

# On the irregularity of $\pi$ -permutation graphs, Fibonacci cubes, and trees

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## Abstract

The irregularity of a graph  $G$  is the sum of  $|\deg(u) - \deg(v)|$  over all edges  $uv$  of  $G$ . In this paper, this invariant is considered on  $\pi$ -permutation graphs, Fibonacci cubes, and trees. An upper bound on the irregularity of  $\pi$ -permutations graphs is given and  $\pi$ -permutation graphs that attain the equality are characterized. The concept of the irregularity is extended to arbitrary edge subsets and applied to permutation edges of  $\pi$ -permutation graphs. An exact formula for the irregularity of Fibonacci cubes is proved. An upper bound on the irregularity of trees in terms of the diameter is given and trees that attain the equality are characterized.

**Key words:** irregularity of graph;  $\pi$ -permutation graph; Fibonacci cube; tree

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## 1 Introduction

The degree of a vertex  $v$  of a graph  $G = (V(G), E(G))$  is denoted by  $\deg_G(v)$ . Graphs in which all vertices have the same degree, that is, regular graphs, are in the center of interest of the graph theory community. If  $G$  is not regular, then it is

called *irregular*, cf. [5], and one is interested in how irregular it is. For this sake let the *imbalance*  $\text{imb}_G(e)$  of an edge  $e = uv \in E(G)$  be defined by

$$\text{imb}_G(e) = |\deg_G(u) - \deg_G(v)|.$$

The imbalance of an edge is thus a local measure of non-regularity of a given graph, cf. [7], where Ramsey problems with repeated degrees were investigated. To measure graph's global non-regularity, different approaches have been proposed; they are nicely presented in two recent papers [4,31]. One of the most natural such measures is the *irregularity*  $\text{irr}(G)$  of  $G$  (lately also called the *Albertson index*) defined as [6]:

$$\text{irr}(G) = \sum_{uv \in E(G)} |\deg_G(u) - \deg_G(v)| = \sum_{e \in E(G)} \text{imb}_G(e).$$

Let us explicitly mention some of the papers in which the irregularity has been studied. In [29] related extremal problems are proved; the paper [40] reports several bounds on irregularity; the paper [15] gives a spectral bound for graph irregularity that improves a bound from [40]; in [20] bipartite graphs having maximum possible irregularity are determined; the irregularity of some graph families that are important in chemistry is reported in [3]; for the irregularity of cacti see [25]; see also [30] for the role of irregularity indices used as molecular descriptors. We also point out that graphs in which  $\text{imb}_G(e) = 1$  holds for all edges have been recently investigated in [18] and named stepwise irregular graphs.

In this paper we focus on the irregularity of three classes of graphs:  $\pi$ -permutation graphs, Fibonacci cubes, and trees. In the next section we give an upper bound on the irregularity of  $\pi$ -permutation graphs which is, roughly speaking, stronger by a factor of 4 than the corresponding bound for general graphs. We also characterize the  $\pi$ -permutation graphs that attain the equality. In Section 3 we extend the concept of the irregularity to arbitrary edge subsets and prove a related upper bound for the irregularity of permutation edges in  $\pi$ -permutation graphs. In Section 4 we prove an exact formula for the irregularity of Fibonacci cubes. In the final section we prove an upper bound for the irregularity of trees in terms of the diameter and characterize the graphs that attain the equality. We also give bounds for the irregularity of  $\pi$ -permutation graphs over trees.

In the rest of the introduction we give some further, basic definitions used in this paper. All graphs in this paper are simple and connected. The order (= number of vertices) and the size (= number of edges) of a graph  $G = (V(G), E(G))$  are denoted with  $n(G)$  and  $m(G)$ , respectively. If  $W \subseteq V(G)$ , then  $\langle W \rangle$  denotes the subgraph of  $G$  induced by  $W$ . The minimum and the maximum degree of vertices from  $G$  are denoted by  $\delta(G)$  and  $\Delta(G)$ , respectively. A vertex  $v \in V(G)$  with  $\deg_G(v) = n(G) - 1$  is a *universal vertex* of  $G$ . The distance  $d_G(u, v)$  between

vertices  $u$  and  $v$  of a graph  $G$  is the number of edges on a  $u, v$ -geodesic. The *diameter*  $\text{diam}(G)$  of  $G$  is the length of a longest geodesic in  $G$ . For a positive integer  $n$  we will denote the set  $\{1, \dots, n\}$  with  $[n]$ .

