Coloring Graph Bundles

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ABSTRACT

Graph bundles generalize the notion of covering graphs and products of graphs. Several results about the chromatic numbers of graph bundles based on the Cartesian product, the strong product and the tensor product are presented.

1. INTRODUCTION

The *Cartesian product* $G \square H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ and $(a,x)(b,y) \in E(G \square H)$ whenever $ab \in E(G)$ and $x = y$, or $a = b$ and $xy \in E(H)$. The *tensor product* $G \times H$ of graphs G and H is the graph with vertex $V(G) \times V(H)$ and $(a,x)(b,y) \in E(G \times H)$ whenever $ab \in E(G)$ and $xy \in E(H)$. The *strong product* $G \boxtimes H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ and $E(G \boxtimes H) = E(G \times H) \cup E(G \square H)$.

Graph bundles [15,14] generalize the notion of covering graphs and Cartesian products of graphs. The notion follows the definition of fiber bundles and vector bundles that became standard objects in topology [8] as spaces that locally look like a product. Graph bundles corresponding to arbitrary graph products were introduced in [15; cf. also 13]. Let $\circ$ be a graph product operation, and let B and F be graphs. A graph X together with an onto map $p: X \to B$ that maps vertices to vertices and edges to either edges, or to vertices is a $\circ$-*bundle with base* B *and fiber* F if for each edge $e = uv \in E(B)$, the subgraph $p^{-1}(e)$ is isomorphic to the product $e \circ F = K_2 \circ F$ and the two F-layers in $e \circ F$ correspond to $p^{-1}(u)$ and $p^{-1}(v)$, respectively. The graph X is also called the *total graph* of the bundle. Intuitively, a bundle is a graph that is locally isomorphic to the product of the base with the fiber. In particular, the product $B \circ F$ (together with the natural projection on B) is a $\circ$-bundle. Suppose that the product $\circ$ is

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hereditary (as it is the case with all usual graph products), i.e., the product $G \circ F$ restricted to a subgraph H of G is equal to $H \circ F$. Then every bundle with base B and fiber F can be represented as a graph X with vertex set $V(X) = V(B) \times V(F)$ as follows. For each edge $e = uv \in E(B)$ choose an orientation of e (say from u to v) and choose an automorphism of the fiber, $\varphi(e) \in \mathrm{Aut}(F)$. Then define the edges of X as follows. If (u, f) and (v, g) are adjacent in the product $e \circ F$, then let (u, f) and $(v, \varphi(uv)(g))$ be adjacent in X. If (u, f) and (u, g) are adjacent in the product $\{u\} \circ F$, then let (u, f) and (u, g) be adjacent in X. The bundle defined this way will be denoted by $B \circ^{\varphi} F$. Note that the chosen edge orientations are used only implicitly. If one decides to take the other orientation of the edge e, the corresponding "voltage" $\varphi(e)$ should be replaced with the inverse automorphism in order to get the same graph. It is easy to see that all bundles can be obtained by the above construction even if we decide that for a chosen spanning forest T of B, all the "voltages" $\varphi(e)$, $e \in E(T)$, are fixed to be the trivial automorphisms of F. By this description it follows that in the case when B is a forest, or when F is asymmetric ($|\mathrm{Aut}(F)| = 1$), every bundle with base B and fiber F is isomorphic to the product $B \circ F$.

When speaking of a bundle, we will usually mention just its graph since the bundle projection will not be of any combinatorial importance to us. Unless there is a confusion, p will denote the bundle projection corresponding to the bundle in question.

As an example of a bundle that is not isomorphic to the product, let us take the Cartesian bundle whose base is the cycle C_k and fiber the path P_n, $C_k \square^{\varphi} P_n$, where all the automorphisms $\varphi(e)$ except one are trivial, and the remaining voltage is equal to the nontrivial automorphism of P_n. In case $n = 2$, the obtained graph bundle is known as the *Möbius ladder graph*. An example of a nontrivial strong bundle is depicted in Figure 1.

