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On Characterizations with Forbidden Subgraphs*

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1. Introduction

Let **G** denote the class of all finite, simple, undirected, and unlabelled graphs. Throughout the paper, $\leq$ will denote a well-founded partial order in **G**. This means that every strictly descending chain in $(\mathbf{G}, \leq)$ is finite. Obviously, this is equivalent to the requirement that every nonempty subclass **C** of **G** contains a graph which is $\leq$-minimal in **C**.

Example 1. It is easy to see that each of the following four relations $G \leq H$ is a well-founded partial order in **G**:

(i) *induced subgraph*: G is isomorphic to an induced subgraph of H;

(ii) *subgraph*: G is isomorphic to a subgraph of H;

(iii) *topological containment*: a subdivision of G is isomorphic to a subgraph of H;

(iv) *minor*: G is isomorphic to a graph which can be obtained from a subgraph of H by contracting some of its edges, and replacing any resulting multiple edges with simple ones.

Note also that each of the first three relations is a subrelation of the next one on this list.

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Definition 1. *A class of graphs* $\mathbf{C}$ *is hereditary (with respect to* $\leq$*) if* $G \leq H, H \in \mathbf{C}$ *imply* $G \in \mathbf{C}$.

Note that $\mathbf{G}$ is hereditary for any relation $\leq$.

Whenever $G \leq H$, we shall say that G is a *$\leq$-subgraph* of H. For a class of graphs $\mathbf{C}$, let $\min(\mathbf{C})$ denote the set of its $\leq$-minimal elements, and $\overline{\mathbf{C}}$ the class of complements of graphs from $\mathbf{C}$.

For a class of graphs $\mathbf{C}$, let $\mathbf{F}(\mathbf{C})$ be the class of all graphs which have no $\leq$-subgraphs in $\mathbf{C}$. Thus we have defined a mapping

$$\mathbf{F} : 2^{\mathbf{G}} \to 2^{\mathbf{G}}$$

One sees easily that $\mathbf{F}(\mathbf{C})$ is hereditary for every $\mathbf{C}$.

Example 2. Let $\mathbf{C}$ be the class of all cycles, $\mathbf{C} = \{C_k : k \geq 3\}$. Then for each of the relations from Example 1, $\mathbf{F}(\mathbf{C})$ is the class of all forests. Note that for induced subgraphs and for subgraphs $\mathbf{C}$ is an antichain, while for topological containment and for minors $\min(\mathbf{C}) = \{C_3\}$.

Proposition 1.

(i) $F(\mathbf{C}_1 \cup \mathbf{C}_2) = \mathbf{F}(\mathbf{C}_1) \cap \mathbf{F}(\mathbf{C}_2)$

(ii) $\mathbf{F}(\mathbf{C}_1) \subseteq \mathbf{F}(\mathbf{C}_2) \iff$ *for every* $G \in \mathbf{C}_2$, *there is an* $H \in \mathbf{C}_1$ *such that* $H \leq G$

(iii) if $\leq$ *is the induced subgraph relation, then* $\mathbf{F}(\overline{\mathbf{C}}) = \overline{\mathbf{F}(\mathbf{C})}$

We omit the proof which is completely analogous to that given in [1] for the induced subgraph relation.

Example 3. Let $\leq$ be the induced subgraph relation. Define $\mathbf{T} := \mathbf{F}(\{C_k : k \geq 4\})$ and $\mathbf{S} := \mathbf{T} \cap \overline{\mathbf{T}}$. Note that $\mathbf{T}$ is the class of *triangulated graphs* and $\mathbf{S}$ the class of *split graphs* (for more information on these graphs, see [4]). Let us determine the minimal forbidden induced subgraphs for $\mathbf{S}$. Using Proposition 1 (i) and (iii), one sees that $S = \mathbf{F}(\{C_k, \overline{C}_k : k \geq 4\})$. But for $k \geq 6$ the cycle C_k contains $2K_2 = \overline{C}_4$ and $\overline{C}_k$ contains C_4, therefore these graphs are not minimal. As $\overline{C}_5 \simeq C_5$, we conclude that $\mathbf{S} = \mathbf{F}(\{2K_2, C_4, C_5\})$, a result due originally to Földes and Hammer [3].

Proposition 2. *The restricted mapping*

$$\mathbf{F} : \{\text{antichains in } \mathbf{G}\} \to \{\text{ hereditary classes in } \mathbf{G}\}$$

is one-to-one and onto.