## 2 $\pi$ -permutation graphs

Let  $G'$  and  $G''$  be disjoint copies of a graph  $G$  and let  $\pi : V(G') \rightarrow V(G'')$  be a bijection, in other words,  $\pi$  is a permutation on  $V(G)$ . The  $\pi$ -permutation graph  $G^\pi$  of  $G$  has the vertex set  $V(G^\pi) = V(G') \cup V(G'')$  and the edge set  $E(G^\pi) = E(G') \cup E(G'') \cup E_G^\pi$ , where

$$E_G^\pi = \{uv : u \in V(G'), v \in V(G''), v = \pi(u)\}.$$

Hence, a  $\pi$ -permutation graph is obtained from two disjoint copies of a given graph by adding a matching between them. This concept was introduced half a century ago in [11] and further investigated in a series of papers including [10, 14, 17, 19, 39]. We point out that the term “permutation graph” is also frequently used for intersection graphs of the lines representing a permutation; see, for example [13]. In this paper we are only interested in the former variation which will be emphasized by speaking of  $\pi$ -permutation graphs and by the notation  $G^\pi$ .

Let  $G^\pi$  be a  $\pi$ -permutation and  $G'$  and  $G''$  be two isomorphic copies of  $G$  in  $G^\pi$ . If  $u \in V(G)$ , then the vertices corresponding to  $u$  in  $G'$  and  $G''$  will be denoted respectively by  $u'$  and  $u''$ . We begin with the following simple result.

**Proposition 2.1** *If  $G$  is a graph and  $\pi$  is a permutation on  $V(G)$ , then*

$$\text{irr}(G^\pi) = 2 \cdot \text{irr}(G) + \sum_{u \in V(G)} |\deg_G(u) - \deg_G(\pi(u))|.$$

*In particular, if  $\pi$  induces an automorphism of  $G$ , then  $\text{irr}(G^\pi) = 2 \cdot \text{irr}(G)$ .*

**Proof.** Let  $G'$  and  $G''$  be the isomorphic copies of  $G$  in  $G^\pi$ . If  $u \in V(G)$ , then  $\deg_{G^\pi}(u') = \deg_{G^\pi}(u'') = \deg_G(u) + 1$ , and, consequently,  $E(G')$  and  $E(G'')$  each contribute  $\text{irr}(G)$  to  $\text{irr}(G^\pi)$ . For the same reason, each matching edge  $u'\pi(u')$  contributes  $|\deg_G(u) - \deg_G(\pi(u))|$  to  $\text{irr}(G^\pi)$ , and, consequently, the first assertion follows. The second assertion then follows because automorphisms preserve degrees.  $\square$

In [2,36] it was proved in two different ways that if  $G$  is a graph of order  $n = n(G)$ , then

$$\text{irr}(G) \leq \left\lfloor \frac{n}{3} \right\rfloor \left\lceil \frac{2n}{3} \right\rceil \left( \left\lceil \frac{2n}{3} \right\rceil - 1 \right) = \left\lfloor \frac{n}{3} \right\rfloor \left( n - \left\lfloor \frac{n}{3} \right\rfloor \right) \left( n - \left\lfloor \frac{n}{3} \right\rfloor - 1 \right). \quad (1)$$

Moreover, let  $KS_{p,q}$ ,  $p, q \geq 1$ , be the *clique-star graph*, that is, the join of a complete graph  $K_p$  and an edge-less graph  $\bar{K}_q$ . (The *join* of graphs  $G$  and  $H$  is the graph obtained from the disjoint union of  $G$  and  $H$  by adding all possible edges between vertices of  $G$  and vertices of  $H$ .) Then the equality in (1) is attained if and only if  $G$  is  $KS_{\lfloor \frac{n}{3} \rfloor, \lceil \frac{2n}{3} \rceil}$  or, if  $n \equiv 2 \pmod 3$ ,  $G$  is  $KS_{\lceil \frac{n}{3} \rceil, \lfloor \frac{2n}{3} \rfloor}$ , see [2, Theorem 2.2]. This immediately implies the following lemma which we need in the proof of our below result analogous to (1) for  $\pi$ -permutation graphs.

**Lemma 2.2** *If  $G$  is a graph of maximum irregularity among all graphs of order  $n$ , then  $G$  has at most  $\lceil \frac{n}{3} \rceil$  universal vertices.*

**Theorem 2.3** *If  $G$  is a graph of order  $n = n(G)$  and  $\pi$  is a permutation on  $V(G)$ , then*

$$\text{irr}(G^\pi) \leq 2 \left\lceil \frac{n}{3} \right\rceil \left( \left\lfloor \frac{2n}{3} \right\rfloor^2 - 1 \right).$$

*Moreover, the equality holds if and only if  $G = KS_{\lceil \frac{n}{3} \rceil, \lfloor \frac{2n}{3} \rfloor}$ .*

**Proof.** Let a graph  $G$  and a permutation  $\pi$  of  $V(G)$  be selected such that the graph  $G^\pi$  has the maximum irregularity among all permutation graphs over graphs of order  $n$ . Let  $V(G) = U \cup W$ , where  $U = \{u_1, \dots, u_s\}$  is the set of universal vertices of  $G$  and  $W = V(G) \setminus U = \{w_1, \dots, w_{n-s}\}$ . Without loss of generality, we may assume that  $\deg_G(w_1) \leq \dots \leq \deg_G(w_{n-s})$ . Under these assumptions and with Lemma 2.2 in mind, we may assume without loss of generality that

$$\pi(u'_i) = w''_i, \pi(w'_i) = u''_i, i \in [s], \text{ and } \pi(w'_{s+i}) = w''_{n-i+1}, i \in [n-s]. \quad (2)$$