An *n-coloring* of a graph G is a function f from $V(G)$ to $\{1, 2, \ldots, n\}$, such that $xy \in E(G)$ implies $f(x) \neq f(y)$. The smallest number n for which an

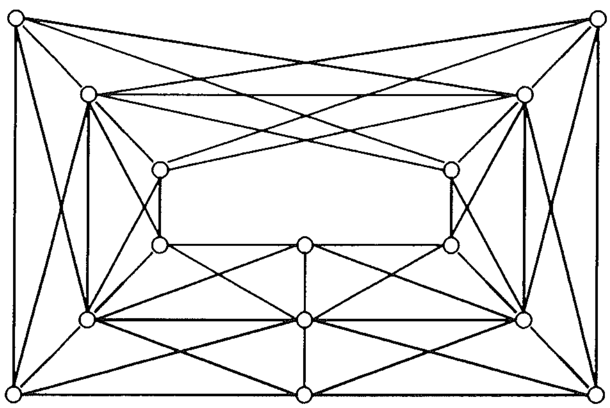

FIGURE 1. The nontrivial strong bundle $C_5 \boxtimes^{\varphi} P_3$.

n-coloring of G exists is the *chromatic number* $\chi(G)$ of G. The size of a largest complete subgraph of graph G will be denoted by $\omega(G)$. Clearly, $\omega(G) \leq \chi(G)$.

In the next section we give some lower and upper bounds for the chromatic number of Cartesian bundles of graphs. Many results are known about the chromatic number of strong products of graphs, [see 10,18,19,20 and also 4,5,9]. In Section 3 we present several basic results about the chromatic numbers of strong bundles of graphs. In the last section we briefly consider colorings of tensor graph bundles. Besides general upper and lower bounds, several constructions are added, showing that these bounds are best possible. Some further results concerning chromatic numbers of graph bundles are given in [11].

2. COLORING CARTESIAN BUNDLES

Sabidussi [16] showed that $\chi(G \square H) = \max\{\chi(G), \chi(H)\}$. This formula does not extend to Cartesian bundles and cannot even be weakened to an inequality (this was observed in [14]). For example, the Cartesian bundle $C_3 \square^{\varphi} C_3$ with $\lambda(\varphi)$ a reflection (see Figure 2) has chromatic number equal to 4. One can check this fact easily by considering the possible 3-colorings of the subbundle $P_3 \square C_3$.

The only general lower bound is the following fact, which is obvious since every Cartesian bundle $B \square^{\varphi} F$ contains copies of F.

Proposition 2.1. We have $\chi(B \square^{\varphi} F) \geq \chi(F)$.

The following example shows that the bound of Proposition 2.1 cannot be improved. Let $F = K_{n,n,\ldots,n}$ be the complete k-partite graph. Then $\chi(F) = k$. It is easy to see that for every graph B there is a bundle $B \square^{\varphi} F$ whose chromatic number is equal to k.

The upper bound $\chi(B \square^{\varphi} F) \leq \chi(B)\chi(F)$ (which also follows from Proposition 3.1 (ii)) is very rough. However, it cannot be improved in general as can be seen from the next theorem.

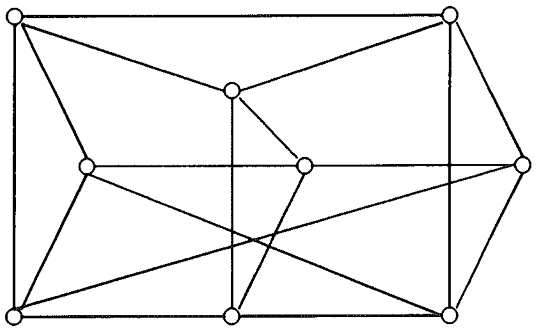

FIGURE 2. The reflective bundle $C_3 \square^{\varphi} C_3$.

Theorem 2.2. Let $m \geq 1$. Then for any $k \geq 1$ there exist a graph B_k with $\chi(B_k) = k$ and voltages φ such that

$$\chi(B_k \,\square^{\varphi}\, K_m) = km\,.$$

Proof. Clearly, $B_1 = K_1$ will do. For $k = 2$ we construct the graph $B_2 = K_{t_1, t_2}$ and voltages φ as follows. Let t_1 be such that any $(2m - 1)$-coloring of $Y = \overline{K}_{t_1} \square K_m$ has at least m fibers that posses identical coloring. Let t_2 be the number of different $(2m - 1)$-colorings of Y. Choose a bijective correspondence between the fibers over $\overline{K}_{t_2}$ and the $(2m - 1)$-colorings of Y. Let f be an arbitrary $(2m - 1)$-coloring of Y and let $H_1, H_2, \dots, H_m$ be copies of K_m in Y that are identically colored by f. Let H be the fiber over $\overline{K}_{t_2}$ corresponding to f. Set $V(H) = V(H_1) = V(H_2) = \cdots = V(H_m) = \{0, 1, \dots, m - 1\}$ and define voltages φ between H_i, $i = 1, 2, \dots, m$ and H in the following way:

