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Simple Explicit Formulas for the Frame-Stewart Numbers*

Sandi Klavžar and Uroš Milutinović

Department of Mathematics, PEF, University of Maribor, Koroška cesta 160, 2000 Maribor Slovenia
{sandi.klavzar, uros.milutinovic}@uni-mb.si

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Abstract. Several different approaches to the multi-peg Tower of Hanoi problem are equivalent. One of them is Stewart's recursive formula

$$S(n, p) = \min\{2S(n_1, p) + S(n - n_1, p - 1) \mid n_1, n - n_1 \in \mathbb{Z}^+\}.$$

In the present paper we significantly simplify the explicit calculation of the Frame-Stewart's numbers $S(n, p)$ and give a short proof of the domain theorem that describes the set of all pairs (n, n_1), such that the above minima are achieved at n_1.

Keywords: multi-peg Tower of Hanoi problem, Frame-Stewart numbers, recursion

1. Introduction

The problem of finding the smallest number of moves in the Multi-peg Tower of Hanoi problem, that is, in the Tower of Hanoi problem with more than three pegs, has been posed by Dudeney [5] in 1908. (The classical Tower of Hanoi problem with three pegs goes back to Lucas [2].) In 1941 two solutions for the multi-peg version appeared, one due to Frame [7] and the other to Stewart [15]. However, already in the editorial note [6] following [15] it was pointed out that the solutions miss an argument that the proposed algorithms are indeed optimal. Proving that these "presumed optimal solutions" are optimal became a notorious open problem.

Bode and Hinz [1] verified that for four pegs and up to 17 disks the Frame-Stewart's approach agrees with the optimal solution. Recently Szegedy [16] proved that for p pegs at least $2^{C_p n^{1/(p-2)}}$ moves are needed. This bound is optimal up to a constant factor in the exponent for fixed p. For the definition of the problem as well as for more information on the history of it we refer to [8, 9]. The Tower of Hanoi has many connections to different areas of mathematics. For instance, a closer look to the Tower

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of Hanoi problem enabled Hinz and Schief [10] to compute the average distance of the Sierpiński gasket, while in [4] it is shown that the graphs of the classical Tower of Hanoi problem contain (in a sense) unique 1-perfect codes.

Stewart's presumed optimal solution for $n \geq p \geq 4$ is given by the recursive scheme:

$$S(n,p) = \min\left\{2S(n_1,p) + S(n-n_1,p-1) \mid n_1 \in \{1,\ldots,n-1\}\right\} \quad (n \geq 2,\ p \geq 4), \tag{1.1}$$

$$S(1,p) = 1 \quad (p \geq 3), \tag{1.2}$$

$$S(n,3) = 2^n - 1 \quad (n \geq 1). \tag{1.3}$$

In [11] it is proved that seven different approaches to the multi-peg Tower of Hanoi problem are equivalent, including Stewart's and Frame's approaches. We thus call the numbers defined by the recursion scheme (1.1)–(1.3) *Frame-Stewart numbers*. Our main result, stated and proved in Section 3, gives a simple explicit expression for these numbers. More precisely, let $p \geq 3$ and let $0 \leq m \leq \binom{p-3+k}{p-3}$. Then

$$S\left(\binom{p-3+k}{p-2} + m,\ p\right) = (P_p(k)+m)2^k + (-1)^p,$$

where $P_p(k)$ is the following polynomial of degree $p-3$:

$$P_p(k) = (-1)^{p-1}\sum_{i=0}^{p-3}(-1)^i\binom{i-1+k}{i}.$$

We believe that this could be a step towards the solution of the Multi-peg Tower of Hanoi problem because the simpler representation of $S(n,p)$ can make the study more accessible. Another result which may lead to the solution of the problem is a short direct proof of the domain theorem of the recursion scheme (1.1)–(1.3), which explicitly describes the set of values n_1 for which the minimum in (1.1) is attained (see Theorem 2.7). In particular, understanding the fact that n_1 is uniquely determined for certain values of n, may give a necessary insight into the problem. The problem of when n_1 is uniquely determined has been treated by Cull and Ecklund in [3].

