Extremal independent set enumeration in bipartite graphs via Young matrices

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Abstract

Enumerating independent sets in graphs under structural constraints is one of the central topics in extremal combinatorics. Let $i(G)$ denote the number of independent sets in a graph G. In this paper, we study the problem of determining, among all bipartite graphs with fixed order and size, those that maximize $i(G)$, let's call them optimal graphs. We show that every optimal graph must be a Ferrers graph. By introducing a weight function on the Young matrix of a Ferrers graph, we explicitly compute $i(G)$ and reformulate the extremal problem as a matrix-based optimization. Using refined compression and operations that preserve the Ferrers property, we prove that every optimal graph is lex bipartite. Our approach yields exact formulas for $i(G)$ in extremal cases and elucidates the structural properties of optimal Ferrers graphs.

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1 Introduction

An *independent set* in a graph is a set of vertices that induces no edges. For a graph G, let $\mathcal{I}(G)$ denote the family of all independent sets and write $i(G) = |\mathcal{I}(G)|$. The problem of enumerating independent sets is a classical topic in extremal graph theory and is closely related to combinatorial enumeration [2, 6, 22], statistical physics (through the hard-core model) [3, 7, 14, 25], and the complexity of approximate counting [9]. Although computing $i(G)$ exactly is generally computationally intractable [21], the extremal problem of determining the maximum possible value of $i(G)$ under structural constraints has received considerable attention.

A fundamental question in this direction is to determine which graphs maximize the number of independent sets under various restrictions. The extremal study of independent sets was initiated by Prodinger and Tichy [20], who proved that the star $K_{1,n-1}$ maximizes the number of independent sets among all trees on n vertices. A separate influential direction was proposed by Granville (see [2]), who asked which d-regular graphs on n vertices maximize the number of independent sets. Alon [2] and Kahn [14] conjectured that, when n is divisible by $2r$, the unique extremal graph is the disjoint union of $n/(2r)$ copies of $K_{r,r}$. Kahn proved the conjecture for bipartite graphs using entropy methods, and Zhao [25] later extended it to all regular graphs through a combinatorial reduction.

Subsequent developments further strengthened our understanding of extremal theory of independent sets. Davies, Jenssen, Perkins, and Roberts [7] obtained sharper results using the occupancy method, while Sah, Sawhney, Stoner, and Zhao [22] extended the extremal characterization to irregular graphs, showing that complete bipartite structures continue to play a central role. Degree-constrained versions of the problem have also been extensively studied. Sapozhenko [23] considered graphs with prescribed minimum degree, and Galvin [11] proved that for sufficiently large n, the graph $K_{\delta,n-\delta}$ uniquely maximizes the number of independent sets among n-vertex graphs of minimum degree δ. Further developments can be found in [1, 8, 17, 26].

Another natural direction is to impose restrictions on both the number of vertices and edges. Let $\mathcal{G}_{n,e}$ denote the family of graphs with n vertices and e edges. Using the Kruskal–Katona theorem [15, 18], Cutler and Radcliffe [5] proved that the lex graph maximizes the number of independent sets among all graphs in $\mathcal{G}_{n,e}$. This result suggests a natural bipartite analogue: *given integers n and e, which bipartite graphs with n vertices and e edges maximize the number of independent sets?* Unlike the unrestricted setting, where lexicographic constructions immediately arise, the bipartite case requires a finer structural analysis.

In this paper, we answer this question by developing a structural approach based on compression and Ferrers graphs. We first show that every extremal bipartite graph with prescribed order and size can be transformed into a Ferrers graph without decreasing the number of independent sets. This reduction allows us to represent the problem through Young matrices and to convert the enumeration of independent sets into a matrix optimization problem. Finally, by analyzing the structure of optimal Young matrices, we prove that every extremal bipartite graph is a lex bipartite graph, which is defined as follows, cf. [17].

Definition 1.1 *Let $n = r + \ell$ with $r \leq \ell$ and $e \leq \ell r$, and let $e = k\ell + c$, $0 \leq c < \ell$. The* lex bipartite graph *$K_{\ell,r}(c)$ with e edges and partite sets L and R of size ℓ and r, respectively, is the bipartite graph in which k vertices in R have degree ℓ, and possibly one vertex in R has degree c. In the case when $c = 0$, we may simplify the notation $K_{\ell,r}(0)$ to $K_{\ell,r}$.*

The paper is organized as follows. In the rest of this section, additional definitions required are given. In Section 2, we introduce Ferrers graphs and Young matrices and establish a reduction of the extremal problem to Ferrers graphs. In Section 3, we develop a framework for computing the number of independent sets in Ferrers graphs and derive a recursive description based on binary sequences. By introducing the notion of the weight of a Young matrix, we reformulate the extremal problem as a matrix optimization problem. In Section 4, we analyze optimal Young matrices using two structural operations and prove that every extremal bipartite graph is a lex bipartite graph.