Again, we omit the proof which can be found (for the induced subgraph relation) in [1].

2. Closedness under derivation rules

Definition 2. *A derivation rule of length k in* **G** *is a $(k+1)$-ary relation in* **G**, *denoted by "$\vdash$".*

Instead of "$(G_1, G_2, \dots, G_k, G) \in \vdash$" we shall write "$G_1, G_2, \dots, G_k \vdash G$" and say that $G_1, G_2, \dots, G_k$ derive G.

Example 4. Here are some examples of derivation rules:

(i) *complementation:* $G \vdash \overline{G}$.

(ii) *disjoint union:* $G_1, G_2 \vdash G_1 \cup G_2$.

(iii) *join:* $G_1, G_2 \vdash G_1 + G_2$.

(iv) *substitution:* $G_1, G_2 \vdash G$, where G is obtained in the following way: Choose a vertex v of G_1 and replace it with G_2. Connect each vertex of G_2 with an edge to each former neighbour of v in G_1.

(v) *substitution with graphs from* **H** where **H** is some fixed hereditary class of graphs: $G \vdash G'$, where G' is obtained in the following way: Choose a graph $H \in \mathbf{H}$ and perform substitution on the graphs G and H (see (iv) above).

(vi) *identification of subgraphs from* **H** where **H** is some fixed hereditary class of graphs: $G_1, G_2 \vdash G$, where G is obtained in the following way: Choose subgraphs $H_1 \le G_1$ and $H_2 \le G_2$ such that $H_1 \simeq H_2 \in H$. Choose an isomorphism between H_1 and H_2 and identify the corresponding vertices of H_1 and H_2.

When $\mathbf{H} = \mathbf{G}$ this derivation rule is called simply *subgraph identification.*

Derivation rules (i) and (v) are of length 1 while all the others in this example are of length 2. Note that disjoint union is a special case of identification of subgraphs from **H** — with **H** containing but the empty graph.

Definition 3. *A derivation rule $\vdash$ is consistent (with $\le$) provided that the following holds: If $G_1, G_2, \dots, G_k \vdash G$ and $H \le G$, then there exist graphs $H_1, H_2, \dots, H_k$ such that $H_i \le G_i$ and $H_1, H_2, \dots, H_k \vdash H$.*

Definition 4. [7] *A derivation rule $\vdash$ is an amalgamation (with respect to $\le$), if $G_1, G_2, \dots, G_k \vdash G$ implies that all $G_i \le G$.*

Example 5. Let $\le$ be the induced subgraph relation. Then it is easy to see that all the derivation rules listed in Example 4 are consistent, and all but (i) are also amalgamations. If $\le$ is the subgraph relation, or the relation of topological containment, or the minor relation, then only (ii) and (vi) are consistent, while again all but (i) are amalgamations.

Let $\vdash_2$ be a derivation rule of length 2, and **H** a class of graphs. Then one can define a derivation rule $\vdash_1$ of length 1 in the following way: $G \vdash_1 G'$ if $G, H \vdash_2 G'$, for some $H \in \mathbf{H}$. It is easy to see that $\vdash_1$ is consistent (an amalgamation), whenever **H** is hereditary and $\vdash_2$ is consistent (an amalgamation). An example: if $\vdash_2$ is "substitution" then $\vdash_1$ is "substitution with graphs from **H**".

Definition 5. *A class of graphs* $\mathbf{C}$ *is closed under* $\vdash$ *if* $G_1, G_2, \ldots, G_k \in \mathbf{C}$ *and* $G_1, G_2, \ldots, G_k \vdash G$ *imply* $G \in \mathbf{C}$.

Definition 6. *Let* $\mathbf{C}$ *be a class of graphs, and* $\vdash$ *a derivation rule of length* k. *A graph* G *is reducible over* $\mathbf{C}$ *(with respect to* $\vdash$*) if there are graphs* $G_1, G_2, \ldots, G_k$ *from* $\mathbf{C}$ *such that all* $G_i \neq G$ *and* $G_1, G_2, \ldots, G_k \vdash G$. *A graph* G *is reducible (with respect to* $\vdash$*) if it is reducible over* $\mathbf{G}$.

Theorem 1. *Let* $\vdash$ *be consistent, and let* $\mathbf{A}$ *be an antichain. Then* $\mathbf{F}(\mathbf{A})$ *is closed under* $\vdash$ *if and only if no element from* $\mathbf{A}$ *is reducible over* $\mathbf{F}(\mathbf{A})$.

