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Y-Compatible and Strict Y-Compatible Functions

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Abstract—Let $Y \in \mathbb{R}^n$. A function $f : \mathbb{R}^n \to \mathbb{R}^k$ is *Y-compatible*, if for any $Z \in \mathbb{R}^n$, $Z \le Y$ if and only if $f(Z) \le f(Y)$ and is *strict Y-compatible*, if for any $Z \in \mathbb{R}^n$, $Z < Y$ if and only if $f(Z) < f(Y)$. It is proved that for any $Y \in \mathbb{R}^n$, $n \ge 2$, there is no Y-compatible polynomial function $f : \mathbb{R}^n \to \mathbb{R}^k$, $1 \le k < n$. It is also proved that for a differentiable strict Y-compatible map f, $J_f(Y) = 0$, where $J_f(Y)$ denote the Jacobian matrix of the mapping f in Y. These problems arose in studying data compression of analog signatures.

Keywords—Functions of several variables, Compatible functions, Strict compatible functions, Jacobian matrix, Data compression.

1. INTRODUCTION

This work was initiated by the problems of storage and processing of measured response data of analog circuits normally used by the fault dictionary techniques in fault localization [1,2]. We explore the possibility of data compression of a series of real numbers representing given response data. In particular, we are looking for some data compression function that would enable us to determine for any two given responses $y_1, y_2, \dots, y_n$ and $z_1, z_2, \dots, z_n$ whether $z_i \le y_i$ holds for all i merely on the basis of their compressed response data (i.e., signatures). If such data compression function existed, regions that characterize the response of a circuit could be simply described by the signatures of their margins. Besides, it would also be possible to determine from the signature of the response if the operation of a circuit-under-test lies in the given region or not. More details on the state-of-the-art and practical background on this issue are given in [3].

The terminology and notions used here, in general, follow [4]. Let $X = (x_1, x_2, \dots, x_n)$ and $Y = (y_1, y_2, \dots, y_n)$ be two vectors from $\mathbb{R}^n$. Then $X < Y$ means that $x_i < y_i$ holds for all i, and $X \le Y$ means that $x_i \le y_i$. Let $Y \in \mathbb{R}^n$. We call a function $f : \mathbb{R}^n \to \mathbb{R}^k$ *Y-compatible*, if for any $Z \in \mathbb{R}^n$, $Z \le Y$ if and only if $f(Z) \le f(Y)$. A function $f : \mathbb{R}^n \to \mathbb{R}^k$ is *strict Y-compatible*, if for any $Z \in \mathbb{R}^n$, $Z < Y$ if and only if $f(Z) < f(Y)$. Note that when we talk about Y-compatibility and strict Y-compatibility, Y is an arbitrary but fixed vector.

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The rest of this note is organized as follows. In the next section, we consider Y-compatible functions and prove that there are no such polynomial function when the dimension of the codomain is smaller than the one of the domain. In Section 3, we prove that for a strict Y-compatible map f, the Jacobian matrix of the mapping f in Y is equal to the zero matrix.

2. COMPATIBLE FUNCTIONS

Consider, for example, the following function $g : \mathbb{R}^n \to \mathbb{R}$:

$$g(X) = \begin{cases} (-1)^{n+1} (x_1 - y_1)(x_2 - y_2) \cdots (x_n - y_n), & X \le Y, \\ \max\{x_1 - y_1, x_2 - y_2, \dots, x_n - y_n\}, & \text{otherwise.} \end{cases}$$

It is easy to see that g is continuous and Y-compatible. Note that the definition of g involves Y. Furthermore, we would like that a Y-compatible function is a polynomial. In the rest of the section, we are going to show that there are no such functions for $k < n$. For the proof, we need the following simple observation.

LEMMA 2.1. *Let p be a real polynomial of n variables. If there exists a neighbourhood $\mathcal{U} \subset \mathbb{R}^n$ such that $p(X) = c$ on $\mathcal{U}$, then $p \equiv c$.*

PROOF. As p is a constant function on $\mathcal{U}$, all partial derivatives of p are equal to 0 on $\mathcal{U}$. Since the expansion of p into the Taylor series is finite, the result follows.

THEOREM 2.2. *For any $Y \in \mathbb{R}^n$, $n \ge 2$, there is no Y-compatible polynomial function $f : \mathbb{R}^n \to \mathbb{R}^k$, $1 \le k < n$.*

PROOF. For $m < n$, let $Y_m = (y_m, y_{m+1}, \dots, y_n)$, and let $S = \{Z \in \mathbb{R}^{n-1};\ Z < Y_2\}$.

Suppose on the contrary that $f : \mathbb{R}^n \to \mathbb{R}^k$ is Y-compatible, $f = (f_1, f_2, \dots, f_k)$. We claim that at least one of the following holds.