For a positive integer k, we use $[k]$ to denote the set $\{1, \ldots, k\}$. Let $G = (V(G), E(G))$ be a graph and let $X \subseteq V(G)$. The set $N_G(X)$ consists of all vertices that have a neighbor in X. In the special case $X = \{x\}$, we write $N_G(x)$ for $N_G(\{x\})$, that is, the set of neighbors of x. We may omit the subscripts when the context is clear. Vertices x and y of G are *twins* if $N(x) = N(y)$. (They are often referred to as "false twins" in the literature, but since we are not interested in "true twins" here, we will simply call them "twins.") The relation of being a twin is an equivalence relation of $V(G)$, its equivalence classes are called *twin classes.* Finally, we write sequences such as $1^{x_1}0^{y_1}1^{x_2}\cdots$ to denote x_1 consecutive 1s, followed by y_1 consecutive 0s, then x_2 consecutive 1s, and so on.

2 Ferrers graphs and Young matrices

In this section, we introduce Ferrers graphs and their associated Young matrices. We begin with the definitions that will be used later, and then show that any extremal bipartite graph maximizing the number of independent sets must be a Ferrers graph.

A particularly important class of bipartite graphs in extremal combinatorics is the family of Ferrers graphs, also known as difference graphs [13], threshold bipartite graphs [19], or chain graphs [24]. These graphs have a rigid structure and can be defined in several equivalent ways. We now give a definition that best suits our needs.

Definition 2.1 *Let $n \geq 2$ and let $\sigma = (\sigma_i)_{i=1}^n$, $\sigma_i \in \{0, 1\}$, be a binary sequence with $\sigma_1 = 0$. Then the* Ferrers graph *of σ is a bipartite graph $\Gamma(\sigma) = (X, Y, E(G))$ constructed as follows. First set $X = Y = \emptyset$. Then, for each $i \in [n]$ in the natural order, add vertex i into X if $\sigma_i = 1$, otherwise add vertex i into Y. So X is the set of indices i such that $\sigma_i = 1$, while $Y = [n] \setminus X$. Finally, for each vertex $i \in X$, add edges between i and all the vertices $j \in Y$, $j < i$.*

This definition admits also disconnected Ferrers graphs. In particular, if σ is a constant binary sequence, then $\Gamma(\sigma)$ is an edgeless graph. The following equivalent description will be very useful to us.

Lemma 2.2 [13] *A graph is a Ferrers graph if and only if is bipartite and the neighborhoods of vertices in one partite set can be linearly ordered by inclusion.*

Note that if the neighborhoods of vertices in one partite set of a bipartite graph can be linearly ordered by inclusion, then the same property holds for the neighborhoods of vertices in the other partite set. Hence Lemma 2.2 can be also restated by saying that a graph is a Ferrers graph if and only if it is bipartite and the neighborhoods of vertices in each of the partite sets can be linearly ordered by inclusion.

A third equivalent definition of Ferrers graphs reads as follows. A bipartite graph $G = (X, Y, E(G))$, where $X = \{x_1, \ldots, x_p\}$ and $Y = \{y_1, \ldots, y_q\}$, is a *Ferrers graph*, if the following holds:

(i) if $\{x_i, y_j\} \in E(G)$, then $\{x_\ell, y_k\} \in E(G)$ for every $\ell \in [i]$ and every $k \in [j]$;

(ii) $\{x_p, y_1\} \in E(G)$.

If we want Ferrers graphs to correspond precisely to Ferrers diagrams, then we add $\{x_1, y_q\} \in E(G)$ to the condition (ii) above. This way, we ensure that Ferrers graphs are connected.

We next introduce a compression operation, a classical transformation widely used in extremal graph theory to analyze and optimize graph parameters (see [4,10,16]). Let u, v be two vertices of G. The *compression* of G from u to v, denoted $G_{u\to v}$, is obtained by removing every edge uw with $w \in N(u) \setminus (N(v) \cup \{v\})$ and adding the edge vw for each such w. All other edges remain unchanged. For any two distinct vertices x and y of a graph G, it was shown in [5] that $i(G) \leq i(G_{x\to y})$. The following result precisely determines when equality holds for bipartite graphs.