In the next section we give basic definitions and recall results needed for the proof of the above identity given in Section 3. The paper is concluded with the proof of the domain theorem.

2. Basic Definitions and Results

Definition 2.1. *For $p \geq 3$, let*

$$h_p(x) = \binom{p-3+x}{p-2},\ x \in \mathbb{R},\ x \geq 0.$$

On nonnegative reals these functions are strictly increasing and therefore they have inverses, which are strictly increasing on nonnegative reals as well.

Definition 2.2. *For $p \geq 3$, let $g_p = h_p^{-1}$, and let*

$$f_p(x) = \lceil g_p(x) \rceil .$$

Remark 2.3. We shall use $h_2(x) = 1$ — this definition coincides with $\binom{2-3+x}{2-2}$. There are no g_2 or f_2.

Proposition 2.4. *For any $p \geq 3$, $k \geq 1$,*

$$h_p(k) - h_p(k-1) = h_{p-1}(k).$$

Proof. This property is just the basic property of binomial coefficients. ∎

The next theorem will be crucial for our present work, see [11] for the proof. In fact, the formula appeared already in [7], but has been treated rather heuristically, and has been presented as a statement about the smallest number of moves in the Multi-peg Tower of Hanoi problem.

Theorem 2.5. *Let $n \geq 1$ and $p \geq 3$. Then*

$$S(n, p) = \sum_{k=1}^{n} 2^{f_p(k)-1}.$$ ∎

From Theorem 2.5 it follows that for $n \geq 2$ and $p \geq 3$

$$S(n, p) = S(n-1, p) + 2^{f_p(n)-1}. \tag{2.1}$$

This formula will be the main ingredient of the proof given in Section 4, where we shall simultaneously prove the following two results.

Theorem 2.6. *Let $p \geq 4$ and $k \geq 2$. Then $n_1 = h_p(k-1)$ is the only value of n_1 for which*

$$S(h_p(k), p) = 2S(n_1, p) + S(h_p(k) - n_1, p-1)$$

holds true.

Theorem 2.6 has been implicitly proved in [11] (within the proof of Theorem 5.1), but to make this paper independent of [11] and since we can prove it along the way while proving Theorem 2.7, we have included the argument. Note that the claim of Theorem 2.6 holds for $p = 3$ as well, if one defines $S(1, 2) = 1$ and $S(k, 2) = 0$, for $k \neq 1$, since obviously $S(k, 3) = 2S(k-1, 3) + S(1, 2)$.

Theorem 2.7. *Let $p \geq 4$, $k \geq 1$. Let Σ_k be the set of all integer pairs (n, n_1), for which $S(n, p) = 2S(n_1, p) + S(n - n_1, p-1)$ and $h_p(k) \leq n \leq h_p(k+1)$, and let Π_k be the set of all integer pairs in the parallelogram in the (n, n_1)-plane, bounded by the lines $n_1 = h_p(k-1)$, $n_1 = h_p(k)$, $n_1 = n + h_p(k-1) - h_p(k)$, $n_1 = n + h_p(k) - h_p(k+1) = n - h_{p-1}(k+1)$. Then $\Sigma_k = \Pi_k$.*

Theorem 2.7 is illustrated for $p = 4$ and $p = 5$ in Figures 1 and 2, respectively. It was proved by a (double) induction on p and n in [12,13]. In [12] Majumdar established the truth of the induction basis, that is, for $p = 4$, while in [13] he followed with a general argument. In fact, only one half of the proof is written down — for the "left" and the "bottom" part of the parallelogram. Thus, the whole argument along these lines would contain quite several pages. In Section 4 we give an alternative, complete, and short proof of Theorem 2.7. Along the way we also prove Theorem 2.6.