(i) There exist $j \in \{1, 2, \dots, k\}$, $Z \in S$, and a neighbourhood $\mathcal{U} \subset S$ of Z in $\mathbb{R}^{n-1}$ such that

$$f_j(y_1, X) - f_j(Y) = 0, \qquad \text{for every } X \in \mathcal{U}.$$

(ii) There exists $X \in \mathbb{R}^{n-1}$, $X < Y_2$, such that for all $i = 1, 2, \dots, k$,

$$f_i(y_1, X) - f_i(Y) < 0.$$

Suppose that (i) is not true. Let $Z \in S$, $Z < Y_2$, and let $\mathcal{U}_1 \subset S$ be a neighbourhood of Z. As (i) does not hold, there is $X_1 \in \mathcal{U}_1$, such that $f_1(y_1, X_1) - f_1(Y) < 0$. As f is continuous, there is a neighbourhood $\mathcal{U}_2$ of X_1 such that $\mathcal{U}_2 \subset \mathcal{U}_1$ and $f_1(y_1, W) - f_1(Y) < 0$ for every $W \in \mathcal{U}_2$. Then in $\mathcal{U}_2$, we can choose a vector X_2 such that $f_2(y_1, X_2) - f_2(Y) < 0$. If we proceed in this manner, we obtain nested neighbourhoods $\mathcal{U}_k \subset \mathcal{U}_{k-1} \subset \cdots \subset \mathcal{U}_1$. Finally, we choose $X \in \mathcal{U}_k$ such that $f_k(y_1, X) - f_k(Y) < 0$. Since $\mathcal{U}_k \subset \mathcal{U}_i$ for $i = 1, 2, \dots, k-1$, we also have $f_i(y_1, X_2) - f_i(Y) < 0$. Furthermore, as $X \in \mathcal{U}_1 \subset S$, it follows $X < Y$. Thus, (ii) holds, which proves the claim.

Suppose that (ii) holds. Then $f(y_1 + \delta, X) \le f(Y)$ for $\delta > 0$ small enough, therefore, f is not Y-compatible. Therefore, (i) must hold. Thus, there exists j such that $f_j(y_1, X) - f_j(Y) = 0$, for each X in some neighbourhood in $S \subset \mathbb{R}^{n-1}$. Consider $f_j(y_1, X) - f_j(Y)$ as a function of variables $x_2, x_3, \dots, x_n$. By Lemma 2.1, we have

$$f_j(y_1, X) - f_j(Y) = 0, \qquad X \in \mathbb{R}^{n-1}. \tag{1}$$

Set $g_i(Z) = f_i(y_1, Z)$, for $Z \in \mathbb{R}^{n-1}$ and $i = 1, 2, \dots, k$. Note that for all i, $g_i(Y_2) = f_i(Y)$, and by (1), $g_j(Z) = g_j(Y_2)$. We next want to show that Y-compatibility of f implies Y_2-compatibility of $g : \mathbb{R}^{n-1} \to \mathbb{R}^{k-1}$, $g = (g_1, \dots, g_{j-1}, g_{j+1}, \dots, g_k)$. Let $Z \in \mathbb{R}^{n-1}$ and let $Z \le Y_2$. Then $(y_1, Z) \le Y$ and $f(y_1, Z) \le f(Y)$. We conclude $g(Z) \le g(Y_2)$. Conversely, assume that $g(Z) \leq g(Y_2)$, i.e., $f_i(y_1, Z) \leq f_i(Y)$ for $i \neq j$. Since by (1) for $i = j$, $f_j(y_1, Z) = f_j(Y)$, we have $f(y_1, Z) \leq f(Y)$ which in turn implies $(y_1, Z) \leq Y$, and therefore, $Z \leq Y_2$.

By the above argument, an existence of Y-compatible function $f : \mathbb{R}^n \to \mathbb{R}^k$ implies an existence of Y_2-compatible function $g : \mathbb{R}^{n-1} \to \mathbb{R}^{k-1}$. We continue this procedure and finally end up with a $Y_k = (y_k, y_{k+1}, \dots, y_n)$-compatible real mapping h. Again, at least one of (i) and (ii) holds for h. If (ii) holds, then as above h is not Y_k-compatible. And if $h(y_k, X) - h(Y_k) = 0$ holds for $X \in \mathbb{R}^{n-k}$, we can easily choose X, which violates the compatibility. This final contradiction completes the proof.

Note that the theorem is clearly no longer true for $k = n$, as can be seen by considering the identity map. We also point out that by similar, but a little more careful argument, one can obtain the same result also for rational functions.