Proof. ($\Longrightarrow$) Let $G \in \mathbf{A}$, $G_1, G_2, \ldots, G_k \in \mathbf{F}(\mathbf{A})$, and $G_1, G_2, \ldots, G_k \vdash G$. By definition of $\mathbf{F}$, $G \notin \mathbf{F}(\mathbf{A})$. It follows that $\mathbf{F}(\mathbf{A})$ is not closed under $\vdash$.

($\Longleftarrow$) Let $G_1, G_2, \ldots, G_k \in \mathbf{F}(\mathbf{A}), G_1, G_2, \ldots, G_k \vdash G$ and $G \notin \mathbf{F}(\mathbf{A})$. By definition of $\mathbf{F}$, there is some $H \in \mathbf{A}$ such that $H \leq G$. By consistency of $\vdash$ with $\leq$, there are $H_1, H_2, \ldots, H_k$ such that $H_i \leq G_i$ and $H_1, H_2, \ldots, H_k \vdash H$. By heredity, $H_i \in \mathbf{F}(\mathbf{A})$, and hence $H_i \neq H$, for all i. Thus H is reducible over $\mathbf{F}(\mathbf{A})$. ∎

Theorem 2. *Let* $\vdash$ *be a consistent amalgamation, and let* $\mathbf{A}$ *be an antichain. Then* $\mathbf{F}(\mathbf{A})$ *is closed under* $\vdash$ *if and only if no element from* $\mathbf{A}$ *is reducible.*

Proof. By Theorem 1, $\mathbf{F}(\mathbf{A})$ is closed under $\vdash$ iff graphs from $\mathbf{A}$ are not reducible over $\mathbf{F}(\mathbf{A})$. Since we required that a reducible graph reduces to graphs different from itself (cf. Def. 5), for amalgamations reducibility of graphs from $\mathbf{A}$ over $\mathbf{F}(\mathbf{A})$ is equivalent to reducibility (over $\mathbf{G}$). ∎

Example 6. [1] Let $\leq$ be the induced subgraph relation, and $\mathbf{A}$ an antichain. Using Theorem 1, one can easily verify that $\mathbf{F}(\mathbf{A})$ is closed under complementation if and only if $\mathbf{A}$ is closed under complementation.

Definition 7. *Let* S *be a set of vertices of the graph* G. S *is* contractible *if vertices of* S *have identical neighbourhoods in* $G - S$, *and* separating *if* $G - S$ *is disconnected.*

Corollary 1. *Let* $\mathbf{A}$ *be an antichain with respect to* $\leq$.

(i) If disjoint union is a consistent amalgamation, then $\mathbf{F}(\mathbf{A})$ *is closed under disjoint union if and only if all graphs from* A *are connected.*

(ii) If join is consistent amalgamation, then $\mathbf{F}(\mathbf{A})$ *is closed under join if and only if all graphs from* $\mathbf{A}$ *have connected complements.*

(iii) If substitution is a consistent amalgamation, then $\mathbf{F}(\mathbf{A})$ *is closed under substitution if and only if no graph* G *from* $\mathbf{A}$ *contains a contractible set* S *with* $1 < |S| < |V(G)|$.

(iv) *If substitution with graphs from* **H** *is a consistent amalgamation, then* **F(A)** *is closed under substitution with graphs from* **H** *if and only if no graph from* **A** *contains a contractible set* S *with* $|S| > 1$ *which induces a graph from* **H**.

(v) *If identification of subgraphs from* **H** *is a consistent amalgamation, then* **F(A)** *is closed under identification of subgraphs from* **H** *if and only if no graph from* **A** *contains a separating set which induces a graph from* **H**.

Proof. (i) and (ii) are proved in [7] (for the induced subgraph relation only, but the general proof is the same).

(iii) Obviously, a graph which is reducible with respect to substitution contains a contractible set. The condition on cardinality ensures that both "factors" to which such a graph reduces are distinct from it. Hence the assertion follows from Theorem 2.

(iv) This derivation rule is very similar to substitution except that it is of length 1, hence the contractible set is no more required to be a proper subset of the vertex set. However, it must now induce a graph belonging to **H**.