Let  $G'$  be the spanning subgraph of  $G$  obtained from  $G$  by removing all the edges of  $\langle W \rangle$ . If  $e = uw \in E(G)$ , where  $u \in U$  and  $w \in W$ , then

$$\text{imb}_{G'}(e) = \text{imb}_G(e) + \deg_{\langle W \rangle}(w). \quad (3)$$

If  $e \in E(\langle W \rangle)$ , then  $\text{imb}_G(e) \leq n - s - 3$  and consequently

$$\sum_{e \in E(\langle W \rangle)} \text{imb}_G(e) \leq \frac{1}{2} \sum_{w \in W} \deg_{\langle W \rangle}(w)(n - s - 3). \quad (4)$$

Setting

$$X = \sum_{e \in E_G^\pi} \text{imb}_{(G')^\pi}(e) - \text{imb}_{G^\pi}(e),$$

and having in mind that  $E_G^\pi = E_{G'}^\pi$ , we can estimate as follows:

$$\begin{aligned}
X &= 2 \sum_{i=1}^s (\text{imb}_{(G')^\pi}(u'_i w''_i) - \text{imb}_{G^\pi}(u'_i w''_i)) \\
&+ 2 \sum_{i=1}^{n-s} |\text{imb}_{(G')^\pi}(w'_{s+i} w''_{n-i+1}) - \text{imb}_{G^\pi}(w'_{s+i} w''_{n-i+1})| \\
&\geq 2 \sum_{i=1}^s \text{deg}_{\langle W \rangle}(w_i) - 2 \sum_{i=1}^{n-s} |\text{deg}_{\langle W \rangle}(w_{s+i}) - \text{deg}_{\langle W \rangle}(w_{n-i+1})| \\
&\geq 2 \left( \sum_{i=1}^s \text{deg}_{\langle W \rangle}(w_i) - \sum_{i=s+1}^{n-s} \text{deg}_{\langle W \rangle}(w_i) \right) \tag{5}
\end{aligned}$$

From (3)-(5) we get

$$\begin{aligned}
\text{irr}((G')^\pi) - \text{irr}(G^\pi) &\geq 2 \sum_{i=1}^{n-s} \text{deg}_{\langle W \rangle}(w_i) s - \sum_{i=1}^{n-s} \text{deg}_{\langle W \rangle}(w_i) (n-s-3) \\
&+ 2 \left( \sum_{i=1}^s \text{deg}_{\langle W \rangle}(w_i) - \sum_{i=s+1}^{n-s} \text{deg}_{\langle W \rangle}(w_i) \right) \\
&\geq 2 \left( s - \frac{1}{2}(n-s-3) - 1 \right) \sum_{i=1}^{n-s} \text{deg}_{\langle W \rangle}(w_i) \\
&= (3s - n + 1) \sum_{i=1}^{n-s} \text{deg}_{\langle W \rangle}(w_i).
\end{aligned}$$

Since  $G^\pi$  has maximum irregularity and the expression  $3s - n + 1$  is positive for  $s \geq \lceil \frac{n}{3} \rceil$ , we infer that  $\sum_{i=1}^{n-s} \text{deg}_{\langle W \rangle}(w_i) = 0$ . Hence  $G$  is a clique-star graph  $KS_{s, n-s}$ . For a fixed value of  $s$  we have

$$\max\{\text{irr}(KS_{s, n-s}^\pi) : \pi \text{ is a permutation}\} = 2s((n-s)^2 - 1).$$

If  $f(s) = 2s((n-s)^2 - 1)$ , then  $f(s)$  is maximized at  $s = \lceil \frac{n}{3} \rceil$ . Therefore we conclude that

$$\text{irr}(G^\pi) \leq 2 \left\lceil \frac{n}{3} \right\rceil \left( \left\lceil \frac{2n}{3} \right\rceil^2 - 1 \right).$$

□

Note that if  $n = n(G)$ , then  $n(G^\pi) = 2n$ . Hence the upper bound of Theorem 2.3 bounds  $\text{irr}(G^\pi)$  from the above with, roughly,  $\frac{1}{27}n(G^\pi)^3$ . On the other hand, the general bound (1) yields, roughly,  $\frac{4}{27}n(G^\pi)^3$ . Hence Theorem 2.3 improves the general

upper bound in the case of  $\pi$ -permutation graphs by, roughly speaking, again a factor of 4.

