Roots of Cube Polynomials of Median Graphs

Boštjan Brešar,[1] Sandi Klavžar,[2] and Riste Škrekovski[3,4]

[1]FEECS, UNIVERSITY OF MARIBOR
SMETANOVA 17
2000 MARIBOR, SLOVENIA
E-mail: bostjan.bresar@uni-mb.si

[2]DEPARTMENT OF MATHEMATICS
PEF, UNIVERSITY OF MARIBOR
KOROŠKA CESTA 160, 2000 MARIBOR, SLOVENIA
E-mail: sandi.klavzar@uni-mb.si

[3]DEPARTMENT OF MATHEMATICS
UNIVERSITY OF LJUBLJANA
JADRANSKA 19, 1111 LJUBLJANA, SLOVENIA

[4]PACIFIC INSTITUTE FOR THE MATHEMATICAL SCIENCES (PIMS)
BURNABY BC, CANADA V5A 1S6
E-mail: skreko@kam.mff.cuni.cz

Abstract: The cube polynomial $c(G,x)$ of a graph G is defined as $\sum_{i \geq 0} \alpha_i(G)x^i$, where $\alpha_i(G)$ denotes the number of induced i-cubes of G, in particular, $\alpha_0(G) = |V(G)|$ and $\alpha_1(G) = |E(G)|$. Let G be a median graph. It is proved that every rational zero of $c(G,x)$ is of the form $-((t+1)/t)$ for some integer $t > 0$ and that $c(G,x)$ always has a real zero in the interval $[-2,-1)$. Moreover, $c(G,x)$ has a p-multiple zero if and only if G is the Cartesian product of p trees all of the same order. Graphs of acyclic cubical complexes

are characterized as the graphs G for which $c(H, -2) = 0$ holds for every 2-connected convex subgraph H of G. Median graphs that are Cartesian products are also characterized. © 2006 Wiley Periodicals, Inc. J. Graph Theory 52: 37–50, 2006

1.   INTRODUCTION

Many different graph polynomials have been introduced in the literature, cf. [11]. One of the most fundamental questions about such polynomials is what can be said about their roots (zeros) and, moreover, can such an information be used to better understand (classes of) graphs. The zeros of the chromatic polynomial have probably been studied most extensively so far. The emphasis of the research is on the location of real zeros, in particular on the determination of zero-free intervals, see e.g. [6,16,26]. For zeros of some other important graph polynomials cf. [9,12,13].

One of the central roles in this article is reserved for median graphs. By now, this class of graphs has been extensively investigated and a rich structure theory is available, see the recent survey [17] and the books [14,20]. For instance, there is an interesting connection between median graphs and (bipartite or non-bipartite) triangle-free graphs which in particular implies that, intuitively speaking, there are as many median graphs as there are triangle-free graphs, see [15].

The cube polynomial introduced in [8] is closely related to median graphs. It has been basically introduced to present a general approach for obtaining the so-called Euler-type formulas for metrically defined classes of graphs, samples of them being presented in [7,18]. Along the way, several general properties of the cube polynomial were obtained, cf. Theorem 2.2 below.

The cube polynomial for median graphs has already found the following application in genetics [5]. Considering a so-called phantom mutation process, one wishes to identify the telltale patterns of the process. For this sake frequent mutations are filtered out and represented by their (quasi-)median network in order to visualize the character conflicts. The latter can be numerically expressed by the *cube spectrum*, which is an alternative name for the cube polynomial.

In this article, we study roots of the cube polynomial of median graphs. This in particular enables us to algebraically characterize several subclasses of median graphs: graphs of acyclic cubical complexes, median graphs that are Cartesian products, and Cartesian products of trees of the same order. So it is relevant to add that the original motivation for introducing median graphs arose in algebra *via* the so-called median algebras. Avann [1] established the relationship between median algebras and median graphs (called unique ternary distance graphs there) as well as later did Nebeský [22]. For more information on median algebras (and their relations to median graphs) see [4,20].