$$\varphi(H_i, H)(j) = (i + j - 1) \pmod m, \ j = 0, 1, \dots, m - 1\,.$$

Then the coloring f on $Y \subset K_{t_1, t_2} \square^{\varphi} K_m$ cannot be extended to a $(2m - 1)$-coloring of $K_{t_1, t_2} \square^{\varphi} K_m$. Since we repeat the above construction for every $(2m - 1)$-coloring of Y (and define all the other voltages arbitrarily), it follows that $\chi(B_2 \square^{\varphi} K_m) > 2m - 1$. Thus $\chi(B_2 \square^{\varphi} K_m) = 2m$.

Assume that $B_s = K_{t_1, t_2, \dots, t_s}$, $s \geq 2$, is a graph with $\chi(B_s \square^{\varphi} K_m) = sm$. We construct the graph $B_{s+1} = K_{pt_1, pt_2, \dots, pt_s, t_{s+1}}$ with $\chi(B_{s+1} \square^{\psi} K_m) = (s + 1)m$ as follows.

Note first that the complete s-partite graph $K_{pt_1, pt_2, \dots, pt_s}$ contains p vertex-disjoint copies of the complete s-partite graph $K_{t_1, t_2, \dots, t_s}$. Let ψ be extended from $B_s \square^{\varphi} K_m$ to $K_{pt_1, pt_2, \dots, pt_s} \square^{\psi} K_m$ in a natural way, i.e., on every copy of $K_{t_1, t_2, \dots, t_s}$, ψ is identical with φ and the other voltages are arbitrary. Let t_{s+1} be the number of $((s + 1)m - 1)$-colorings of $K_{pt_1, pt_2, \dots, pt_s} \square^{\psi} K_m$. Choose p large enough, so that in any $((s + 1)m - 1)$-coloring, say f, of $K_{pt_1, pt_2, \dots, pt_s} \square^{\psi} K_m$, a particular coloring, say g, of a copy of $K_{t_1, t_2, \dots, t_s} \square^{\varphi} K_m$ repeats at least m times. By the induction assumption, g uses at least sm colors. As for the graph B_2 we now construct voltages ψ between fibers over $\overline{K}_{t_{s+1}}$ and identically colored fibers (for every coloring) in such a way that in the fiber layer over $\overline{K}_{t_{s+1}}$ corresponding to f, m new colors must be used. Therefore f cannot be extended to an $((s + 1)m - 1)$-coloring of $B_{s+1} \square^{\psi} K_m$. As the same holds for any coloring, the proof is complete. ∎

Let $V(B) = V_1 \cup V_2 \cup \cdots \cup V_k$ be a partition of the vertex set of a graph B. Then

$$\chi(B \square^{\varphi} F) \leq \sum_{i=1}^{k} \chi(\langle V_i \rangle \square^{\varphi} F) \qquad (1)$$

where $\langle V_i \rangle$ denotes the subgraph of B induced by V_i. Note that if we partition $V(B)$ into its color classes with respect to an optimal coloring, we obtain the (trivial) bound $\chi(B \,\square^{\varphi}\, F) \leq \chi(B)\chi(F)$. However, in some cases (1) can be used to give better upper bounds. Consider, for example, the wheel W_n. It is easy to partition its vertex set into two parts, each inducing a path. If F contains at least one edge, then $\chi(P_t \,\square^{\varphi}\, F) = \chi(P_t \,\square\, F) = \chi(F)$. Hence, $\chi(W_n \,\square^{\varphi}\, F) \leq 2\chi(F)$. Similarly, we see that $\chi(K_n \,\square^{\varphi}\, F) \leq \lceil n/2 \rceil \chi(F)$, if F contains at least one edge. In a special case we can further improve this bound. We need the following well-known result (see, for example, [1, Theorem 10.5.1]). Recall that the *deficiency* of a bipartite graph $G = (X \cup Y, E)$ is defined as

$$\max_{A \subseteq X}\{|A| - |\{y \in Y; xy \in E \text{ for some } x \in A\}|\}.$$

Theorem 2.3. The size of a maximum matching in a bipartite graph $G = (X \cup Y, E)$ is $|X| - d$, where d is the deficiency of G.

