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# Graphs which locally mirror the hypercube structure

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## Abstract

Let  $Q_n$  be the  $n$ -cube and let  $Q_n^k$  be the subgraph of  $Q_n$  induced by the vertices at distance  $\leq k$  from a given vertex.  $Q_n^k$ -like graphs are introduced as graphs in which for any vertex  $u$  the set of vertices at distance  $\leq k$  from  $u$  induces a  $Q_n^k$ . Two characterizations of  $Q_n^k$ -like graphs are given and an  $O(d|V(G)|^2)$  recognition algorithm is presented, where  $d$  is the degree of a given  $d$ -regular graph  $G$ . Several examples of  $Q_n^k$ -like graphs are also listed. © 1999 Published by Elsevier Science B.V. All rights reserved.

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## 1. Introduction

Hypercubes form a class of graphs with amazingly much structure and have been studied extensively. However, not many graphs are hypercubes. It is therefore not surprising that they have been generalized in several directions. One can consider a class of graphs which contains hypercubes as a (very) special case. Median graphs and partial cubes form well-known bipartite generalizations of hypercubes, while Hamming graphs,  $(0, \lambda)$ -graphs and distance monotone graphs present some of the non-bipartite generalizations, see [2,3,8,10,11,13,14].

In interconnection network design theory, a hypercube is taken as a starting topology and then by some modification one tries to improve the topological prop-

erties of the network. Here we mention generalized hypercubes [4], incomplete hypercubes [12], and supercubes [16]. In this note we propose a class of graphs which contains hypercubes as a special case yet locally these graphs are hypercube like. Such graphs could be useful, for instance, when we plan an interconnection network in which a typical message is expected to be sent not far away because then a simple hypercube-like local routing using the Hamming distance could be used.

In the rest of this section we give necessary definitions. In the next section we introduce these graphs, characterize them in two ways and give an  $O(d \cdot |V(G)|^2)$  recognition algorithm, where  $d$  is the degree of a given graph. We conclude with several examples.

All graphs considered are connected and simple. The distance  $d(u, v)$  between vertices  $u$  and  $v$  of a graph  $G$  will be the usual shortest path distance. For a vertex  $u$  of a graph  $G$ , let

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$$N_k(u) = \{w \mid d(u, w) = k\} \quad \text{and}$$

$$N_{\leq k}(u) = \{w \mid d(u, w) \leq k\}.$$

The interval  $I(u, v)$  between vertices  $u$  and  $v$  of a graph  $G$  contains the vertices of  $G$  that lie on shortest paths between  $u$  and  $v$ . Let  $N_i(u, x) = I(u, x) \cap N_i(u)$ .

The hypercube  $Q_n$  of dimension  $n$  ( $n$ -cube for short) is a graph whose vertices are elements of the power set of an  $n$ -set, and two vertices are adjacent if the symmetric difference of the corresponding sets has exactly one element. Alternatively, the vertices of  $Q_n$  are all sequences of length  $n$  over  $\{0, 1\}$  and two vertices are adjacent if the corresponding sequences differ in precisely one coordinate. For any  $k$ , we denote by  $Q_n^k$  the subgraph of  $Q_n$  induced by the vertices represented by subsets of size at most  $k$ .

A connected graph  $G$  is a  $(0, 2)$ -graph if any two distinct vertices in  $G$  have exactly 2 common neighbors or none at all. It was proved in [13] that  $(0, 2)$ -graphs are regular.

Finally, for a graph  $G$ , let the *odd girth* of  $G$ ,  $og(G)$ , be the length of a shortest odd cycle of  $G$ , if there is one; otherwise the odd girth is defined to be infinite.

## 2. Characterizing and recognizing $Q_n^k$ -like graphs

Let  $k \geq 2$ . Then  $G$  is a  $Q_n^k$ -like graph if, for any vertex  $u$  of  $G$  the set  $N_{\leq k}(u)$  induces a  $Q_n^k$ . Thus, looking around from any vertex  $u$  to distance  $\leq k$ , the structure is the same as in the  $n$ -cube  $Q_n$ . Clearly,  $Q_n$  is a  $Q_n^k$ -like graph for every  $k \geq 2$ . Note also that  $Q_n^k$ -like graphs are  $n$ -regular.