### 3 Irregularity of edge subsets and $\pi$ -permutation graphs

As a variant of the irregularity measure, Abdo, Brandt, and Dimitrov [1] suggested to consider the imbalance over all pairs of vertices, in this way introducing the *total irregularity*  $\text{irr}_t(G)$  of a graph  $G$  with

$$\text{irr}_t(G) = \sum_{\{u,v\} \subseteq V(G)} |\deg_G(u) - \deg_G(v)|.$$

The total irregularity has been compared with the irregularity in [12] see also [9, 35, 38]. For our purposes, however, it is useful to extend the concept of irregularity from the sum of the imbalances of all the edges of a graph to arbitrary edge subsets. More precisely, if  $F \subseteq E(G)$ , then let

$$\text{irr}_G(F) = \sum_{f \in F} \text{imb}(f).$$

Note that with this notation  $\text{irr}(G) = \text{irr}_G(E(G))$  and that Proposition 2.1 reads as:

$$\text{irr}(G^\pi) = 2 \cdot \text{irr}(G) + \text{irr}_{G^\pi}(E_G^\pi).$$

Hence,  $\text{irr}_{G^\pi}(E_G^\pi)$  is of special interest and in our next result we give a sharp upper bound for it.

**Theorem 3.1** *If  $G$  is a graph of order  $n$  and  $\pi$  a permutation on  $V(G)$ , then*

$$\text{irr}_{G^\pi}(E_G^\pi) \leq 2 \left\lfloor \frac{n-1}{2} \right\rfloor \left\lceil \frac{n-1}{2} \right\rceil.$$

*Moreover, equality holds if and only if  $G = KS_{\lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n+1}{2} \rceil}$  or  $G = KS_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n+1}{2} \rfloor}$ .*

**Proof.** Let  $G^\pi$  be a permutation graph that has maximum  $\text{irr}_{G^\pi}(E_G^\pi)$  among all  $G$  of order  $n$  and all permutations on  $V(G)$ . Moreover, among such graphs assume that  $G$  has maximum number of universal vertices. Let  $U = \{u_1, \dots, u_s\}$  and  $W = \{w_1, \dots, w_{n-s}\}$  be the sets of its universal and its non-universal vertices, respectively, defined just as in the proof of Theorem 2.3. Then (2) applies also to  $\pi$ .

We claim that  $\deg_W(w_i) = 0$  for  $i \in [n-s]$ . On the contrary, suppose that  $\deg_W(w_{n-s}) \geq 1$ . Let  $w_p$  and  $w_q$  be two adjacent vertices in  $W$ , where  $w_{n-s}$

is adjacent to  $w_p$  and non-adjacent to  $w_q$ . Consider the following transformation: remove the edge  $w_p w_q$ , then add an edge between  $w_{n-s}$  and  $w_q$ . Let  $H$  be the newly obtained graph. Then we have  $\deg_H(w_{n-s}) = \deg_G(w_{n-s}) + 1$  and  $\deg_H(w_p) = \deg_G(w_p) - 1$ . Moreover, the degrees of the other vertices remain the same. If  $\pi(w'_{n-s}) = w'_q$ , then  $\text{imb}_H(w'_{n-s}\pi(w'_{n-s})) = \text{imb}_G(w'_{n-s}\pi(w'_{n-s})) + 2$  otherwise  $\text{imb}_H(w'_{n-s}\pi(w'_{n-s})) = \text{imb}_G(w'_{n-s}\pi(w'_{n-s})) + 1$ . Also for  $w_q$ , we have  $\text{imb}_H(w'_q\pi(w'_q)) \geq \text{imb}_G(w'_q\pi(w'_q)) - 1$ . Therefore  $\text{irr}_H(E_H^\pi) \geq \text{irr}_G(E_G^\pi)$ . So we apply the above transformation until  $w_{n-s}$  is adjacent to all vertices of  $W$ . As we have assumed that  $G$  has the largest possible number of universal vertices, we have a contradiction.

Hence  $\deg_W(w_{n-s}) = 0$  and consequently  $\deg_W(w_i) = 0$ ,  $i \in [n-s]$ . This implies that  $\text{irr}_{G^\pi}(E_G^\pi) = \sum_{i=1}^s \text{imb}(u_i w_i) = 2(n-1-s)s$ . By Lemma 2.2,  $s \leq \lfloor \frac{n}{2} \rfloor$ . Hence  $\text{irr}_{G^\pi}(E_G^\pi)$  is maximized when  $s = \lfloor \frac{n-1}{2} \rfloor$  or  $s = \lceil \frac{n-1}{2} \rceil$  and then  $G = KS_{p,q}$  where  $p = \lfloor \frac{n-1}{2} \rfloor$  or  $p = \lceil \frac{n-1}{2} \rceil$  and  $q = n-p$ .  $\square$

We close this section with a certain sub-additivity result on  $\text{irr}_{G^\pi}(E_G^\pi)$ , where  $\pi\alpha$  denotes the composition of the permutations  $\pi$  and  $\alpha$ .