Theorem 2.4. For any $m \geq 2$, $n \geq 2$, and an arbitrary voltage assignment φ, we have

$$\chi(K_m \,\square^{\varphi}\, K_n) \leq m + n - 2.$$

Proof. The result holds for $m = 2$ since $K_2 \,\square^{\varphi}\, K_n$ is isomorphic to $K_2 \,\square\, K_n$.

Assume that the result is true for all k, $2 \leq k \leq m$, and consider a bundle $H_{m+1} = K_{m+1} \,\square^{\varphi}\, K_n$. Let v be a vertex of K_{m+1}. By the induction hypothesis, there is an $(m + n - 2)$-coloring, say c, of $H_m = (K_{m+1} \,\square^{\varphi}\, K_n) \backslash p^{-1}(v)$. Construct a bipartite graph G with a vertex partition $X \cup Y$ as follows. Let $X = V(p^{-1}(v))$, $Y = \{1, 2, \ldots, m + n - 2\}$, and $u \in X$ is adjacent to $y \in Y$ if and only if no vertex $w \in H_m$ with $c(w) = y$ is adjacent to u. Roughly speaking, u is adjacent to all colors y that can be used to color the vertex u when extending the coloring c of H_m to H_{m+1}. Observe that the degree of u in G is at least $n - 2$ for any $u \in X$. Hence the deficiency of G is at most 2. We distinguish two cases.

Case 1. The deficiency of G is 0 or 1.

By Theorem 2.3, G has a matching of size n or $n - 1$. Therefore, an $(m + n - 2)$-coloring of H_m can be easily extended to a $(m + n - 1)$-coloring of H_{m+1}.

Case 2. The deficiency of G is 2.

In this case, all the vertices in X have the same neighborhood in Y. Thus the coloring c is using exactly m colors. If $n \geq 3$, then we can recolor one of the vertices of H_m by a new color to get an $(m + n - 2)$-coloring whose corresponding deficiency will be 1. If $n = 2$, let $p^{-1}(v) = \{u_1, u_2\}$. The coloring c uses every color among $1, 2, \ldots, m$ exactly twice—once on a neighbor of u_1 and once on a neighbor of u_2. Pick vertices $w, w' \in H_m$ with $c(w) = c(w')$ such that u_1 is adjacent to w and u_2 to w'. Recoloring w' and u_1 with $m + 1$ and u_2 with $c(w)$ completes the proof. ∎

3. COLORING STRONG BUNDLES

Proposition 3.1. For any strong bundle $X = B \boxtimes^{\varphi} F$,

(i) $\omega(X) \leq \omega(B)\omega(F)$,
(ii) $\chi(X) \leq \chi(B)\chi(F)$.

Proof. (i) Let Q be a clique of X, $|Q| = \omega(X)$. Since $p(Q)$ is a complete subgraph of B, $|p(Q)| \leq \omega(B)$. Furthermore, $|Q \cap p^{-1}(v)| \leq \omega(F)$, for all $v \in V(B)$, and the result follows.

(ii) Let $\chi(B) = n$ and let $\{C_1, C_2, \ldots, C_n\}$ be the color classes of a coloring of B with $\{1, 2, \ldots, n\}$. Let $\chi(F) = m$ and, for each $v \in V(B)$ let f_v be a coloring of $p^{-1}(v)$ with $\{1, 2, \ldots, m\}$. It is straightforward to verify that $g\colon X \rightarrow \{1, 2, \ldots, nm\}$ defined by

$$g(x) = f_v(x) + (i - 1)m, \quad x \in p^{-1}(v),\ v \in C_i,$$

is a coloring of X. ∎

It follows from Proposition 3.1 that if $\omega(B) = \chi(B)$, $\omega(F) = \chi(F)$, and $\omega(X) = \omega(B)\omega(F)$, then $\chi(X) = \chi(B)\chi(F)$. However, it is not hard to find examples where $\omega(B) = \chi(B)$ and $\omega(F) = \chi(F)$ but $\chi(X) < \chi(B)\chi(F)$.

Consider the bundle $K_n \boxtimes^{\varphi} C_4$ with $\varphi(e)$ being a cyclic shift by 2 for every edge e. Color every C_4 the same to see that $\chi(K_n \boxtimes^{\varphi} C_4) = 4$. The example can be readily extended to bundles with any nontrivial base. These examples show that the inequality (ii) of Proposition 3.1 can be arbitrarily weak.