If  $G$  is a  $Q_n^k$ -like graph, then, for each vertex  $u$ , we can provide all vertices in  $N_{\leq k}(u)$  with the *subset representation* with respect to  $u$ . More precisely, we represent  $u$  by  $\emptyset$ , and  $x \in N_i(u)$  ( $1 \leq i \leq k$ ) we represent by  $N_1(u, x)$ , which is a uniquely determined  $i$ -subset of  $N(u)$ .

In the next theorem we characterize  $Q_n^k$ -like graphs in two different ways. These characterizations could be deduced from results 4.3.6 and 4.3.7 of [6]. However, as our direct proof is quite short, we include it to make this note self-contained.

**Theorem 2.1.** *For any  $k \geq 2$  and any connected graph  $G$ , the following statements are equivalent.*

- (i)  $G$  is a  $Q_n^k$ -like graph,

- (ii) for any two vertices  $u$  and  $v$  of  $G$  with  $d(u, v) \leq k$ , the interval  $I(u, v)$  induces the hypercube  $Q_{d(u,v)}$  and  $og(G) \geq 2k + 3$ ,
- (iii)  $|I(u, v) \cap N(v)| = d(u, v)$ , for any  $u$  and  $v$  with  $d(u, v) \leq k$ , and  $og(G) \geq 2k + 3$ .

**Proof.** (i)  $\Rightarrow$  (ii)  $\Rightarrow$  (iii) These implications follow because intervals in hypercubes induce hypercubes.

(iii)  $\Rightarrow$  (i) First note that  $G$  is a  $(0, 2)$ -graph, thus  $G$  is regular of degree  $n$ , say. Let  $u$  be an arbitrary vertex of  $G$ . We will prove by induction on  $t$  that  $N_{\leq t}(u)$  induces a  $Q_n^t$ , for  $t \leq k$ . The statement is clearly true for  $t = 2$ , since  $G$  is a  $(0, 2)$ -graph with  $og(G) \geq 2k + 3$ .

Suppose now that for some  $t$ , with  $2 \leq t < k$ , the set  $N_{\leq t}(u)$  induces a hypercube  $Q_n^t$ . In particular, each vertex  $x$  in  $N_t(u)$  has  $t$  neighbors in  $N_{t-1}(u)$ . Since  $og(G) \geq 2k + 3 > 2t + 3$ , the other  $n - t$  neighbors of  $x$  lie in  $N_{t+1}(u)$ . Let  $w$  be an arbitrary vertex in  $N_{t+1}(u)$ . By the assumption,  $w$  has  $t + 1$  neighbors in  $N_t(u)$ , say  $v_1, v_2, \dots, v_{t+1}$ . We claim that for each  $i$  and  $j$ ,  $i \neq j$ ,  $v_i$  and  $v_j$  have their second common neighbor  $x$  in  $N_{t-1}(u)$ . Consider the  $t$  neighbors of  $v_i$  in  $N_{t-1}(u)$ . Each of them has a unique second common neighbor with  $w$  in  $N_t(u) \cap (N(w) \setminus \{v_i\})$ . Hence  $v_i$  and  $v_j$  have a common neighbor in  $N_{t-1}(u)$ .