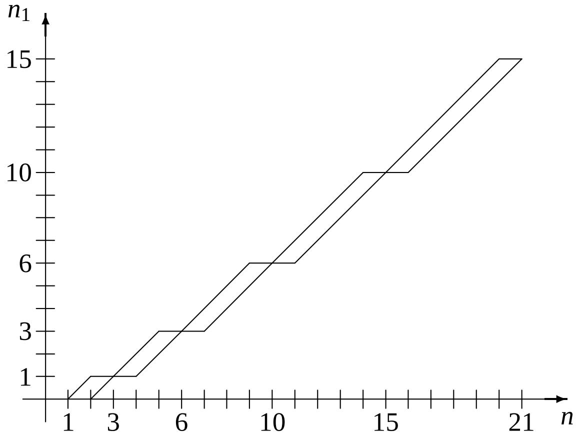

Figure 1: Sets Σ_1, Σ_2, Σ_3, Σ_4, Σ_5 for $p = 4$.

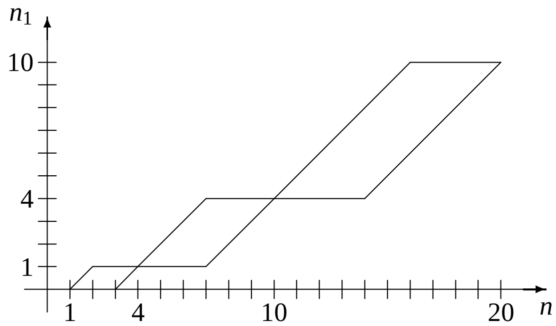

Figure 2: Sets Σ_1, Σ_2, Σ_3 for $p = 5$.

3. Explicit Formulas

Here is the main result of this paper.

Theorem 3.1. *Let $p \geq 3$, $k \geq 1$, and $0 \leq m \leq h_{p-1}(k+1)$. Then*

$$S(h_p(k) + m,\, p) = (P_p(k) + m)2^k + (-1)^p, \tag{3.1}$$

where P_p is the following polynomial of degree $p - 3$:

$$P_p(k) = (-1)^{p-1} \sum_{i=0}^{p-3} (-1)^i h_{i+2}(k). \tag{3.2}$$

Formula (3.2) is in a way the most natural explicit presentation of the polynomials P_p, since the polynomials $h_2, h_3, \ldots, h_{p-1}$ form a standard basis for the space of all integer-valued polynomials of degree $\le p-3$ in the following sense: They are integer-valued and any integer-valued polynomial of degree $\le p-3$ can be written in the unique way as a linear combination of $h_2, h_3, \dots, h_{p-1}$, and the linear combination has integer coefficients only. An analogous result is given by Pólya and Szegő in [14, p. 129] for $\binom{x}{0}, \binom{x}{1}, \dots, \binom{x}{p-3}$. Our claim may be proved analogously, or deduced from the Pólya-Szegő's by noting that $\binom{x}{i-2} - h_i(x)$ is an integer-valued polynomial of degree $i-3$.

In the rest of the section we prove Theorem 3.1. Since the points $(h_p(k+1), h_p(k))$ show special behavior — the uniqueness of the choice of n_1 — we first concentrate on them. Let us introduce a shorthand notation for $S(h_p(k), p)$.

Definition 3.2. *For $p \ge 3$, $k \ge 1$, let $N(k, p) = S(h_p(k), p)$.*

For $p = 3$, $k \ge 1$, $N(k,p) = S(k, p)$, since $h_3(k) = k$. Therefore, in this case no condensation occurs, but it is reasonable to treat it this way, since all the results obtained in this section remain valid for $p = 3$ (with the above interpretation), too.

The recursive formula of Theorem 2.6 for $n_1 = h_p(k-1)$ (recall that by Proposition 2.4 we have $h_p(k) - h_p(k-1) = h_{p-1}(k)$), written as

$$S(h_p(k+1), p) = 2S(h_p(k), p) + S(h_{p-1}(k+1), p-1),$$

now becomes

$$N(k+1, p) = 2N(k, p) + N(k+1, p-1).$$

It holds for $k \ge 1$, $p \ge 3$.

Taking into account $N(1, p) = S(1, p) = 1$ and $N(k, 3) = S(k, 3) = 2^k - 1$, and using standard methods of solving linear difference equations (obtained by fixing $p = 4, 5$), one easily gets the following explicit formulas, for $k \ge 1$:

$$N(k, 3) = 2^k - 1,$$
$$N(k, 4) = (k-1)2^k + 1,$$
$$N(k, 5) = (k(k-1)+2)2^{k-1} - 1.$$

More generally:

Lemma 3.3. *For any $p \ge 3$ and any $k \ge 1$,*

$$N(k, p) = P_p(k) \cdot 2^k + (-1)^p, \tag{3.3}$$

where P_p is a polynomial of degree $p-3$.