Lemma 2.3 *Let G be a bipartite graph, and let x, y lie in the same partite set. If $A = N(x) \setminus N(y)$ and $B = N(y) \setminus N(x)$, then $i(G) = i(G_{x\to y})$ if and only if $A = \emptyset$ or $B = \emptyset$.*

Proof. Let $G = (X, Y, E(G))$ be a bipartite graph, and let x and y be two vertices in the same part of G. Without loss of generality, assume that $x, y \in X$. Let $\mathcal{I}(G)$ and $\mathcal{I}(G_{x\to y})$ denote the families of all independent sets of G and $G_{x\to y}$, respectively.

Let $S \subseteq V(G)$. Since the compression $G_{x\to y}$ is obtained by removing all edges xu with $u \in A$ and adding all edges yu with $u \in A$, we infer that if $\{x, y\} \subseteq S$, then $S \in \mathcal{I}(G)$ if and only if $S \in \mathcal{I}(G_{x\to y})$. Analogously, if $S \cap \{x, y\} = \emptyset$, then $S \in \mathcal{I}(G)$ if and only if $S \in \mathcal{I}(G_{x\to y})$. Furthermore, if $S \cap \{x, y\} = \{x\}$ and $S \in \mathcal{I}(G)$, then $S \in \mathcal{I}(G_{x\to y})$. Consider now the families

$$\mathcal{A} = \{I \in \mathcal{I}(G) : \ y \in I, \ I \cap A \neq \emptyset\},$$
$$\mathcal{B} = \{J \in \mathcal{I}(G_{x\to y}) : \ x \in J, \ J \cap A \neq \emptyset\}.$$

Define $\phi : \mathcal{A} \to \mathcal{B}$ by $\phi(I) = I \triangle \{x, y\}$, i.e., replacing y by x. Then ϕ is well-defined and injective, hence $|\mathcal{A}| \leq |\mathcal{B}|$. Since

$$i(G_{x \to y}) - i(G) = |\mathcal{B}| - |\mathcal{A}|,$$

the equality $i(G) = i(G_{x \to y})$ holds if and only if ϕ is bijective. Thus every $J \in \mathcal{B}$ must admit a preimage $I = (J - \{x\}) \cup \{y\} \in \mathcal{I}(G)$, which is equivalent to $N(y) \cap (J \setminus \{x\}) = \emptyset$.

If $A = \emptyset$ or $B = \emptyset$, then the condition is trivially satisfied. In the first case no edges are modified, while in the second case the compression does not create unmatched configurations, so ϕ is bijective.

Conversely, assume that $A \neq \varnothing$ and $B \neq \varnothing$. Choose $a \in A$ and $b \in B$. Then $y \sim b$ and $x \nsim b$, while $x \sim a$ and $y \nsim a$. Consider $J = \{x, a, b\} \in \mathcal{B}$. However, the corresponding set $I = \{y, a, b\}$ is not independent in G because y is adjacent to b. Hence J has no preimage under ϕ, showing that the bijection fails. Therefore $|\mathcal{A}| < |\mathcal{B}|$, and thus $i(G) < i(G_{x \to y})$. $\square$

Let $\mathcal{F}_{n,e}$ denote the family of Ferrers graphs with n vertices and e edges; then $\mathcal{F}_{n,e} \subset \mathcal{G}_{n,e}$.

Theorem 2.4 *Let $G \in \mathcal{G}_{n,e}$ be a bipartite graph. Then there exists a Ferrers graph H from $\mathcal{F}_{n,e}$ such that*

$$i(G) \leq i(H)\,.$$

Moreover, equality holds if and only if G is already a Ferrers graph.

Proof. Let $X = \{x_1, \ldots, x_p\}$ and $Y = \{y_1, \ldots, y_q\}$ denote the bipartition of G, where $p+q = n$. Order the vertices of X and Y by non-increasing degree. Apply vertex compressions iteratively to all pairs of vertices in X and then in Y to make their neighborhoods nested. By Lemma 2.2, the resulting graph H is Ferrers. Moreover, vertex compression does not decrease the number of independent sets [5, Lemma 3.1], so $i(G) \leq i(H)$.

