed, say $u_i \in V(P_\infty^n)$, then this notation will be extended as $u_i = (u_{i,1},\ldots,u_{i,n})$. From the Distance Lemma [6, Lemma 12.2] it follows that

$$d_{P_\infty^n}(u,v) = \sum_{i=1}^{n} |u_i - v_i| \,. \tag{1}$$

From here it is not difficult to deduce that a vertex $w \in V(P_\infty^n)$ lies on a shortest u,v-path in P_∞^n if and only if $\min\{u_i, v_i\} \le w_i \le \max\{u_i, v_i\}$ holds for every $i \in [n]$.

A sequence of real numbers is *monotone* if it is monotonically increasing or monotonically decreasing. The celebrated Erdős-Szekeres result on monotone sequences read as follows (cf. also [2, Theorem 1.1]).

Theorem 2 [4] *For every $n \ge 2$, every sequence $(a_1,\ldots,a_N)$ of real numbers with $N \ge (n-1)^2 + 1$ elements contains a monotone subsequence of length n.*

2 Proof of Theorem 1

Theorem is obviously true for $n = 1$ and was proved for $n = 2$ in [11, Corollary 3.2].

Let now $n \ge 3$ and let $U^{(1)} = \{u_1,\ldots,u_{2^{2^{n-1}}+1}\}$ be a set of vertices of P_∞^n of cardinality $2^{2^{n-1}} + 1$. We may without loss of generality assume that the first coordinates of the vertices from $U^{(1)}$ are ordered, that is, $u_{1,1} \le u_{2,1} \le$

$\cdots \leq u_{2^{2^{n-1}}+1,1}$. By Theorem 2, there exists a subset $U^{(2)}$ of $U^{(1)}$ of cardinality $2^{2^{n-2}}+1$, such that the second coordinates of the vertices from $U^{(2)}$ form a monotone (sub)sequence. Inductively applying this argument we arrive at a set $U^{(n)} \subset U^{(n-1)}$ of cardinality $2^{2^{n-n}}+1=3$, in which the n^{th} coordinates of the three vertices form a monotone (sub)sequence. As $U^{(n)} \subset U^{(n-1)} \subset \cdots \subset U^{(1)}$, the induction argument yields that for every $i \in [n-1]$, the i^{th} coordinates of the vertices from $U^{(n)}$ likewise form a monotone (sub)sequence. if $U^{(n)} = \{u, v, w\}$, where $u_1 \leq v_1 \leq w_1$, this implies (having (1) in mind) that v lies on a shortest u,w-path. We conclude that $\text{gp}(P_\infty^n) \leq 2^{2^{n-1}}$.

To prove the other inequality we are going to inductively construct a general position set $X^{(n)} = \{x_1^{(n)}, \ldots, x_{2^{2^{n-1}}}^{(n)}\}$ for $n \geq 2$ as follows. Set $X^{(2)} = \{(1,2),(2,1),(3,4),(4,3)\}$, where $x_1^{(2)} = (1,2)$, $x_2^{(2)} = (2,1)$, $x_3^{(2)} = (3,4)$, and $x_4^{(2)} = (4,3)$. Suppose now that $X^{(n-1)}$ is defined for some $n \geq 3$, and construct $X^{(n)}$ as follows. Set first

- $x_{i,1}^{(n)} = i$, $i \in [2^{2^{n-1}}]$.

Next, write each $i \in [2^{2^{n-1}}]$ as $i = p \cdot 2^{2^{n-2}} + r$, where $0 \leq p < 2^{2^{n-2}}$ and $r \in [2^{2^{n-2}}]$, and set

- $x_{i,j}^{(n)} = \left(x_{(p+1),j}^{(n-1)} - 1\right) \cdot 2^{2^{n-2}} + x_{r,j}^{(n-1)}$ for each $j \in \{2, \ldots, n-1\}$, and
- $x_{i,n}^{(n)} = x_{(p+1)2^{2^{n-2}}-r+1,n-1}^{(n)}$.