Proof. We shall prove the claim by induction on p. As $P_3(k) = 1$, it is obviously true for $p = 3$. By the inductive assumption, the recursive formula

$$N(k, p+1) = 2N(k-1, p+1) + N(k, p)$$

becomes

$$N(k, p+1) = 2N(k-1, p+1) + P_p(k) \cdot 2^k + (-1)^p,$$

for any $k \geq 2$. Multiplying

$$N(k+1-i,\,p+1) = 2N(k-i,\,p+1) + P_p(k+1-i)\cdot 2^{k-i+1} + (-1)^p,$$

by 2^{i-1}, and summing up the resulting identities for $i = 1, 2, \ldots, k-1$, one gets

$$\begin{aligned} N(k,\,p+1) &= 2^{k-1}N(1,\,p+1) + 2^k(P_p(2)+P_p(3)+\cdots+P_p(k)) + (-1)^p(2^{k-1}-1) \\ &= 2^k((1+(-1)^p)/2 + P_p(2)+P_p(3)+\cdots+P_p(k)) + (-1)^{p+1}. \end{aligned}$$

Hence, the statement of the theorem holds true for

$$P_{p+1}(k) = (1+(-1)^p)/2 + P_p(2)+P_p(3)+\cdots+P_p(k).$$

Using Euler-Maclaurin summation formula for Σm^i for $0 \leq i \leq p-3$, we see that there is such a polynomial and that its degree is $p-2$. ∎

Combining Lemma 3.3 with Theorem 2.5 we can directly calculate all the values of $S(n,\,p)$, as follows.

$$\begin{aligned} S(h_p(k),\,p) &= P_p(k)2^k + (-1)^p, \\ S(h_p(k)+1,\,p) &= S(h_p(k),\,p) + 2^{f_p(h_p(k)+1)-1} = (P_p(k)+1)2^k + (-1)^p, \\ &\vdots \\ S(h_p(k)+m,\,p) &= S(h_p(k)+m-1,\,p) + 2^{f_p(h_p(k)+m)-1} = (P_p(k)+m)2^k + (-1)^p, \\ &\vdots \\ S(h_p(k+1),\,p) &= P_p(k)2^{k-1} + (-1)^p + (h_p(k+1)-h_p(k))2^k \\ &= (P_p(k)+h_{p-1}(k+1))2^k + (-1)^p. \end{aligned}$$

By this we have (in particular) proved statement (3.1) of Theorem 3.1.

Since $S(h_p(k+1),\,p) = P_p(k+1)2^{k+1} + (-1)^p$, the last line of the above calculations gives the following recursive formula:

$$2P_p(k+1) = P_p(k) + h_{p-1}(k+1), \quad p \geq 3,\ k \geq 1. \tag{3.4}$$

Summing up the identities (3.4) $2P_p(i) = P_p(i-1) + h_{p-1}(i)$, multiplied by 2^{i-k-1}, for $i = 2, 3, \ldots, k$, we get

$$P_p(k) = \frac{P_p(1)}{2^{k-1}} + \frac{h_{p-1}(2)}{2^{k-1}} + \frac{h_{p-1}(3)}{2^{k-2}} + \cdots + \frac{h_{p-1}(k-1)}{2^2} + \frac{h_{p-1}(k)}{2}. \tag{3.5}$$

Using $h_{p-1}(\ell) = h_p(\ell) - h_p(\ell-1)$ this identity can be rewritten as

$$P_p(k) = \frac{P_p(1)}{2^{k-1}} - \frac{h_p(1)}{2^{k-1}} - \frac{h_p(2)}{2^{k-1}} - \cdots - \frac{h_p(k-1)}{2^2} + \frac{h_p(k)}{2}.$$