Finally, by Lemma 2.3, equality holds if and only if every vertex compression step is trivial, that is, the neighborhoods of vertices in each partite set are already nested. This is exactly the defining property of Ferrers graphs, so $i(G) = i(H)$ if and only if G itself is a Ferrers graph. $\square$

By Theorem 2.4 it suffices to restrict our attention to Ferrers graphs in our study. Let us say that a Ferrers graph $G \in \mathcal{F}_{n,e}$ is *optimal* if

$$i(G) = \max\{\, i(G') : \; G' \in \mathcal{F}_{n,e} \,\}\,.$$

To further analyze the structure of optimal Ferrers graphs, we introduce a matrix representation that allows compressions to be studied at the matrix level.

Let $G = (X, Y, E(G))$ be a Ferrers graph, and let X_i, $i \in [k]$, (resp. Y_i, $i \in [k]$) be the twin equivalence classes of X (resp. Y). Then

$$X = \cup_{i=1}^{k} X_i \quad \text{and} \quad Y = \cup_{i=1}^{k} Y_i.$$

Moreover, in view of Lemma 2.2 we may without loss of generality assume that

$$N(X_1) \supseteq \cdots \supseteq N(X_k) \quad \text{and} \quad N(Y_1) \supseteq \cdots \supseteq N(Y_k)\,.$$

Applying Lemma 2.2 once more, we obtain the neighborhoods for each X_i and Y_i:

$$N(x) = \cup_{j=1}^{k-i+1} Y_j \text{ for all } x \in X_i \quad \text{and} \quad N(y) = \cup_{j=1}^{k-i+1} X_j \text{ for all } y \in Y_j.$$

Thus, G is uniquely determined by the collections $(X_1, \ldots, X_k; Y_1, \ldots, Y_k)$. Let $|X_i| = x_i$ and $|Y_i| = y_i$ for $i \in [k]$. Then we say that $(x_1, \ldots, x_k; y_1, \ldots, y_k)$ is the *vertex-vector* of G (see [12]).

Definition 2.5 *The* Young matrix *associated with $G \in \mathcal{F}_{n,e}$ is the 0–1 matrix M whose rows and columns are indexed by the vertices of X and Y, respectively, in the above order. An entry of M is equal to 1 if the corresponding vertices are adjacent in G, and 0 otherwise.*

We now describe a compression operation on Young matrices associated with Ferrers graphs. Let M be such a Young matrix. An entry (i, j) of M is called an *outer-corner* if $m_{ij} = 1$ and $m_{i+1,j} = m_{i,j+1} = 0$, and an entry (k, ℓ) of M is an *inner-corner* if $m_{k\ell} = 0$ and $m_{k-1,\ell} = m_{k,\ell-1} = 1$. Assume that (i, j) is an outer-corner and (k, ℓ) is an inner-corner satisfying $i < k$ and $j < \ell$. The matrix $M_{(i,j)\to(k,\ell)}$ is obtained from M by replacing m_{ij} with 0, replacing $m_{k\ell}$ with 1, and leaving all other entries unchanged. The resulting matrix is again a Young matrix. This operation corresponds to moving the edge ij of the associated Ferrers graph to $k\ell$.

3 Independent sets in Ferrers graphs

In this section, we develop a framework for computing the number of independent sets in Ferrers graphs and derive a recursive description based on binary sequences. By introducing the weight of a Young matrix, we reformulate the extremal problem as a matrix optimization problem, which provides a structural characterization of optimal graphs.

The following theorem provides a recursive formula for the number of independent sets in Ferrers graphs when a new vertex is added, that is, a binary sequence is extended by one bit.

Theorem 3.1 *Let $\sigma = (\sigma_i)_{i=1}^n$, where $\sigma_1 = 0$ and $\sum_{i=1}^n \sigma_i = k$. If $G = \Gamma(\sigma)$ and $G' = \Gamma((\sigma_i)_{i=1}^{n+1})$, then*

$$i(G') = \begin{cases} 2i(G); & \sigma_{n+1} = 0\,, \\ i(G) + 2^k; & \sigma_{n+1} = 1\,. \end{cases}$$

Proof. Let v be vertex added to G to obtain G'. Let $I(G)$ be an arbitrary independent set of G. Clearly, $I(G)$ remains an independent set in G'. We now consider the contribution of the new vertex v.

If $\sigma_{n+1} = 0$, then v is isolated in G'. Each independent set $I(G)$ thus gives rise to exactly two distinct independent sets in G': one containing v and one not. Therefore, $i(G') = 2i(G)$.