Theorem 3.2. Let $X = B \boxtimes^{\varphi} F$ be a strong bundle and let $\omega(B) = k$. Then $\omega(X) = \omega(B)\omega(F)$ if and only if there exist a clique Q *of* B, $V(Q) = \{v_1, v_2, \ldots, v_k\}$, *and cliques* $Q_1, Q_2, \ldots, Q_k$, $|Q_i| = \omega(F)$, $Q_i \subseteq p^{-1}(v_i)$, *such that for every* $e = v_i v_j \in E(Q)$, $\varphi(e)Q_i = Q_j$.

Proof. Suppose $\omega(X) = \omega(B)\omega(F)$. Let Q_X be a clique of X, $|Q_X| = \omega(X)$ and let $Q = p(Q_X)$. Clearly, Q is a clique of size $\omega(B)$. We claim that Q together with the cliques $\{Q_v = Q_X \cap p^{-1}(v) \mid v \in V(Q)\}$ fulfills the condition of the theorem. Assume not. Let there be an edge $e = uv \in E(Q)$ such that $\varphi(e)Q_u \neq Q_v$. Let $y \in V(Q_v \backslash \varphi(e)Q_u)$. As Q_X is complete, y is adjacent to every vertex in Q_u. Furthermore, y is adjacent to vertices in Q_u by "tensor" edges. Hence y is adjacent to every vertex in $\varphi(e)Q_u$. It follows that $\{y\} \cap \varphi(e)Q_u$ induce a complete graph, a contradiction.

Conversely, let there be cliques as in the theorem. Then they induce a clique of size $\omega(B)\omega(F)$. Proposition 3.1 (i) completes the proof. ∎

The above theorem immediately implies the following well-known result for the strong product of graphs.

Corollary 3.3. For any graphs G and H, $\omega(G \boxtimes H) = \omega(G)\omega(H)$.

We next improve the upper bound of Proposition 3.1 (ii) in two special cases.

Theorem 3.4. Let B be a graph, $\chi(B - v) < \chi(B)$ for some $v \in V(B)$. Then for any graph F,

$$\chi(B \boxtimes^{\varphi} F) \le \chi(B)\chi(F) - 2\chi(F) + \chi(K_2 \boxtimes F).$$

Proof. Let $\chi(F) = m$ and let $\chi(B) = n$. Let c be an $(n - 1)$-coloring of $B - v$ and let $C_1, C_2, \ldots, C_{n-1}$ be the corresponding color classes. For $i = 1, 2, \ldots, n - 1$ let $\tilde{C}_i = \bigcup_{u \in C_i} p^{-1}(u)$. Note that if $u, w \in C_i$, then no vertex from $p^{-1}(u)$ is adjacent to a vertex from $p^{-1}(w)$. Hence we may color $\tilde{C}_i$, $1 \le i \le n - 2$ using the colors $(i - 1)m + 1, \ldots, im$. Let $C_{n-1} = A \cup B$, where A consists of the vertices in C_{n-1} adjacent to v and B of the remaining vertices. Let $\tilde{A}$ and $\tilde{B}$ be the corresponding partition of $\tilde{C}_{n-1}$. Next, we color $\tilde{B}$ using a new set of m colors. Finally, it is easy to see that we can color $p^{-1}(v)$ and $\tilde{A}$ using $\chi(K_2 \boxtimes F)$ colors, and m of these colors may coincide with those used to color $\tilde{B}$. Since $\chi(K_2 \boxtimes F) \ge \chi(F) = m$, we got a coloring of $B \boxtimes^{\varphi} F$ using only $m(n - 2) + \chi(K_2 \boxtimes F)$ colors. ∎

Theorem 3.5. For any $k \ge 2$ and any graph F,

$$\chi(K_2 \boxtimes F) \le \chi(C_{2k+1} \boxtimes^{\varphi} F) \le 2\chi(F) + \left\lceil \frac{\chi(F)}{k} \right\rceil .$$

Proof. The lower bound is trivial since the subbundle over an edge is isomorphic to the strong product of the fiber F with K_2.