Equation (3.5) for subscript $p+1$ reads as

$$P_{p+1}(k) = \frac{P_{p+1}(1)}{2^{k-1}} + \frac{h_p(2)}{2^{k-1}} + \frac{h_p(3)}{2^{k-2}} + \cdots + \frac{h_p(k-1)}{2^2} + \frac{h_p(k)}{2}.$$

Summing the last two equations and taking into account $P_p(1) = (1-(-1)^p)/2$, $P_{p+1}(1) = (1+(-1)^p)/2$ (obtained from (3.3)), and $h_p(1) = 1$, we get the following recursion — this time in p:

$$P_{p+1}(k) + P_p(k) = h_p(k), \quad p \geq 3,\ k \geq 1. \tag{3.6}$$

Summation of the above identities (3.6) $P_j(k) = -P_{j-1}(k) + h_{j-1}(k)$, multiplied by $(-1)^{p-j}$, for $j = 4, \dots, p$, yields

$$P_p(k) = (-1)^{p-1} \sum_{i=1}^{p-3} (-1)^i h_{i+2}(k) - (-1)^{p-2} P_3(k).$$

Taking into account $P_3(k) = 1$ and $h_2(k) = 1$ we finally obtain

$$P_p(k) = (-1)^{p-1} \sum_{i=0}^{p-3} (-1)^i h_{i+2}(k),$$

and the proof of Theorem 3.1 is complete.

We conclude the section with the following consequence of Theorem 3.1.

Corollary 3.4. *Let* $p \geq 3$ *and* $k \geq 1$. *Then*

$$S(h_p(k+1), p) = (P_p(k) + h_{p-1}(k+1))2^k + (-1)^p.$$

4. Proof of Theorems 2.6 and 2.7

Recall that Σ_k denotes the set of all pairs (n, n_1), for which $S(n, p) = 2S(n_1, p) + S(n - n_1, p-1)$ and $h_p(k) \leq n \leq h_p(k+1)$, $k \geq 1$, and that Π_k is the parallelogram in the (n, n_1)-plane, cf. Figures 1 and 2.

First note that $n_1 = 0$ means $S(n, p) = S(n, p-1)$ (using the convention $S(0, p) = 0$), and from Theorem 2.5 we see that it is equivalent to $1 = h_{p-1}(1) \leq n \leq h_{p-1}(2) = p-2$. Note that this exactly corresponds to the bottom side of the parallelogram Π_1. Therefore in the rest of the proof we shall deal either with this side, or with $n_1 \in \{1, 2, \dots, n-1\}$.

We proceed by induction on k.

We shall prove $\Sigma_k \subseteq \Pi_k$ by induction on n (from the above range). We shall treat the basis of the induction (when $k = 1$) and the inductive step (when we deduce that our claims are true for certain $k > 1$ from the assumption that it is true for $k-1$) in a uniform manner.

Clearly, $(1, 0)$ is the only point belonging to Σ_1 having the first coordinate equal 1. Using this observation if $k = 1$, or by the inductive assumption, we infer that $(h_p(k), h_p(k-1))$ is the only point of Σ_k for which the first coordinate equals $h_p(k)$. It obviously belongs to Π_k, being one of its vertices.

Let $S(n, p) = 2S(n_1, p) + S(n - n_1, p-1)$, $h_p(k) < n \leq h_p(k+1)$, and assume that all points from Σ_k with the first coordinate less than n belong to Π_k.

First note that $h_p(k) < n \leq h_p(k+1)$ is equivalent to $k < g_p(n) \leq k+1$, and this is equivalent to $f_p(n) = k+1$.

If $n_1 > h_p(k)$, then $g_p(n_1) > k$, and finally $f_p(n_1) > k$. Hence, using (2.1), we obtain

$$\begin{aligned}
2S(n_1, p) + S(n - n_1, p-1) &= 2(S(n_1 - 1, p) + 2^{f_p(n_1)-1}) + S(n - n_1, p-1) \\
&> 2S(n_1 - 1, p) + S(n - n_1, p-1) + 2^k \\
&\geq S(n-1, p) + 2^k \\
&= S(n-1, p) + 2^{f_p(n)-1} \\
&= S(n, p).
\end{aligned}$$

By this, it is proved that $n_1 \leq h_p(k)$.