If $\sigma_{n+1} = 1$, then v is adjacent to all vertices of G encoded by 0. In this case, every independent set consisting solely of the k vertices encoded by 1 in G, together with v, forms an independent set in G'. Since there are 2^k such independent sets in the subgraph induced by these k vertices, it follows that in this case we have $i(G') = i(G) + 2^k$. □

Definition 3.2 *Let $\sigma = (\sigma_i)_{i=1}^n$ be a binary sequence. Set $\sigma^{(0)}$ to be the empty sequence and for $i \in [n]$ let*

$$\sigma^{(i)} = \sigma_1 \dots \sigma_i.$$

Define $F_{\sigma^{(i)}} : \mathbb{N} \to \mathbb{N}$ inductively by

$$F_{\sigma^{(0)}}(k) = k$$

and, for $i = 0, \dots, n-1$,

$$F_{\sigma^{(i+1)}}(k) = \begin{cases} 2F_{\sigma^{(i)}}(k); & \sigma_{i+1} = 0, \\ F_{\sigma^{(i)}}(k) + 2^{p_i}; & \sigma_{i+1} = 1, \end{cases}$$

where $p_i = \sum_{j=1}^{i} \sigma_j$.

Lemma 3.3 *For all σ the function F_σ is a strictly increasing function with integer coefficients. Moreover, if $G = \Gamma(\sigma)$, then $i(G) = F_\sigma(1)$.*

Proof. The first statement follows easily by induction on $|\sigma|$ using Theorem 3.1. The base case, where G consists of a single vertex, is trivial. Let $\sigma' = \sigma_1 \dots \sigma_{n-1}$, where $\sigma_1 = 0$ and $\sum_{i=1}^{n-1} \sigma_i = k$, and let $G' = \Gamma(\sigma')$.

Assume now that $G = \Gamma(\sigma)$ and $\sigma = \sigma_1 \dots \sigma_n$. In the following, we will consider two cases. If $\sigma_n = 0$, then G is obtained from G' by adding one isolated vertex to G'. By Theorem 3.1, we have $i(G) = 2i(G')$. By the inductive hypothesis, $i(G') = F_{\sigma'}(1)$. It follows from Definition 3.2 that

$$i(G) = 2i(G') = 2F_{\sigma'}(1) = F_\sigma(1)\,.$$

If $\sigma_n = 1$, then G is obtained from G' by adding a vertex, say v, adjacent to all vertices encoded by 0 in G'. This generates 2^k new independent sets containing v in G. Again by Theorem 3.1 and Definition 3.2, we have

$$i(G) = i(G') + 2^k = F_{\sigma'}(1) + 2^k = F_\sigma(1)\,.$$

In both cases, the relation $i(G) = F_\sigma(1)$ holds, completing the proof. □

We next analyze the effect of removing a single edge from a Ferrers graph.

Lemma 3.4 *Let $G \in \mathcal{F}_{n,e}$ be a Ferrers graph and M be its associated Young matrix. If (i,j) is an outer-corner of M, and G^* denotes the graph obtained from G by removing the edge ij, then*

$$i(G^*) - i(G) = 2^{n-i-j}.$$

Proof. Let $\sigma = 0^{x_1}1^{y_1}\cdots 0^{x_k}1^{y_k}\cdots 0^{x_\ell}1^{y_\ell}$, where $x_1 \geq 1$, and let $G = \Gamma(\sigma)$ be the corresponding Ferrers graph, where $\sum_{t=1}^{\ell}(x_t + y_t) = n$. Since an outer-corner of the Young matrix corresponds to an occurrence of the pattern 01 in the binary sequence, removing the edge ij associated with this outer-corner changes the corresponding factor 01 in σ into 10. Moreover, removing an outer-corner from a Young matrix produces another Young matrix. Hence, G^* is also a Ferrers graph, and its binary sequence can be written as

$$\sigma^* = 0^{x_1}1^{y_1}\cdots 0^{x_k-1}101^{y_k-1}\cdots 0^{x_\ell}1^{y_\ell}\,,$$

where

$$i = \sum_{t=1}^{k} x_t \quad \text{and} \quad j = \sum_{t=k}^{\ell} y_t\,.$$

By Lemma 3.3, define

$$a = F_{0^{x_1}1^{y_1}\ldots 0^{x_k-1}}(1).$$

Using Definition 3.2, we obtain

$$\begin{aligned}
&F_{0^{x_1}1^{y_1}\ldots 0^{x_k-1}10}(1) - F_{0^{x_1}1^{y_1}\ldots 0^{x_k-1}01}(1)\\
&= F_{10}(a) - F_{01}(a)\\
&= F_0\left(a + 2^{\sum_{t=1}^{k-1} y_t}\right) - F_1(2a)\\
&= \left(2a + 2^{\sum_{t=1}^{k-1} y_t + 1}\right) - \left(2a + 2^{\sum_{t=1}^{k} y_t}\right)\\
&= 2^{\sum_{t=1}^{k-1} y_t}.
\end{aligned}$$