**Theorem 3.2** *If  $\pi$  and  $\alpha$  are permutations on  $V(G)$ , then*

$$\text{irr}_{G^{\pi\alpha}}(E_G^{\pi\alpha}) \leq \text{irr}_{G^\pi}(E_G^\pi) + \text{irr}_{G^\alpha}(E_G^\alpha).$$

**Proof.** Set  $\beta = \pi\alpha$ . Then, having in mind that the degree of each vertex of  $G^\beta$  (as well as of  $G^\pi$  and  $G^\alpha$ ) is by 1 larger than the degree of its corresponding vertex in  $G$ , we can estimate as follows:

$$\begin{aligned} \text{irr}_{G^\beta}(E_G^\beta) &= \sum_{v \in V(G)} \text{imb}_{G^\beta}(v'\beta(v')) = \sum_{v \in V(G)} |\deg_{G^\beta}(\beta(v')) - \deg_{G^\beta}(v')| \\ &= \sum_{v \in V(G)} |\deg_G(\beta(v)) - \deg_G(v)| \\ &= \sum_{v \in V(G)} |\deg_G(\beta(v)) - \deg_G(\alpha(v)) + \deg_G(\alpha(v)) - \deg_G(v)| \\ &\leq \sum_{v \in V(G)} (|\deg_G(\beta(v)) - \deg_G(\alpha(v))| + |\deg_G(\alpha(v)) - \deg_G(v)|) \\ &= \sum_{v \in V(G)} |\deg_{G^\pi}(\pi(v)) - \deg_{G^\pi}(v)| \\ &\quad + \sum_{v \in V(G)} |\deg_{G^\alpha}(\alpha(v)) - \deg_{G^\alpha}(v)| \\ &= \text{irr}_{G^\pi}(E_G^\pi) + \text{irr}_{G^\alpha}(E_G^\alpha) \end{aligned}$$

and we are done.  $\square$

## 4 Fibonacci cubes

Fibonacci cubes were introduced by Hsu [21] as an interconnection network model. Afterwards they have been studied from different perspectives; the developments until 2013 are summarized in the survey article [23]. Among the subsequent developments on Fibonacci cubes we point out the studies of the structure of their induced hypercubes [16,27,33] and to investigations of their domination invariants [8,22,32]. Moreover, Fibonacci cubes can be recognized in linear time [37]. In this section we add to the literature on the Fibonacci cubes their irregularity.

A *Fibonacci string* of length  $n$  is a binary string  $b_1 \dots b_n$  with  $b_i \cdot b_{i+1} = 0$  for  $1 \leq i < n$ , that is, a binary string that contains no consecutive 1s. The *Fibonacci cube*  $\Gamma_n$ ,  $n \geq 1$ , is the graph whose vertices are all Fibonacci strings of length  $n$ , two vertices being adjacent if they differ in a single coordinate. It is well-known that  $|V(\Gamma_n)| = F_{n+2}$ , where  $F_0 = 0$ ,  $F_1 = 1$ , and  $F_{n+2} = F_{n+1} + F_n$ ,  $n \geq 0$ , are the Fibonacci numbers.

**Theorem 4.1** *If  $n \geq 1$ , then*

$$\text{irr}(\Gamma_n) = \frac{2}{5} \left( (n-1)F_n + 2nF_{n-1} \right).$$

**Proof.** We proceed by induction on  $n$ . By a direct computation we see that  $\text{irr}(\Gamma_1) = 0$ ,  $\text{irr}(\Gamma_2) = 2$ ,  $\text{irr}(\Gamma_3) = 4$ ,  $\text{irr}(\Gamma_4) = 10$ , and  $\text{irr}(\Gamma_5) = 20$ , hence the stated formula holds for  $n \leq 5$ . From now on assume that  $n \geq 6$ .

Define the following subsets of vertices of  $\Gamma_n$ :

$$\begin{aligned} A_n &= \{00b_3 \dots b_n : b_3, \dots, b_n \in \{0, 1\}\}, \\ B_n &= \{10b_3 \dots b_n : b_3, \dots, b_n \in \{0, 1\}\}, \\ C_n &= \{010b_4 \dots b_n : b_4, \dots, b_n \in \{0, 1\}\}. \end{aligned}$$

The sets  $A_n$ ,  $B_n$ , and  $C_n$  are disjoint and  $V(\Gamma_n) = A_n \cup B_n \cup C_n$ . In addition, the subgraphs  $\langle A_n \rangle$ ,  $\langle B_n \rangle$ , and  $\langle C_n \rangle$  are isomorphic to  $\Gamma_{n-2}$ ,  $\Gamma_{n-2}$ , and  $\Gamma_{n-3}$ , respectively. Since each vertex from  $B_n$  has exactly one neighbor outside  $B_n$  (more precisely in  $A_n$ ), we see that

- $\text{irr}_{\Gamma_n}(E(\langle B_n \rangle)) = \text{irr}(\Gamma_{n-2})$ .

Similarly, since each vertex from  $C_n$  has exactly one neighbor outside  $C_n$  (more precisely in  $A_n$ ), we get

- $\text{irr}_{\Gamma_n}(E(\langle C_n \rangle)) = \text{irr}(\Gamma_{n-3})$ .