3. STRICT-COMPATIBLE FUNCTIONS

The function g defined in Section 2 can be made strict Y-compatible simply by changing $X \leq Y$ to $X < Y$. For another example, consider the following function $h : \mathbb{R}^n \to \mathbb{R}$ defined by

$$h(X) = \begin{cases} -\prod_{i=1}^n (x_i - y_i)^2, & X < Y, \\ \prod_{i=1}^n (x_i - y_i)^2, & \text{otherwise}. \end{cases}$$

It is easy to see that h is continuous, differentiable, and strict Y-compatible. However, h cannot be made Y-compatible. Note also that $\frac{\partial h}{\partial x_i}(Y) = 0$ for all i.

In the rest of this section, we assume that all functions are differentiable. Let $J_f(Y)$ be the Jacobian matrix of the mapping f in Y.

LEMMA 3.1. *Let $Y \in \mathbb{R}^n$, and let $f : \mathbb{R}^n \to \mathbb{R}^k$ be strict Y-compatible. Then for $D = J_f(Y)$, we have:*

(i) *for all $H \in \mathbb{R}^n$, $H < 0$ implies $DH \leq 0$;*
(ii) *for all $H \in \mathbb{R}^n$, $DH < 0$ implies $H < 0$.*

PROOF. As f is differentiable, for any $H \in \mathbb{R}^n$, we can write

$$f(Y + H) = f(Y) + DH + ||H||\, G(H),$$

where

$$\lim_{||H|| \to 0} G(H) = 0.$$

For any $t > 0$, we thus have

$$f(Y + tH) = f(Y) + t(DH + ||H||\, G(tH)). \tag{2}$$

Since f is strict Y-compatible for any $H < 0$, (2) gives

$$0 > f(Y + tH) - f(Y) = t(DH + ||H||\, G(tH)).$$

Now if (i) would not hold, say $(DH)_i > 0$ for some i, then for t small enough, the i^{th} component of $DH + ||H||\, G(tH)$ would still be positive, a contradiction. Thus, (i) holds.

Suppose next, $DH < 0$. Then by (2), $f(Y + tH) - f(Y) < 0$ holds for t small enough. Since f is strict Y-compatible, $tH < 0$. But this implies $H < 0$ and the proof is complete.

LEMMA 3.2. *Let D be a $k \times n$ matrix, $k < n$, and let $K \in \mathbb{R}^n$ be a vector such that $DK < 0$. Then there exists $H \in \mathbb{R}^n$, such that $DH < 0$ and $H \not< 0$.*

PROOF. Let $DK = W < 0$. Then, $DH = W$ can be interpreted as a consistent system of k linear equations with n variables $h_1, h_2, \dots, h_n$. Let $m = \operatorname{rank}(D)$. Since $m \leq k < n$, there exists an $m \times m$ submatrix with nontrivial determinant. We may, without loss of generality, suppose that this is the upper left submatrix. Then, the system $DH = W$ is equivalent to the system

$$h_i = g_i\left(h_{m+1}, h_{m+2}, \dots, h_n, w_1, w_2, \dots, w_k\right), \qquad i = 1, 2, \dots, m.$$

Thus, we can arbitrarily choose components $h_{m+1}, h_{m+2}, \dots, h_n$, and then calculate components $h_1, h_2, \dots, h_m$. In this way, we can obtain H such that $DH < 0$, but $H \not< 0$.

THEOREM 3.3. *For a strict Y-compatible map f, $J_f(Y) = 0$.*

PROOF. By Lemma 3.1 (i), $DH \leq 0$ for any $H < 0$. If for some H, $DH < 0$, then Lemma 3.2 contradicts 3.1 (ii). Thus, $DH = 0$ for all $H < 0$. Now let H be an arbitrary vector and define vectors H' and H'' in the following way:

$$H'_i = \begin{cases} -H_i, & H_i > 0, \\ 2H_i, & H_i < 0, \\ -1, & H_i = 0, \end{cases} \quad \text{and} \quad H''_i = \begin{cases} H_i, & H_i < 0, \\ -2H_i, & H_i > 0, \\ 1, & H_i = 0. \end{cases}$$

Then, $H' < 0$ and $H'' < 0$. Note also that $H = H' - H''$. Therefore, $DH = D(H' - H'') = D(H') - D(H'') = 0$. Hence, $DH = 0$ for any $H \in \mathbb{R}^n$, which is only possible if $J_f(Y) = 0$.

4. CONCLUSION

The presented proof of nonexistence of Y-compatible polynomial function indicates the limits in data compression of analog signatures. In the future, we may expect solutions which may only to a given (but still acceptable) degree satisfy the condition stated in the definition of Y-compatible polynomial function. Such is the case in digital signature analysis [5], where the probability of two different responses of equal length having the same signature is usually neglected in practice.

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