Set  $\{1, 2, \dots, \hat{\ell}, \dots, k\} = \{1, 2, \dots, k\} \setminus \{\ell\}$ . Without loss of generality let  $v_1$  be represented by  $\{\hat{1}, 2, \dots, t + 1\}$  and let  $v_2$  be represented by  $\{1, \hat{2}, 3, \dots, t + 1\}$ . Consider the representation of  $v_\ell$ , for  $\ell \neq 1, 2$ . Without loss of generality let the common neighbor of  $v_1$  and  $v_\ell$  in  $N_{t-1}(u)$  be represented by  $A = \{\hat{1}, 2, \dots, \hat{\ell}, \dots, t + 1\}$ . Let the common neighbor of  $v_2$  and  $v_\ell$  in  $N_{t-1}(u)$  be represented by  $B = \{1, \hat{2}, \dots, \hat{\ell}_1, \dots, t + 1\}$ . Then  $v_\ell$  is represented by a  $t$ -subset which is contained in  $A \cup B$ . Hence  $\ell = \ell_1$  and  $v_\ell$  is represented by  $\{1, 2, \dots, \hat{\ell}, \dots, t + 1\}$ . Thus the neighbors of  $w$  in  $N_t(u)$  are represented by  $\{1, 2, \dots, \hat{\ell}, \dots, t + 1\}$ ,  $\ell = 1, 2, \dots, t + 1$ . It follows that we can represent  $w$  by  $\{1, 2, \dots, t + 1\}$ . Furthermore, vertices  $w$  and  $w'$  from  $N_{t+1}(u)$  have at most one common neighbor in  $N_t(u)$ , otherwise the interval between any two common neighbors would contain  $w$  and  $w'$  and some vertex from  $N_{t-1}(u)$  which is impossible. This implies that  $w$  and  $w'$  are represented by different  $(t + 1)$ -sets.

To complete the proof we must show that all  $(t + 1)$ -sets occur in  $N_{t+1}(u)$ . By the induction hypothesis, there are  $\binom{n}{t}$  vertices in  $N_t(u)$ , each having  $n - t$

neighbors in  $N_{t+1}(u)$ . Since a vertex from  $N_{t+1}(u)$  is adjacent to  $t + 1$  vertices in  $N_t(u)$  there are

$$\frac{n-t}{t+1} \binom{n}{t} = \binom{n}{t+1}$$

vertices in  $N_{t+1}(u)$ . Hence every  $(t + 1)$ -subset is used to represent a vertex in  $N_{t+1}(u)$ .  $\square$

For  $k = 2$ , Theorem 2.1 gives us:

**Corollary 2.2.** *Let  $G$  be a connected graph with  $og(G) \geq 7$ . Then  $G$  is a  $(0, 2)$ -graph if and only if  $G$  is a  $Q_n^2$ -like graph for some  $n$ .*

**Proposition 2.3.** *For a given graph  $G$  one can decide in  $O(n|V(G)|^2)$  time and  $|V(G)|^2$  space whether  $G$  is a  $Q_n^k$ -like graph.*

**Proof.** Consider the following procedure.

- (1) If  $G$  is not an  $n$ -regular connected graph then reject.
- (2) Compute the distance matrix of  $G$ .
- (3) For each pair of vertices  $u, v \in V(G)$  with  $d(u, v) \leq k$ , do the following:
  - (a) if  $|\{w|vw \in E(G), d(w, u) = d(v, u) - 1\}| \neq d(u, v)$ , then reject;
  - (b) if  $|\{w|vw \in E(G), d(w, u) = d(v, u) + 1\}| \neq n - d(u, v)$ , then reject.
- (4) If  $G$  was not rejected, then  $G$  is a  $Q_n^k$ -like graph.

Clearly, the first condition of Theorem 2.1(iii) is verified in step (3a). Since in step 1  $G$  was tested for  $n$ -regularity, step (3b) then ensures that  $og(G) \geq 2k + 3$ . Thus the procedure correctly recognizes  $Q_n^k$ -like graphs.

Step 1 can clearly be performed within the given time and space bounds.