(v) A graph is reducible with respect to identification of subgraphs from **H** iff it contains a separating set which induces a graph belonging to **H**. Hence the assertion follows from Theorem 2. ∎

Corollary 2. *Let* $\leq$ *be such that subgraph identification is a consistent amalgamation, and let* **A** *be an antichain. Then* **F(A)** *is closed under subgraph identification if and only if* **A** *contains only complete graphs.*

Proof. From Corollary 1(v) it follows that **F(A)** is closed under subgraph identification iff no graph from **A** contains any separating set whatsoever. But as soon as a graph contains two nonadjacent vertices it contains a separating set (namely, the set consisting of the remaining vertices). Hence in this case all graphs from **A** must be complete. ∎

Note that for the relations of Example 1 an antichain of complete graphs contains at most one element.

These results can be used in two different ways. First, given an antichain **A** and a derivation rule $\vdash$, one can ask whether **F(A)** is closed under $\vdash$.

Example 7. Triangulated graphs (cf. Example 3) are closed under disjoint union (as all C_k, for $k \geq 4$, are connected), but not under join (as $\overline{C}_4$ is not connected). They are closed under identification of complete subgraphs (as C_k, for $k \geq 4$, contain no clique-cutsets). They are also closed under substitution with complete graphs (as none of C_k, for $k \geq 4$, contains a nontrivial contractible clique), but not under substitution (as two nonconsecutive vertices of C_4 form a contractible set). And they are obviously not closed under complementation.

Second, knowing that **F(A)** is closed under a derivation rule $\vdash$, it is possible to infer certain properties of graphs from **A**.

Example 8. It is easy to see that the class of perfect graphs (which is hereditary for the induced subgraph relation) is closed under disjoint union, join, and identification of complete subgraphs. It is also closed under substitution and complementation [6]. It follows that minimal imperfect graphs are connected, have connected complements, contain no clique-cutsets, contain no contractible sets other than singletons and the whole vertex set, and that their complements are also minimal imperfect graphs.

3. Forbidden subgraphs for closures

Definition 8. *Let* $\mathbf{C}$ *be a class of graphs, and* $\vdash$ *a derivation rule. A finite sequence of graphs is called a construction sequence over* $\mathbf{C}$ *if every graph from the sequence either belongs to* $\mathbf{C}$ *or is derived by* $\vdash$ *from some of its predecessors in the sequence. The closure of* $\mathbf{C}$ *under* $\vdash$*, denoted by* $cl(\mathbf{C})$*, is the class of all graphs which appear as the final term of some construction sequence over* $\mathbf{C}$.

Proposition 3. *If* $\vdash$ *is a consistent derivation rule, then the closure of a hereditary class is hereditary.*

Proof. Let $\mathbf{C}$ be a hereditary class, $G \in \mathrm{cl}(\mathbf{C})$, and $H \leq G$. We want to prove that $H \in \mathrm{cl}(\mathbf{C})$ as well. The proof is by induction on the length ℓ of a shortest construction sequence for G.

Base: If $\ell = 1$, G belongs to $\mathbf{C}$ and so does H. Hence $H \in \mathrm{cl}(\mathbf{C})$.

Induction step: Suppose that $G_1, G_2, \ldots, G_k \vdash G$ where $G_1, G_2, \ldots, G_k$ precede G in a shortest construction sequence for G. As $\vdash$ is consistent, there are $H_1, H_2, \ldots, H_k$ such that $H_i \leq G_i$ and $H_1, H_2, \ldots, H_k \vdash H$. By the induction hypothesis all H_i belong to $\mathrm{cl}(\mathbf{C})$. Hence $H \in \mathrm{cl}(\mathbf{C})$ as well. ■

Knowing minimal forbidden $\leq$-subgraphs for some hereditary class, how can we find minimal forbidden $\leq$-subgraphs for its closure under some consistent derivation rule? Here is an answer for consistent amalgamations.

Theorem 3. *If* $\vdash$ *is a consistent amalgamation, then*

$$\mathrm{cl}(\mathbf{F}(\mathbf{A})) = \mathbf{F}(\min\{\text{irreducible graphs not in } \mathbf{F}(\mathbf{A})\})$$

Proof. Let $G \in \mathrm{cl}(\mathbf{F}(\mathbf{A}))$. Then, by the definition of closure, either G is reducible or it belongs to $\mathbf{F}(\mathbf{A})$. By Proposition 3, the same holds for all of its $\leq$-subgraphs. Hence G contains no irreducible graph not in $\mathbf{F}(\mathbf{A})$ as a $\leq$-subgraph.