Let $V(C_{2k+1}) = \{v_0, v_1, \ldots, v_{2k}\}$. Let $\chi(F) = m$. Let A, B, and C be pairwise disjoint sets of colors, $|A| = |B| = m$, $|C| = \lceil m/k \rceil$. Note that $|A \cup B \cup C| = 2m + \lceil m/k \rceil$. Partition A into subsets $A_1, A_2, \ldots, A_k$ and B into subsets $B_1, B_2, \ldots, B_k$. In addition, if $m = qk + r$, $0 \le r < k$, then let the sets $A_1, \ldots, A_r$, $B_1, \ldots, B_r$ be of size $\lceil m/k \rceil$ and $A_{r+1}, \ldots, A_k$, $B_{r+1}, \ldots, B_k$ of size $\lfloor m/k \rfloor$. We color $C_{2k+1} \boxtimes^{\varphi} F$ in the following way. For

$i = 0, 1, \ldots, k$, color the vertices of $p^{-1}(v_{2i})$ with the colors in the sets

$$A_1, A_2, \ldots, A_{k-i}, B_{k-i+1}, B_{k-i+2}, \ldots, B_k$$

and for $i = 1, 2, \ldots, k$, color $p^{-1}(v_{2i-1})$ with the sets

$$B_1, B_2, \ldots, B_{k-i}, C, A_{k-i+2}, \ldots, A_k\,.$$

By construction, there is at least $\chi(F)$ colors in every fiber. Furthermore, any two consecutive fibers (including the closeup from $p^{-1}(v_{2k})$ to $p^{-1}(v_0)$) have no color in common. It follows that we constructed a coloring of $C_{2k+1} \boxtimes^{\varphi} F$. ∎

Let G be a graph and let Φ be a subgroup of Aut(G). For $k \geq \chi(G)$ we define the graph $G(\Phi, k)$ in the following way. Label G to obtain G'. The vertices of $G(\Phi, k)$ are all different k-colorings of G' and two vertices are adjacent if and only if there is a $\varphi \in \Phi$, such that the corresponding coloring of $K_2 \boxtimes^{\varphi} G'$ is proper. If $\Phi = \{id\}$ then we get the k-coloration graph of G defined in [19]. Note also that we may have loops in $G(\Phi, k)$.

Example. Let $G = C_4$ and let its consecutive vertices be labeled with a, b, c, and d. Let $\Phi = \{id, (ac)(bd)\}$. There are $4! = 24$ different 4-colorings of C_4 that use all four colors. Let f be a 4-coloring of C_4, say $f(a) = 1$, $f(b) = 2$, $f(c) = 3$, and $f(d) = 4$. It is easy to see that f is adjacent only to itself and to the coloring g defined by $g(a) = 3$, $g(b) = 4$, $g(c) = 1$, and $g(d) = 2$. It follows that $C_4(\Phi, 4)$ contains an induced subgraph isomorphic to 12 disjoint copies of K_2, each having a loop at both vertices. In addition $C_4(\Phi, 4)$ contains 48 vertices corresponding to all 3-colorings of C_4 that form 12 disjoint 4-cycles and 12 vertices corresponding to all 2-colorings of C_4 that form 3 disjoint 4-cycles.

Theorem 3.6. Let Φ be a subgroup of Aut(F) and $B \boxtimes^{\varphi} F$ a bundle where $\varphi(e) \in \Phi$, for each $e \in E(B)$. If $\chi(B \boxtimes^{\varphi} F) \leq k$, then there exists a graph homomorphism $g: B \to F(\Phi, k)$. Conversely, let $g: B \to F(\Phi, k)$ be a graph homomorphism. Then there exists ψ such that $\chi(B \boxtimes^{\psi} F) \leq k$ and $\psi(e) \in \Phi$ for every $e \in E(B)$.

Proof. Choose a labeling of F. We may assume that φ is trivial on a spanning subgraph T of B.

Let $\chi(B \boxtimes^{\varphi} F) \leq k$, $\varphi(e) \in \Phi$, $e \in E(B)$. Let f be a k-coloring of $B \boxtimes^{\varphi} F$. Define a mapping $g: V(B) \to F(\Phi, k)$ in the following way $v \in V(B) \mapsto x \in V(F(\Phi, k))$, where x is the vertex corresponding to the coloring of $p^{-1}(v)$ and the given labeling. Clearly, g is a homomorphism.

Conversely, let $g: B \to F(\Phi, k)$ be a homomorphism. Define ψ and a k-coloring of $B \boxtimes^{\psi} F$ in the following way. For $v \in V(B)$, color $p^{-1}(v)$

with the coloring $g(v)$, and for the edge $uv \in E(B)$, let $\psi(uv)$ be an element of Φ that justifies the adjacency of $g(u)$ and $g(v)$ in $F(\Phi, k)$. ∎

Corollary 3.7 [19]. $\chi(G \boxtimes H) \leq k$ if and only if there exists a homomorphism $G \rightarrow H(\{id\}, k)$.