Using Dijkstra's algorithm, step 2 can be computed in  $O(|V(G)||E(G)|)$  time and  $O(|V(G)|^2)$  space. Since  $G$  is a  $n$ -regular graph and

$$\sum_{v \in V(G)} d(v) = 2|E(G)|,$$

it follows that

$$O(|E(G)|) = O(n|V(G)|).$$

For step 3 we must compare at most  $O(|V(G)|^2)$  pairs of vertices. Let  $u, v$  be such a pair. Then we go through the adjacency list of  $v$  and count the number

of vertices  $w$  with  $d(w, u) = d(v, u) - 1$  and with  $d(w, u) = d(v, u) + 1$ . Because each distance can be accessed in  $O(1)$  time, the time needed for one such pair is  $O(n)$ . Thus the time complexity of step 3 is  $O(n|V(G)|^2)$ .  $\square$

### 3. Examples of $Q_n^k$ -like graphs

The Cartesian product  $G \square H$  of graphs  $G$  and  $H$  is the graph with vertex set  $V(G) \times V(H)$  and  $(u, u')(v, v')$  is an edge of  $G \square H$  whenever  $uv \in E(G)$  and  $u' = v'$ , or  $u = v$  and  $u'v' \in E(H)$ .

The Cartesian product of graphs is a natural construction, in particular it is quite frequent in the interconnection network design theory. Let us give some examples. First of all,  $Q_n$  is the Cartesian product of  $n$  copies of  $K_2$ . In addition, the so-called  $k$ -ary  $n$ -cube, cf. [1,5], is just the Cartesian product of  $n$  copies of the  $k$ -cycle. For one more example we point out that the cube-connected cycles, cf. [9,15], are spanning subgraphs of the Cartesian product of a cycle with a hypercube.

For our graphs the Cartesian product is important for the following reason.

**Proposition 3.1.** *Let  $G$  be a  $Q_n^k$ -like graph and let  $H$  be a  $Q_{n'}^{k'}$ -like graph. Then  $G \square H$  is a  $Q_{n+n'}^{\min\{k, k'\}}$ -like graph.*

**Proof.** If  $(u, u')$  and  $(v, v')$  are vertices of  $G \square H$ , then

$$I_{G \square H}((u, u'), (v, v')) = I_G(u, u') \times I_H(v, v').$$

Then for any vertices  $u, v$  in  $G$  with  $d(u, v) = s \leq k$  and any vertices  $u', v'$  in  $H$  with  $d(u', v') = s' \leq k'$ , the subgraph induced by  $I_{G \square H}((u, u'), (v, v'))$  is  $Q_{ss'}$ . Without loss of generality let  $k' \leq k$ . Then it follows that for any two vertices of  $G \square H$  with distance at most  $k'$  the interval between them induces a hypercube. Clearly,  $og(G \square H)$  is at least  $2k' + 3$ . Thus, from Theorem 2.1(ii) the result follows.  $\square$

**Corollary 3.2.** *Let  $G$  be a  $Q_n^k$ -like graph. Then for any  $m \geq 1$ ,  $G \square Q_m$  is a  $Q_{n+m}^k$ -like graph.*

A folded  $n$ -cube  $FQ_n$  is obtained from  $Q_n$  by identifying antipodal vertices. There is a distinction

between the even and the odd case. For  $n = 2k$ , the graph  $FQ_{2k}$  can be described as follows. It is obtained from  $Q_{2k}^k$  by identifying vertices corresponding to disjoint  $k$ -sets. It is denoted by  $\frac{1}{2}Q_{2k}$ . Note that  $\frac{1}{2}Q_{2k}$  is a bipartite graph. For  $n = 2k + 1$ ,  $FQ_{2k+1}$  can be obtained from  $Q_{2k+1}^k$  by connecting vertices corresponding to disjoint  $k$ -sets. The graph  $FQ_{2k+1}$  was named *extended odd graph* in [14] and denoted by  $E_{k+1}$ .

The graph  $\frac{1}{2}Q_{2k+2}$  is a  $Q_{2k+2}^k$ -like graph, and  $E_{k+2}$  is a  $Q_{2k+3}^k$ -like graph. These two facts can be most easily observed using the definition by identifying antipodal vertices.

Finally, we wish to add that it was recently proved [7] that the connectivity of an  $(0, 2)$ -graph equals its degree. Thus a  $Q_n^k$ -like graph has connectivity  $n$ , an important property for interconnecting networks.

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