Roughly speaking, for the j^{th} coordinate, where $j \in \{2, \ldots, n-1\}$, we partition the sequence $(x_{i,j}^{(n)})_{i=1}^{2^{2^{n-1}}}$ into $2^{2^{n-2}}$ blocks each of $2^{2^{n-2}}$ values and sort the blocks as well as the values inside the blocks according to the values $(x_{i,j}^{(n-1)})_{i=1}^{2^{2^{n-2}}}$. The values of the n^{th} coordinate is then obtained from the values of the $(n-1)^{\text{th}}$ coordinate by reversion the sequence in each of the $2^{2^{n-2}}$ blocks, while keeping the sequence of the blocks. For example, the coordinates of the vertices from $X^{(3)}$ are shown in Table 1.

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$x_{i,1}^{(3)}$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$x_{i,2}^{(3)}$	6	5	8	7	2	1	4	3	14	13	16	15	10	9	12	11
$x_{i,3}^{(3)}$	7	8	5	6	3	4	1	2	15	16	13	14	11	12	9	10

Table 1: Coordinates of the vertices from $X^{(3)}$.

To complete the proof it suffices to show that for each $n \geq 2$, the set $X^{(n)}$ forms a general position set of P_∞^n. We proceed by induction on n, the base case $n = 2$ being clear. Suppose now that $X^{(n-1)}$ is a general position set of P_∞^{n-1} and consider $X^{(n)}$. Partition the set $[2^{2^{n-1}}]$ into $2^{2^{n-2}}$ blocks $\{1, \ldots, 2^{2^{n-2}}\}$, $\{2^{2^{n-2}} + 1, \ldots, 2^{2^{n-2}+1}\}$, ... Let $x_i^{(n)}$, $x_j^{(n)}$, and $x_k^{(n)}$ be pairwise different vertices from $X^{(n)}$ and consider the following three cases, where again write each number $m \in [2^{2^{n1}}]$ as $m = p_m \cdot 2^{2^{n-2}} + r_m$, where $0 \le p_m < 2^{2^{n-2}}$ and $r_m \in [2^{2^{n-2}}]$.

If i, j, k belong to the same block, that is, if $p_i = p_j = p_k$, then by the induction hypothesis $n-1$ coordinates assure that $x_i^{(n)}$, $x_j^{(n)}$, and $x_k^{(n)}$ do not lie on a common geodesic. If p_i, p_j, and p_k are pairwise different, then we can consider their blocks. Considering the whole blocks as single contracted vertices, the induction hypothesis assures that these contracted vertices do not lie on a common geodesic, which in turn implies then also $x_i^{(n)}$, $x_j^{(n)}$, and $x_k^{(n)}$ do not lie on a common geodesic. In the last case assume without loss of generality that $p_i = p_j < p_k$. Further assuming without loss of generality that $r_i < r_j$, we have $x_{i,1}^{(n)} < x_{j,1}^{(n)} < x_{k,1}^{(n)}$. Hence if $x_i^{(n)}$, $x_j^{(n)}$, and $x_k^{(n)}$ lie on a common geodesic, then necessarily $x_j^{(n)}$ lies between $x_i^{(n)}$ and $x_k^{(n)}$. If $x_{i,n-1}^{(n)} < x_{j,n-1}^{(n)}$, then $x_{i,n}^{(n)} > x_{j,n}^{(n)}$, so $x_i^{(n)}$, $x_j^{(n)}$, and $x_k^{(n)}$ do not lie on a common geodesic. Analogously we see that the same conclusion follows if $x_{i,n-1}^{(n)} > x_{j,n-1}^{(n)}$. Hence in each case the vertices $x_i^{(n)}$, $x_j^{(n)}$, and $x_k^{(n)}$ do not lie on a common geodesic and we are done.