Consider now the edges  $uv$  from  $G[A_n]$ , where  $u = 00u_3 \dots u_n$  and  $v = 00v_3 \dots v_n$ . If  $u_3 = v_3$ , then  $u$  and  $v$  have the same number of neighbors outside  $A_n$ , Hence the irregularity of  $uv$  in  $\Gamma_n$  is equal to the irregularity of the corresponding edge in  $G[A_n] = \Gamma_{n-2}$ . Suppose now that  $u = 000u_4 \dots u_n$  and  $v = 001v_4 \dots v_n$ . In this case  $v_4 = 0$  and hence also  $u_4 = 0$ , so that  $u = 0000u_5 \dots u_n$  and  $v = 0010v_5 \dots v_n$ . The irregularity of such an edge in  $\Gamma_n$  is by 1 larger than the irregularity of the corresponding edge in  $\Gamma_{n-2}$ . Since there are precisely  $F_{n-2}$  such edges, we get

- $\text{irr}_{\Gamma_n}(E(\langle A_n \rangle)) = \text{irr}(\Gamma_{n-2}) + F_{n-2}$ .

We still need to consider the edges between  $A_n$  and  $B_n$  and between  $A_n$  and  $C_n$ . (Note that there are no edges between  $B_n$  and  $C_n$ .)

Consider first the edges  $uv$  between  $A_n$  and  $B_n$ , in which case  $u = 00u_3 \dots u_n$  and  $v = 10u_3 \dots u_n$ . Among them, the edges where  $u = 0000u_5 \dots u_n$  contribute  $F_{n-2}$ , the edges where  $u = 0010u_5 \dots u_n$  contribute nothing, and the edges where  $u = 00010u_6 \dots u_n$  contribute  $F_{n-3}$ . Hence

- the contribution of the edges between  $A_n$  and  $B_n$  is  $F_{n-2} + F_{n-3}$ .

Consider next the edges  $uv$  between  $A_n$  and  $C_n$ , in which case  $u = 000u_4 \dots u_n$  and  $v = 010u_3 \dots u_n$ . Among them, the edges where  $u = 0000u_5 \dots u_n$  contribute  $2F_{n-2}$  and the edges where  $u = 00010u_6 \dots u_n$  contribute  $F_{n-3}$ . Hence

- the contribution of the edges between  $A_n$  and  $C_n$  is  $2F_{n-2} + F_{n-3}$ .

We have thus considered all the edges of  $\Gamma_n$ . Putting together the above itemized contributions, we infer that

$$\begin{aligned} \text{irr}(\Gamma_n) &= 2 \cdot \text{irr}(\Gamma_{n-2}) + \text{irr}(\Gamma_{n-3}) + 4F_{n-2} + 2F_{n-3} \\ &= 2 \cdot \text{irr}(\Gamma_{n-2}) + \text{irr}(\Gamma_{n-3}) + 2F_n. \end{aligned}$$

Using the induction assumption we thus get

$$\begin{aligned} \text{irr}(\Gamma_n) &= \frac{4}{5} \left( (n-3)F_{n-2} + 2(n-2)F_{n-3} \right) + \\ &\quad \frac{2}{5} \left( (n-4)F_{n-3} + 2(n-3)F_{n-4} \right) + 2F_n \\ &= \frac{2}{5} \left( (n-1)F_n + 2nF_{n-1} \right), \end{aligned}$$

where the last equality follows by a lengthy but straightforward computation using the definition of the Fibonacci numbers.  $\square$

In [28] it is proved that  $m(\Gamma_n) = (nF_{n+1} + 2(n+1)F_n)/5$ . Hence, Theorem 4.1 has the following interesting consequence.

**Corollary 4.2** *If  $n \geq 2$ , then*

$$\text{irr}(\Gamma_n) = 2 \cdot m(\Gamma_{n-1}).$$

To conclude the section we remark that in [34] the so-called boundary enumerator polynomials  $D_k$  of hypercubes in Fibonacci cubes are considered and that the proof of  $D_1$  has similarities with our proof of Theorem 4.1. Namely,  $D_1$  is related with  $\sum_{uv \in E(\Gamma_n)} (\deg(u) + \deg(v) - 2)$  and the proof in [34] uses the same decomposition as we do here. Both proofs were produced independently thought (we are adding this remark while preparing a revised version).

## 5 Trees

The irregularity of trees has been already investigated. In [26] the irregularity of trees (and of unicyclic graphs) with given matching number was studied, while in [24] trees with minimum/maximum irregularity among the trees with given degree sequence and among the trees with given branching number were investigated. In the main result of this section we add to these studies the irregularity of trees of a given diameter and characterize the trees that attain the equality. Connecting the present section with Sections 2 and 3 we also find lower and upper bounds for the irregularity of the  $\pi$ -permutation graph of an arbitrary tree.