For convenience, let $a^* = F_{01}(a)$. Then the above equality implies that

$$F_{10}(a) = a^* + 2^{\sum_{t=1}^{k-1} y_t}.$$

By Lemma 3.3, we have $i(G) = F_\sigma(1)$ and $i(G^*) = F_{\sigma^*}(1)$. By repeatedly applying Definition 3.2 to the remaining suffix, we obtain

$$\begin{aligned}
&i(G^*) - i(G)\\
&= F_{\sigma^*}(1) - F_\sigma(1)\\
&= F_{0^{x_1}1^{y_1}\ldots 0^{x_k-1}101^{y_k-1}\ldots 0^{x_\ell}1^{y_\ell}}(1) - F_{0^{x_1}1^{y_1}\ldots 0^{x_k-1}011^{y_k-1}\ldots 0^{x_\ell}1^{y_\ell}}(1)\\
&= F_{101^{y_k-1}0^{y_{k+1}}\ldots 0^{x_\ell}1^{y_\ell}}(a) - F_{01^{y_k}0^{y_{k+1}}\ldots 0^{x_\ell}1^{y_\ell}}(a)\\
&= F_{1^{y_k-1}0^{y_{k+1}}\ldots 0^{x_\ell}1^{y_\ell}}(a^* + 2^{\sum_{t=1}^{k-1} y_t}) - F_{1^{y_k-1}0^{y_{k+1}}\ldots 0^{x_\ell}1^{y_\ell}}(a^*)\\
&= \cdots = 2^{\sum_{t=1}^{k-1} y_t + \sum_{t=k+1}^{\ell} x_t}\\
&= 2^{n-i-j}\,.
\end{aligned}$$

$\square$

Motivated by the Lemma 3.4, we now introduce the notion of the weight of a Young matrix. Let $M = [m_{ij}]_{p\times q}$ be the Young matrix associated with a Ferrers graph $G \in \mathcal{F}_{n,e}$, where $q \geq p \geq 1$. The *weight* of M, denoted by $w(M)$, is defined as

$$w(M) = 2^n - \sum_{i=1}^{p}\sum_{j=1}^{q} w_{ij},$$

where

$$w_{ij} = \begin{cases} 2^{n-i-j}; & \text{if } m_{ij} = 1\,, \\ 0; & \text{otherwise}\,. \end{cases}$$

Theorem 3.5 *Let $M = [m_{ij}]_{p\times q}$ be the Young matrix associated with a Ferrers graph $G \in \mathcal{F}_{n,e}$. Then*

$$i(G) = w(M)\,.$$

Proof. Let $E(G) = \{f_1, f_2, \ldots, f_e\}$ denote the edge set of G. Each edge f_t corresponds bijectively to a 1-entry (i_t, j_t) of the Young matrix M, where $i_t \in [p]$ and $j_t \in [q]$.

We define a sequence of graphs

$$G_0, G_1, \ldots, G_e = G,$$

where

$$G_t = \big(V(G), \{f_1, f_2, \ldots, f_t\}\big), \qquad t = 0, 1, \ldots, e.$$

Clearly, G_0 is the null graph on n vertices, and thus $i(G_0) = 2^n$.

The 1-entries of M can be ordered so that, for each $t \in [e]$, the entry (i_t, j_t) is an outer-corner of the submatrix corresponding to G_t. In particular, G_{t-1} is obtained from G_t by deleting the edge f_t, and is therefore again a Ferrers graph. Equivalently, G_t is obtained from G_{t-1} by adding the edge f_t.