In the next section, we recall the notations, concepts, and results needed later. In particular we present several basic properties of the cube polynomial. The main result of Section 3 asserts that a rational zero of the cube polynomial of a median

graph G is of the form $-((t+1)/t)$ for some $t \in \mathbb{N}$. We also show that $c(G,x)$ has no zeros in $[-1, \infty)$. Then we characterize graphs of acyclic cubical complexes as those graphs G for which $c(H, -2) = 0$ holds for every 2-connected convex subgraph H of G. In Section 5, we prove that $c(G,x)$ has a real zero in the interval $[-2, -1)$ for every non-trivial median graph G. In the following section, we first characterize median graphs that are Cartesian products using the concept of the cube polynomial. This result, together with several other previously proved results, enables us to prove that for a median graph G, $c(G,x)$ has a p-multiple zero if and only if G is the Cartesian product of p trees all of the same order.

2.   PRELIMINARIES

For $u, v \in V(G)$, let $d_G(u,v)$ denote the length of a shortest path (also called *geodesic*) in G from u to v. A subgraph H of a graph G is an *isometric* subgraph if $d_H(u,v) = d_G(u,v)$ for all $u, v \in V(H)$. A subgraph H of a graph G is *convex* if for any two vertices u, v of H all shortest paths between u and v in G are already in H.

A graph G is a *median graph* if there exists a unique vertex x to every triple of vertices u, v, and w such that x lies simultaneously on a shortest u, v-path, a shortest u, w-path, and a shortest w, v-path. The vertex x is called the *median* of the triple u, v, w. It follows immediately from the definition that median graphs are bipartite. Let $\mathcal{M}$ denotes the class of all median graphs and $\mathcal{M}^*$ the class of all graphs whose connected components are median graphs.

The *Cartesian product* $G \square H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$ and $(a,x)(b,y) \in E(G \square H)$ whenever either $ab \in E(G)$ and $x = y$ or $a = b$ and $xy \in E(H)$. The n-cube Q_n is the Cartesian product of n copies of the complete graph on two vertices K_2.

Edges $e = xy$ and $f = uv$ of a graph G are in the Djoković–Winkler relation Θ [10,25] if

$$d_G(x,u) + d_G(y,v) \neq d_G(x,v) + d_G(y,u).$$

Relation Θ is reflexive and symmetric. If G is bipartite, then Θ can be defined as follows: $e = xy$ and $f = uv$ are in relation Θ if $d(x,u) = d(y,v)$ and $d(x,v) = d(y,u)$. Median graphs embed isometrically into hypercubes, see [19,20], hence by Winkler's result from [25] relation Θ is transitive on median graphs. By $\mathcal{F}(G)$, denote the set of edges consisting of representatives of the Θ classes of a median graph G.

A *cover* of a graph G is a pair of induced subgraphs G_1, G_2 of G such that $V(G) = V(G_1) \cup V(G_2)$. We say that a cover G_1, G_2 is *cubical*, if every induced hypercube of G is contained in at least one of the G_1 and G_2. A cover G_1, G_2 of G is called *convex* if $G_1 \cap G_2$ induces a convex subgraph of G. Note that in a cubical cover there is no edge between a vertex of $G_1 \setminus G_2$ and a vertex of $G_2 \setminus G_1$. Moreover, in a convex cover G_1, G_2 the subgraphs G_1 and G_2 are necessarily convex as well.

Let G be a connected graph. The *expansion* G^* of G with respect to the cubical cover G_1, G_2 of G is the graph constructed as follows. Let G_i^* be an isomorphic copy of G_i. For $i = 1, 2$ and for every vertex u of $G_0 = G_1 \cap G_2$, let u_i be the corresponding vertex in G_i^*. Then G^* is constructed from the disjoint union of G_1^* and G_2^* so that for each u of G_0 the corresponding vertices u_1 and u_2 are joined by an edge. An expansion with respect to the cubical cover G_1, G_2 is called *peripheral* if $G_1 \subseteq G_2$ or $G_2 \subseteq G_1$. Combining Mulder's expansion theorem [19,20] with his [21, Lemma 9], cf. also [14, Lemma 2.36], we have:

Theorem 2.1.   *Let G be a connected graph. Then G is a median graph if and only if G can be obtained from the one vertex graph by a sequence of peripheral convex expansions.*