Let  $T_i(n, d)$ ,  $2 \leq i \leq d$ ,  $n \geq 2$ ,  $d \leq n - 1$ , denote the tree obtained from the path  $P_{d+1}$  by attaching  $n - d - 1$  leaves to the  $i^{\text{th}}$  vertex of  $P_{d+1}$ . Note that  $n(T_i(n, d)) = n$  and that  $\text{diam}(T_i(n, d)) = d$ . Observe also that if  $d = n - 1$ , then for every  $i$  the graph  $T_i(n, d) = T_i(n, n - 1)$  is the path on  $n$  vertices. Recall also that a tree is called a *caterpillar* if, after its leaves are removed, a path graph remains. In other words, a caterpillar is obtained from a path graph by attaching some leaves to its vertices. Thus, the trees  $T_i(n, d)$  belong to the class of caterpillars.

**Theorem 5.1** *If  $T$  is a tree of order  $n \geq 2$  and with  $\text{diam}(T) = d$ , then*

$$\text{irr}(T) \leq (n - d)(n - d + 1).$$

*where the equality holds if and only if  $T \in \{T_2(n, d), \dots, T_d(n, d)\}$ .*

**Proof.** Let  $T$  be a tree of order  $n \geq 2$  and diameter  $d$  such that  $\text{irr}(T)$  is the largest possible.

We claim first that  $T$  is a caterpillar and assume on the contrary that it is not. Let  $P$  be a diametrical path of  $T$ . Since  $T$  is not a caterpillar and the path  $P$  is diametrical,  $P$  contains an inner vertex  $y$  such that the rooted tree  $T_y$ , defined as

the maximal subtree of  $T$  with  $V(T_y) \cap V(P) = \{y\}$ , is of depth  $r \geq 2$ . Clearly,  $n(T_y) \geq 3$ .

Consider the following transformation. Let  $u \in V(T_y)$  be a vertex with  $d_{T_y}(y, u) = r - 2$ . (If  $r = 2$ , then  $u = y$ .) Let  $A = \{w_1, \dots, w_k\}$  be the set of down-neighbors of  $u$  in the rooted tree  $T_y$  and  $B = N_T(u) - A$ . Let  $S_i = N_T(w_i) - \{u\}$ ,  $i \in [k]$ , and  $S = \bigcup_{i=1}^k S_i$ . Set  $s = |S|$  and note that  $s = \sum_{i=1}^k (\deg_T(w_i) - 1)$ . Let now  $T'$  be the tree obtained from  $T$  by removing the edges between  $w_i$  and the vertices of  $S_i$ ,  $i \in [k]$ , and then added edges between vertices of  $S_i$  and  $u$ .

In  $T'$  the vertices  $w_i$ ,  $i \in [k]$ , as well as all the vertices from  $S$  are leaves. The contribution to the irregularity in  $T$  and  $T'$  differ only for the edges incident with  $u$  and  $w_i$ . Therefore, setting  $D = \text{irr}(T') - \text{irr}(T)$ , we have

$$\begin{aligned} D &= \sum_{i=1}^k (\text{imb}_{T'}(uw_i) - \text{imb}_T(uw_i)) + \sum_{i=1}^k \sum_{x \in S_i} (\text{imb}_{T'}(ux) - \text{imb}_T(w_i x)) \\ &\quad + \sum_{z \in B} (\text{imb}_{T'}(uz) - \text{imb}_T(uz)) \\ &= F_1 + F_2 + F_3, \end{aligned}$$

where

$$\begin{aligned} F_1 &= \sum_{i=1}^k [((\deg_T(u) + s) - 1) - |\deg_T(u) - \deg_T(w_i)|], \\ F_2 &= \sum_{i=1}^k \sum_{x \in S_i} [(\deg_T(u) + s - 1) - (\deg_T(w_i) - 1)], \\ F_3 &= \sum_{z \in B} [(\deg_T(u) + s - 1) - |\deg_T(u) - \deg_T(z)|]. \end{aligned}$$

Now we have

$$\begin{aligned}
F_1 &\geq \sum_{i=1}^k 2 \cdot \min\{\deg_T(u) - 1, \deg_T(w_i) - 1\} > 0, \\
F_2 &= \sum_{i=1}^k \sum_{x \in S_i} [(\deg_T(u) + s - 1) - (\deg_T(w_i) - 1)] \\
&= s \cdot \deg_T(u) + s^2 - \sum_{i=1}^k \deg_T(w_i)(\deg_T(w_i) - 1) \\
&= s \cdot \deg_T(u) + s^2 - \sum_{i=1}^k (\deg_T(w_i) - 1)^2 - s \\
&= s \cdot (\deg_T(u) - 1) + s^2 - \sum_{i=1}^k (\deg_T(w_i) - 1)^2, \\
F_3 &= \sum_{z \in B} [(\deg_T(u) + s - 1) - |\deg_T(u) - \deg_T(z)|] \geq -|B| \cdot s.
\end{aligned}$$

Note that  $|B| \in [2]$ . If  $|B| = 1$ , then

$$F_2 + F_3 \geq (s + s^2 - \sum_{i=1}^k (\deg_T(w_i) - 1)^2) - s \geq 0.$$

If  $|B| = 2$ , then  $u = y$  and  $\deg_T(u) \geq 3$ . Therefore,

$$F_2 + F_3 \geq (2s + s^2 - \sum_{i=1}^k (\deg(w_i) - 1)^2 - 2s) \geq 0.$$

Hence if  $S \neq \emptyset$ , then  $D > 0$ . Applying iteratively the above transformation as many times as required, we arrive at a caterpillar.