By Lemma 3.4, for each $t \in [e]$,

$$i(G_{t-1}) - i(G_t) = 2^{\,n-i_t-j_t}\,.$$

Summing over all $t \in [e]$, it follows that

$$i(G) = i(G_0) - \sum_{t=1}^{e} 2^{\,n-i_t-j_t} = 2^n - \sum_{t=1}^{e} 2^{\,n-i_t-j_t}\,.$$

Note that each edge of G corresponds to a unique 1-entry of M, the sum above ranges over all entries (k, ℓ) with $m_{k\ell} = 1$. Hence,

$$\sum_{t=1}^{e} 2^{\,n-i_t-j_t} = \sum_{k=1}^{p}\sum_{\ell=1}^{q} w_{k\ell}\,,$$

where $w_{k\ell} = 2^{n-k-\ell}$ if $m_{k\ell} = 1$, and $w_{k\ell} = 0$ otherwise. Therefore, $i(G) = w(M)$. $\square$

Finally, we adopt an optimization viewpoint. Let $\mathcal{M}_{n,e}$ denote the family of all Young matrices associated with the graphs in $\mathcal{F}_{n,e}$. Since Ferrers graphs and Young matrices are in one-to-one correspondence, Theorem 3.5 shows that the problem of determining those graphs in $\mathcal{F}_{n,e}$ that maximize the number of independent sets is equivalent to the following matrix optimization problem:

$$\max \ w(M)$$

subject to

$$M \in \mathcal{M}_{n,e}\,.$$

4 Characterization of optimal graphs

In this section, we characterize the optimal graphs by studying their associated Young matrices. To this end, we introduce two structural operations on Young matrices and show that these operations force an optimal graph to be a lex bipartite graph.

Lemma 4.1 *Let $M = [m_{ij}]_{p\times q}$ be a Young matrix associated with a graph $G \in \mathcal{F}_{n,e}$. If an outer-corner (i,j) and an inner-corner (s,t) of M are not adjacent and $i + j \le s + t$, then*

$$w(M) \le w(M_{(i,j)\to(s,t)})\,.$$

Moreover, equality holds if and only if $i + j = s + t$.

Proof. By construction, the matrix $M_{(i,j)\to(s,t)}$ is obtained from M by changing the entry $m_{ij} = 1$ to 0 and the entry $m_{st} = 0$ to 1, while all other entries remain unchanged. By the definition of the weight of a Young matrix, it follows that

$$w\big(M_{(i,j)\to(s,t)}\big) - w(M) = w_{ij} - w_{st} = 2^{n-i-j} - 2^{n-s-t}\,.$$

Since $i + j \le s + t$, we have $2^{n-i-j} \ge 2^{n-s-t}$, and hence

$$w(M) \le w\big(M_{(i,j)\to(s,t)}\big)\,.$$

Moreover, equality holds if and only if $i + j = s + t$. $\square$

We define two operations on a Young matrix $M = [m_{ij}]_{p\times q}$ associated with a Ferrers graph G, where $q \ge p \ge 1$. Let t be the largest positive integer such that $e \le t(n - t)$. By definition, $q \le t$.

Diagonal shift. A *diagonal shift* moves each 1-entry (i.e., $m_{ij} = 1$) along the diagonal $i + j =$ const from a lower position to a higher one, that is, to a position previously occupied by a 0-entry, as far upward as possible while preserving the Ferrers property of the matrix. After performing this operation, the first several rows of M are completely filled with q entries equal to 1. Since the weight w_{ij} depends only on the sum $i + j$, the total weight of M is invariant under diagonal shifts.

Row-to-column filling. Suppose that $q < t$. If the last row of M contains c entries equal to 1, where $1 \leq c \leq p - 1$, we remove these 1-entries from the positions $(p, 1), \ldots, (p, c)$ and place them at the positions $(1, q + 1), \ldots, (c, q + 1)$, thereby extending M by one column. While the Young matrix contains a zero row and the number of columns has not yet reached t, we introduce additional columns and repeat the diagonal shift operation. The resulting matrix remains a Young matrix and hence corresponds to a Ferrers graph. Moreover, this operation strictly increases the total weight. By iterating this procedure, the number of columns of M can be increased until it reaches t.

Example 4.2 *Consider a Ferrers graph G with 14 vertices and 36 edges, and let $M = [m_{ij}]_{6\times 8}$ be its Young matrix. Entries equal to 1 are represented by filled circles "•" and entries equal to 0 by empty circles "∘" (see the first panel of Figure 1).*

By successively applying diagonal shifts, compression, and row-to-column filling, we obtain Young matrices M_3 and M_4 (see the remaining panels of Figure 1). Consequently, the graphs $K_{4,9} \cup K_1$ and $K_{4,10}(6)$ attain the maximum number of independent sets among all bipartite graphs with 14 vertices and 36 edges.