Let $B \boxtimes^{\varphi} F$ be a strong bundle and let f: $B' \rightarrow B$ be a homomorphism. Denote by $B' \boxtimes^{\varphi \circ f} F$ the strong bundle where for $e' \in E(B')$, $(\varphi \circ f)(e') = \varphi(f(e'))$.

Proposition 3.8. Let f: $B' \rightarrow B$ be a graph homomorphism. Then

$$\chi(B' \boxtimes^{\varphi \circ f} F) \leq \chi(B \boxtimes^{\varphi} F).$$

Proof. Denote by r the bundle projection of $B' \boxtimes^{\varphi \circ f} F$ to B'. Define g: $B' \boxtimes^{\varphi \circ f} F \rightarrow B \boxtimes^{\varphi} F$ in the following way. If $x \in r^{-1}(v')$, $v' \in B'$, then let $g(x)$ be the vertex corresponding to x in $p^{-1}(f(x))$. It is clear that g is a graph homomorphism. Therefore, $\chi(B' \boxtimes^{\varphi \circ f} F) \leq \chi(B \boxtimes^{\varphi} F)$. ∎

4. COLORING TENSOR BUNDLES

In 1966 Hedetniemi [7] conjectured that for any graphs G and H, $\chi(G \times H) = \min\{\chi(G), \chi(H)\}$]. This conjecture is the most tempting problem connected with product colorings (see, for example, [2,3,6,17]) The conjecture cannot be extended to tensor bundles. For example, the nontrivial bundle $X = C_5 \times^{\varphi} K_2$ consists of two disjoint 5-cycles (see Figure 3), hence $\chi(X) = 3$.

Proposition 4.1. For any tensor bundle $X = B \times {}^{\varphi}F$, $\chi(X) \leq \chi(B)$.

Proof. Let $\chi(B) = m$. Let c be an m-coloring of B and let $C_1, C_2, \ldots, C_m$ be the corresponding color classes. Then $\bigcup_{u \in C_i} p^{-1}(u)$ is an independent set of vertices of X for $i = 1, 2, \ldots, m$. ∎

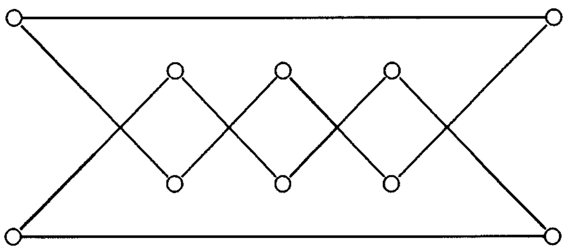

FIGURE 3. The nontrivial $C_5 \times^{\varphi} K_2$ bundle.

Let B be a connected graph and let F consist of connected components K^1, K^2, each isomorphic to K_n. If for every edge uv of B, $\varphi(uv)$ interchanges the two copies K^1, K^2 of $p^{-1}(u)$ and $p^{-1}(v)$, then it is easy to see that $B \times^{\varphi} F$ is bipartite. This example shows that the chromatic number of a tensor bundle can be arbitrarily smaller than the minimum of the chromatic numbers of the base and the fiber. Therefore any lower bounds on $\chi(B \times^{\varphi} F)$ are some interest.

Theorem 4.2. $\min\{m,n\} \leq \omega(K_m \times {}^{\varphi} K_n)$.

Proof. Let $V(K_m) = \{v_1, v_2, \ldots, v_m\}$ and $k = \min\{m,n\}$. We claim that for $i = 1,2,\ldots,k$, the subgraph $X_i = \bigcup_{j=1,2,\ldots,i} p^{-1}(v_j)$ contains a clique of size i. The claim is clearly true for $i = 2$. Assume that the claim holds for $s = k - 1$. Let Q be a corresponding clique in $X_s = |Q| = s$. Every vertex from Q is nonadjacent to exactly one vertex in $p^{-1}(v_k)$. Since $s \leq n - 1$, it follows that the vertices of Q have a common neighbor in $p^{-1}(v_k)$. ∎

From Proposition 4.1 and Theorem 4.2 we have

Corollary 4.3. $\min\{m,n\} \leq \chi(K_m \times {}^{\varphi} K_n) \leq m$.

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