Let now  $T$  be a caterpillar of diameter  $d$  and let  $P$  be its diametrical path. Suppose that  $T \notin \{T_2(n, d), \dots, T_d(n, d)\}$ . Let  $v_1$  and  $v_2$  be vertices with the first and the second largest degree among the vertices of  $P$ , respectively. Note that  $\deg(v_1) \geq \deg(v_2) \geq 3$ . Let  $S_i = N(v_i) - V(P)$ ,  $i \in [2]$ . Remove the edges between  $v_2$  and  $S_2$ , add edges between the vertices of  $S_2$  and  $v_1$ , and denote the obtained tree with  $T'$ . The contribution of the edges incident by vertices  $v_1$  and  $v_2$  is changed in  $T'$  compared with  $T$ , while the contribution of every other edge to the irregularity

is the same. Therefore, setting  $D = \text{irr}(T') - \text{irr}(T)$ , we have

$$\begin{aligned}
D &= \sum_{w \in S_1} (\text{imb}_{T'}(v_1 w) - \text{imb}_T(v_1 w)) + \sum_{w \in S_2} (\text{imb}_{T'}(v_1 w) - \text{imb}_T(v_2 w)) \\
&\quad + \sum_{w \in N(v_1) \cap V(P)} (\text{imb}_{T'}(v_1 w) - \text{imb}_T(v_1 w)) \\
&\quad + \sum_{w \in N(v_2) \cap V(P)} (\text{imb}_{T'}(v_2 w) - \text{imb}_T(v_2 w)) \\
&= |S_1|(\deg_T(v_2) - 2) + (|S_2|((\deg_T(v_1) - 2) - (\deg_T(v_2) - 1))) \\
&\quad + 2(\deg_T(v_2) - 2) + \sum_{w \in N(v_2) \cap V(P)} (\text{imb}_{T'}(v_2 w) - \text{imb}_T(v_2 w)) \\
&\geq |S_1|(\deg_T(v_2) - 2) + |S_2|(\deg_T(v_1) - \deg_T(v_2) - 1) \\
&\quad + 2(\deg_T(v_2) - 2) - 2(\deg_T(v_2) - 2) \\
&= (\deg_T(v_2) - 2)(2\deg_T(v_1) + \deg_T(v_2) - 3) > 0.
\end{aligned}$$

This proves the theorem.  $\square$

To conclude the paper we find bounds for the irregularity of the  $\pi$ -permutation graphs of tree.

**Theorem 5.2** *If  $T$  is a tree of order  $n \geq 3$  and  $\pi$  a permutation on  $V(T)$ , then*

$$4 \leq \text{irr}(T^\pi) \leq 2n(n - 2).$$

*Moreover, the left equality holds if and only if  $T = P_n$  and  $\pi = \text{id}$ , and the right equality holds if and only if  $T = S_n$  and  $\pi$  is a permutation that maps the center of  $S_n$  into a leaf.*

**Proof.** From the fact that path  $P_n$  has the minimum possible irregularity among the trees of order  $n \geq 3$ , that is, irregularity 2, and by Proposition 2.1, we infer that  $\text{irr}(T^\pi) \geq \text{irr}(P_n^\pi)$  with equality holding if and only if  $T = P_n$  and  $\pi = \text{id}$ .

Let  $\pi$  be an arbitrary permutation on  $V(T)$ . Then we have

$$\begin{aligned}
\text{irr}(T^\pi) &= 2 \cdot \text{irr}(T) + \sum_{v \in V(T)} |\deg_{T^\pi}(v') - \deg_{T^\pi}(\pi(v'))| \\
&\leq 2 \cdot \text{irr}(T) + 2 \sum_{v \in V(T)} |\deg_T(v) - \delta(T)| \\
&= 2 \cdot \text{irr}(T) + 2(2m(T) - n(T)\delta(T)) \\
&\leq 2 \cdot \text{irr}(S_n) + 2(n - 2) \\
&= \text{irr}(S_n^{\pi'}).
\end{aligned}$$

In the last equality  $\pi'$  is a permutation on  $V(S_n)$  that maps the center of  $S_n$  to a leaf. Since  $S_n$  is the only tree with maximum irregularity, the right equality holds if and only if  $T = S_n$  and  $\pi$  is a permutation as just described.  $\square$

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