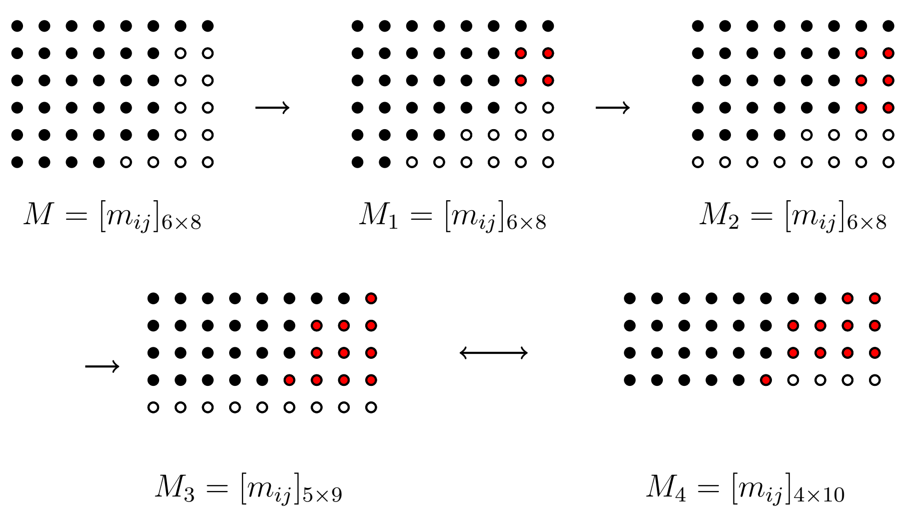

Figure 1: Compression process yielding two optimal graphs on 14 vertices and 36 edges.

Having established that every optimal graph is a Ferrers graph, we now turn to the central question of this work: *among all bipartite graphs on n vertices and e edges, which* *ones maximize the number of independent sets?* The following result provides a complete and explicit characterization of the optimal graphs.

Theorem 4.3 *Let n and e be integers with $0 \le e \le \lfloor n^2/4 \rfloor$, and let t be the largest positive integer satisfying $e \le (n-t)t$. Then every optimal graph of order n and size e is a lex bipartite graph. Moreover, the structure of all such optimal graphs is as follows.*

(i) *If $e = (n-t)(t-1)$, then the optimal graphs are*

$$K_{n-t,t}(c) \quad \textit{and} \quad K_{n-t,t-1} \cup K_1.$$

(ii) *If $e \neq (n-t)(t-1)$, then the unique optimal graph is*

$$K_{n-t,t}(c).$$

Proof. From Theorem 2.4, every optimal graph can be assumed to be a Ferrers graph. Let $G \in \mathcal{F}_{n,e}$ be an optimal graph with Young matrix M. By applying diagonal shifts and row-to-column fillings, and using Lemma 4.1, successive compression operations do not decrease the number of independent sets, and they strictly increase this number whenever a partially filled row exists. Hence, M may be assumed in canonical form: all rows but at most one are completely filled, and any remaining 1-entries appear consecutively from the left.

In this form, the corresponding Ferrers graph admits a bipartition (X, Y) in which all but at most one vertex of X are adjacent to all vertices of Y, and the remaining vertex (if present) is adjacent to an initial segment of Y. It follows that every optimal graph in $\mathcal{F}_{n,e}$ is a lex bipartite graph.

Let t be the largest integer such that $e \le (n-t)t$ and let $b = t(n-t) - e$. We now determine the precise structure of the optimal graphs. First, assume that $b \neq n-t$. Then compression yields a Young matrix with $n-t$ fully filled rows of length t and at most one additional row containing c consecutive 1-entries from the left. Since $0 < b < t$, this configuration uniquely determines the lex bipartite graph $K_{n-t,t}(c)$, and hence is the unique optimal graph.

Second, assume that $b = n-t$. In this case, the canonical Young matrix consists of $n-t$ rows each containing $t-1$ entries equal to 1, together with one zero row, which corresponds to $K_{n-t,t-1} \cup K_1$. Alternatively, applying diagonal shifts and row-to-column fillings produces a matrix with $n-t$ full rows of length t and one partially filled row, corresponding to $K_{n-t,t}(c)$. Hence, there are exactly two non-isomorphic optimal graphs. $\square$

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Declaration of interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Our manuscript has no associated data.

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