We claim next that $n_1 \geq h_p(k-1)$. If $n_1 < h_p(k-1)$, then $n - n_1 > n - h_p(k-1) > h_p(k) - h_p(k-1) = h_{p-1}(k)$. It follows that $g_{p-1}(n-n_1) > k$, and hence $f_{p-1}(n-n_1) \geq k+1$. Therefore, applying (2.1),

$$\begin{aligned}
2S(n_1, p) + S(n - n_1, p-1) &= 2S(n_1, p) + S(n - n_1 - 1, p-1) + 2^{f_{p-1}(n-n_1)-1} \\
&\geq S(n-1, p) + 2^k \\
&= S(n-1, p) + 2^{f_p(n)-1} \\
&= S(n, p).
\end{aligned}$$

From $S(n, p) = 2S(n_1, p) + S(n - n_1, p-1)$ it follows

$$S(n-1, p) = 2S(n_1, p) + S(n-1-n_1, p-1)\,.$$

Since $h_p(k) \leq n-1 < h_p(k+1)$, by the choice of n we get $n_1 \geq h_p(k-1)$. This contradiction proves the claim.

Now, we know that $h_p(k) < n \leq h_p(k+1)$, $h_p(k-1) \leq n_1 \leq h_p(k)$, and $S(n, p) = 2S(n_1, p) + S(n - n_1, p-1)$. Denote by T_1 the triangle consisting of all pairs (n, n_1) satisfying $n > h_p(k)$, $n_1 \leq h_p(k)$, and $n_1 > n + h_p(k-1) - h_p(k)$; also denote by T_2 the triangle consisting of all pairs (n, n_1) satisfying $n \leq h_p(k+1)$, $h_p(k-1) \leq n_1$, and $n_1 < n + h_p(k) - h_p(k+1)$, cf. Figure 3.

If $n_1 > h_p(k-1)$, i.e., $f_p(n_1) = k$ (while $f_p(n) = k+1$), then $S(n, p) = 2S(n_1, p) + S(n - n_1, p-1)$ implies

$$S(n-1, p) + 2^{f_p(n)-1} = 2S(n_1 - 1, p) + 2 \cdot 2^{f_p(n_1)-1} + S(n - n_1, p-1).$$

It follows that $S(n-1, p) = 2S(n_1 - 1, p) + S(n - n_1, p-1)$; it means that $(n-1, n_1 - 1) \in \Sigma_k$. Since by finitely many repetitions of this transformation, any point from T_1 is transformed to a point strictly above $(h_p(k), h_p(k-1))$, not belonging to Σ_k, it follows that $T_1 \cap \Sigma_k = \emptyset$.

If $n_1 < h_p(k)$ (hence $f_p(n_1 + 1) = k$) and $n < h_p(k+1)$ (hence $f_p(n+1) = k+1$),

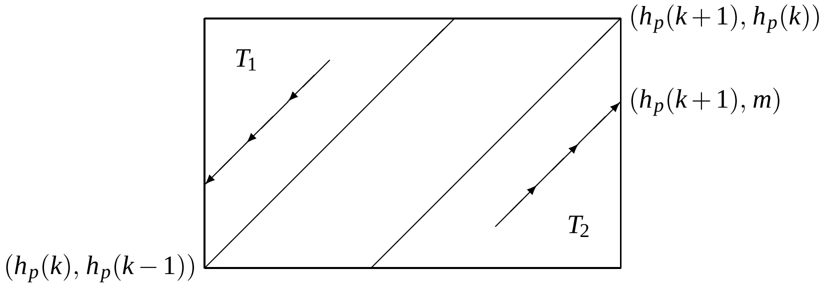

Figure 3: Sets T_1 and T_2.

then we can compute, using (2.1) again:

$$\begin{aligned}S(n+1, p) &= S(n, p) + 2^{f_p(n+1)-1}\\ &= 2S(n_1, p) + S(n-n_1, p-1) + 2^{k+1-1}\\ &= 2(S(n_1, p) + 2^{k-1}) + S(n-n_1, p-1)\\ &= 2(S(n_1, p) + 2^{f_p(n_1+1)-1}) + S(n-n_1, p-1)\\ &= 2S(n_1+1, p) + S(n-n_1, p-1).\end{aligned}$$