Conversely, assume $G \notin \mathrm{cl}(\mathbf{F}(\mathbf{A}))$, and let $\mathbf{S}$ be the set of all those $\leq$-subgraphs of G which do not belong to $\mathrm{cl}(\mathbf{F}(\mathbf{A}))$. As $G \in \mathbf{S}$, $\mathbf{S}$ is not empty, hence it contains a minimal graph H, by well-foundedness of $\leq$. Suppose that H is reducible to "factors" $H_1, H_2, \ldots, H_k$. As $\vdash$ is an amalgamation, $H_i \leq H \leq G$. If all H_i belong to $\mathrm{cl}(\mathbf{F}(\mathbf{A}))$, H itself belongs to $\mathrm{cl}(\mathbf{F}(\mathbf{A}))$, a contradiction; otherwise H is not minimal in $\mathbf{S}$, another contradiction. It follows that H is irreducible. H is not in $\mathrm{cl}(\mathbf{F}(\mathbf{A}))$, hence it is also not in $\mathbf{F}(\mathbf{A})$. Thus G contains an irreducible graph not in $\mathbf{F}(\mathbf{A})$ as a $\leq$-subgraph. ∎

Let $\mathbf{A}$ be an antichain, and $\vdash$ a consistent amalgamation. According to Theorem 3, one can find minimal forbidden $\leq$-subgraphs for $\mathrm{cl}(\mathbf{F}(\mathbf{A}))$ by the following procedure:

Step 1: Take all irreducible graphs from $\mathbf{A}$.

Step 2: Add all minimal irreducible graphs which contain some reducible graph from $\mathbf{A}$, and no irreducible graph from $\mathbf{A}$, as a $\leq$-subgraph.

Example 9. Let $\leq$ be the induced subgraph relation. In [5] it is shown that $\mathbf{U} := \mathbf{F}(\{C_4, \overline{C}_{2n+1} : n \geq 1\})$ is the class of all finite intersection graphs of unbounded intervals on the real line, and that $\mathrm{cl}(\mathbf{U})$, the closure of $\mathbf{U}$ under join, is the class of all finite intersection graphs of halfspaces in $\mathbf{R}^n$, for some $n \geq 2$. Let us determine the minimal forbidden induced subgraphs for $\mathrm{cl}(\mathbf{U})$. Obviously, being irreducible under join is equivalent to having connected complement. Thus Step 1 of the above procedure yields the graphs $\overline{C}_{2n+1}, n \geq 1$. In Step 2 we have to consider all irreducible supergraphs of C_4 which contain none of $\overline{C}_{2n+1}, n \geq 1$, as an induced subgraph. Let G be such a graph, with a, b, c, d (in this order) inducing a C_4. In $\overline{G}$, these vertices induce $2K_2$, with edges ac and bd. As $\overline{G}$ is connected, there exists a path with one endpoint in $\{a, c\}$ and another in $\{b, d\}$. Let $P := v_1 v_2 \ldots v_k$ be a shortest such path. Then $k \geq 3$. For the sake of definiteness, assume that $v_1 = a$ and $v_k = b$ (see Fig. 1). As $\overline{G}$ is bipartite it contains no triangle, hence c is not adjacent to v_2 and d is not adjacent to v_{k-1}. As P is shortest, c is not adjacent to any of $v_3, \ldots v_k$ and d is not adjacent to any of $v_1, \ldots v_{k-2}$. It follows that $cv_1 \ldots v_k d$ is an induced path on at least 5 vertices in $\overline{G}$. Hence G contains $\overline{P}_5$ (the house graph) as an induced subgraph. As $\overline{P}_5$ is itself an irreducible supergraph of C_4 containing none of $\overline{C}_{2n+1}$, for $n \geq 1$, it is the only new minimal forbidden subgraph obtained on Step 2. Thus $\mathrm{cl}(\mathbf{U}) = \mathbf{F}(\{\overline{P}_5, \overline{C}_{2n+1} : n \geq 1\})$.

Another example of an application of this procedure can be found in [2], where the authors determine the minimal forbidden induced subgraphs for line graphs of multigraphs. They use the fact that this class of graphs is exactly the closure under substitution with complete graphs of the class of line graphs of simple graphs.

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Figure 1. $\overline{G}$ contains an induced P_5

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