3 Concluding remarks

Recall that a subgraph H of a graph G is *isometric* if $d_H(u,v) = d_G(u,v)$ holds for each pair of vertices $u, v \in V(H)$. Since $P_{i_1} \,\square\, \cdots \,\square\, P_{i_n}$ is an isometric subgraph of P_∞^n, Theorem 1 immediately implies:

Corollary 3 *If $n \ge 2$, and $i_1, \ldots, i_n \ge 2^{2^{n-1}}$, then* $\mathrm{gp}(P_{i_1} \,\square\, \cdots \,\square\, P_{i_n}) = 2^{2^{n-1}}$.

More generally, if a graph G contains an isometric grid $P_{i_1} \,\square\, \cdots \,\square\, P_{i_n}$, where each $i_j \ge 2^{2^{n-1}}$, then $\mathrm{gp}(G) \ge 2^{2^{n-1}}$. For instance:

Corollary 4 *If $n \ge 2$, and $i_1, \ldots, i_n \ge 2^{2^{n-1}+1}$, then* $\mathrm{gp}(C_{i_1} \,\square\, \cdots \,\square\, C_{i_n}) \ge 2^{2^{n-1}}$.

From [5] we know that $\mathrm{gp}(G \,\square\, H) \ge \mathrm{gp}(G) + \mathrm{gp}(H) - 2$ holds for finite, connected graphs G and H. Since the general position number of a path is 2, Corollary 3 demonstrates that the difference in the inequality can be arbitrary large.

In [8] a formula for the number of general position sets of cardinality 4 in $P_r \,\square\, P_s$ (that is, of largest general position sets) is determined for each $r, s \ge 2$. Because of this result and Corollary 3, an interesting and intriguing problem is to determine number of largest general position sets in $P_{i_1} \,\square\, \cdots \,\square\, P_{i_n}$, where $n \ge 3$ and $i_1, \ldots, i_n \ge 2^{2^{n-1}}$.

Acknowledgements

We acknowledge the financial support from the Slovenian Research Agency (research core funding No. P1-0297 and projects J1-9109, J1-1693, N1-0095, N1-0108).

References

[1] B.S. Anand, S.V. Ullas Chandran, M. Changat, S. Klavžar, E.J. Thomas, Characterization of general position sets and its applications to cographs and bipartite graphs, Appl. Math. Comput. 359 (2019) 84–89.

[2] B. Bukh, J. Matoušek, Erdős-Szekeres-type statements: Ramsey function and decidability in dimension 1, Duke Math. J. 163 (2014) 2243–2270.

[3] H.E. Dudeney, Amusements in Mathematics, Nelson, Edinburgh, 1917.

[4] P. Erdős, G. Szekeres, A combinatorial problem in geometry, Compositio Math. 2 (1935) 463–470.

[5] M. Ghorbani, S. Klavžar, H.R. Maimani, M. Momeni, F. Rahimi-Mahid, G. Rus, The general position problem on Kneser graphs and on some graph operations, Discuss. Math. Graph Theory (2019) doi:10.7151/dmgt.2269.

[6] W. Imrich, S. Klavžar, D.F. Rall, Topics in Graph Theory: Graphs and Their Cartesian Product, A K Peters, Wellesley, MA, 2008.

[7] S. Klavžar, I.G. Yero, The general position problem and strong resolving graphs, Open Math. 17 (2019) 1126–1135.

[8] S. Klavžar, B. Patkós, G. Rus, I.G. Yero, On general position sets in Cartesian grids, arXiv:1907.04535 [math.CO] (July 25, 2019).

[9] J. Körner, On the extremal combinatorics of the Hamming space, J. Combin. Theory Ser A 71 (1995) 112–126.

[10] P. Manuel, S. Klavžar, A general position problem in graph theory, Bull. Aust. Math. Soc. 98 (2018) 177–187.

[11] P. Manuel, S. Klavžar, The graph theory general position problem on some interconnection networks, Fund. Inform. 163 (2018) 339–350.

[12] B. Patkós, On the general position problem on Kneser graphs, arXiv:1903.08056 [math.CO] (19 Mar 2019), also: Ars Math. Contemp., to appear.

[13] S. V. Ullas Chandran, G. Jaya Parthasarathy, The geodesic irredundant sets in graphs, Int. J. Math. Combin. 4 (2016) 135–143.