This means that $(n+1, n_1+1) \in \Sigma_k$. By finitely many repetitions of this transformation, any point from T_2 is transformed to a point $(h_p(k+1), m)$ strictly below $(h_p(k+1), h_p(k))$ and belonging to Σ_k, see Figure 3. From $m < h_p(k)$ it follows $h_p(k+1) - m > h_p(k+1) - h_p(k) = h_{p-1}(k+1)$. Hence $g_{p-1}(h_p(k+1) - m) > k+1$ and therefore $f_{p-1}(h_p(k+1) - m) \geq k+2$. Since $h_p(k-1) < m+1 \leq h_p(k)$, it follows $k-1 < g_p(m+1) \leq k$, and $f_p(m+1) \leq k$. From these inequalities it follows that

$$\begin{aligned}S(h_p(k+1), p) &= 2S(m, p) + S(h_p(k+1) - m, p-1)\\ &= 2S(m, p) + S(h_p(k+1) - m - 1, p-1) + 2^{f_{p-1}(h_p(k+1)-m)-1}\\ &\geq 2S(m, p) + 2\cdot 2^{f_p(m+1)-1} + S(h_p(k+1) - m - 1, p-1)\\ &= 2S(m+1, p) + S(h_p(k+1) - m - 1, p-1),\end{aligned}$$

which contradicts the minimality of $S(h_p(k), p)$ and the definition of Σ_k. It follows that $T_2 \cap \Sigma_k = \emptyset$.

By all these we have proved that Σ_k is a subset of the parallelogram Π_k. Note that this also proves that $h_p(k)$ is the only value of n_1 for which $S(h_p(k+1), p) = 2S(n_1, p) + S(h_p(k+1) - n_1, p-1)$. Thus the inductive proof of Theorem 2.6 is completed.

To complete the proof we must show $\Pi_k \subseteq \Sigma_k$.

First, note that $n_1 = h_p(k-1)$ and $n = h_p(k)$ satisfy $n_1 < h_p(k)$ and $n < h_p(k+1)$. Therefore, since $(h_p(k), h_p(k-1)) \in \Sigma_k$, it follows (as in the proof of $T_2 \cap \Sigma_k = \emptyset$) that

$(h_p(k)+1, h_p(k-1)+1)$, $(h_p(k)+2, h_p(k-1)+2), \ldots \in \Sigma_k$, until we get that all the points of the parallelogram, lying on the line $n_1 = n + h_p(k-1) - h_p(k)$, belong to Σ_k.

Next, note that all points of the parallelogram, except those from the line $n_1 = n + h_p(k) - h_p(k+1)$, satisfy $n_1 > n + h_p(k) - h_p(k+1)$, i.e., $n - n_1 > h_p(k+1) - h_p(k) = h_{p-1}(k+1)$, as well as $n_1 \leq n + h_p(k-1) - h_p(k)$, i.e., $n - n_1 \leq h_p(k) - h_p(k-1) = h_{p-1}(k)$. That means that $f_{p-1}(n-n_1+1) = k+1$. If $h_p(k) \leq n < h_p(k+1)$, then also $f_p(n+1) = k+1$.

Therefore, for such an $(n, n_1) \in \Sigma_k$, for which n and n_1 satisfy these additional conditions, $S(n, p) = 2S(n_1, p) + S(n - n_1, p-1)$ implies

$$\begin{aligned} S(n+1, p) &= 2S(n_1, p) + S(n-n_1, p-1) + 2^{f_p(n+1)-1} \\ &= 2S(n_1, p) + S(n-n_1, p-1) + 2^{f_{p-1}(n-n_1+1)-1} \\ &= 2S(n_1, p) + S(n+1-n_1, p-1), \end{aligned}$$

i.e., it follows that $(n+1, n_1) \in \Sigma_k$.

Thus, we are able to complete our proof, by repeatedly translating points from the left side of the parallelogram Π_k, for which we already know that they belong to Σ_k, to the right for vector $(1, 0)$, keeping them in Σ_k, until we cover all integral points from the parallelogram Π_k.

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