For a graph G, let $\alpha_i(G)$, $i \geq 1$, denotes the number of induced i-cubes of G. So $\alpha_1(G)$ is the number of edges of G. We also set $\alpha_0(G)$ to be the number of vertices of G. Then the *cube polynomial* $c(G, x)$ of G is defined as

$$c(G, x) = \sum_{i \geq 0} \alpha_i(G) x^i .$$

For an edge $e = uv$ of a median graph G let U_e be the subgraph of G induced by the vertices x of G such that there is an edge $f = xy$ with $e \Theta f$ and $d(u, x) < d(u, y)$. Let, in addition, the *derivative* ∂G of a median graph G be the disjoint union of the graphs U_e, $e \in \mathcal{F}(G)$. If $G \in \mathcal{M}^*$, then $G = G_1 \cup G_2 \cup \cdots \cup G_s$, where each G_i is a median graph, so the derivative of G can be defined as $\partial G = \partial G_1 \cup \partial G_2 \cup \cdots \cup \partial G_s$. Moreover, one also defines higher derivatives in the following way. For $k \geq 0$, set $\partial^k G = G$, if $k = 0$, and $\partial^k G = \partial(\partial^{k-1} G)$, if $k \geq 1$. As usual, by $c'(G, x)$ we denote the derivative of $c(G, x)$, and by $c^{(k)}(G, x)$ its k-th derivative.

From [8] we recall the following properties of the cube polynomial to be used in the sequel.

Theorem 2.2.   *Let $c(G, x)$ be the cube polynomial of a graph G.*

  (i) *Let G be the expansion with respect to the cubical cover G_1, G_2 and let $G_0 = G_1 \cap G_2$. Then $c(G, x) = c(G_1, x) + c(G_2, x) + x\, c(G_0, x)$.*
 (ii) *For every median graph G, it holds $c(G, -1) = 1$.*
(iii) *For every graphs G and H, it holds $c(G \square H, x) = c(G, x)c(H, x)$.*
 (iv) *For every median graph G and every integer $k \geq 1$, it holds $c^{(k)}(G, x) = c(\partial^k G, x)$.*
  (v) *For every median graph G, the value $c'(G, -1)$ equals the number of Θ-classes of G.*

We note that Theorem 2.2 (ii) follows from the fact that for every median graph G,

$$\sum_{i \geq 0} (-1)^i \alpha_i(G) = 1 \,,$$

a result first proved in [24, Theorem 4.2.(6)]. It was proved independently in [23], where also Theorem 2.2 (v) was established using a different notation.

3. RATIONAL ROOTS OF THE CUBE POLYNOMIAL

We first observe that in $\mathcal{M}^*$ every rational number smaller than -1 is realizable as a zero of some cube polynomial. To see this, let $s, t \in \mathbb{N}$ with $s \geq t + 1$. Let G be the disjoint union of a tree on $t + 1$ vertices (and t edges) and $s - t - 1$ copies of K_1. Then, $c(G, -\frac{s}{t}) = 0$.

The situation with rational zeros of the cube polynomial is different in the class $\mathcal{M}$. Namely, they are restricted to the interval $[-2, -1)$ as it follows from Theorem 3.3. Moreover, the cube polynomial of a median graph does not have any real zero in the interval $[-1, \infty)$. In fact, more is true:

Proposition 3.1. *Let $G \in \mathcal{M}^*$ be a graph with at least one edge. Then $c(G, x)$ is a strictly increasing function on $[-1, \infty)$.*

Proof. We prove it by induction on the number of edges of G. If G has one edge then it is the disjoint union of a K_2 and $r \geq 0$ copies of K_1. Then $c(G, x) = (r + 2) + x$, so the assertion is clearly true.

Let G be a graph from $\mathcal{M}^*$ with more than one edge and let $G_1, G_2, \ldots, G_k$ be its connected components. Then

$$c(G, x) = \sum_{i=1}^{k} c(G_i, x),$$

and hence, using Theorem 2.2 (iv),

$$c'(G, x) = \sum_{i=1}^{k} c'(G_i, x) = \sum_{i=1}^{k} c(\partial G_i, x) = \sum_{i=1}^{k} \sum_{e \in \mathcal{F}_i} c(U_e, x) \,,$$

where $\mathcal{F}_i$ denotes the set of representatives of the Θ classes of G_i. Now, each U_e is a median graph with fewer edges than G. If $U_e = K_1$ then $c(U_e, x) = 1$. And, if U_e has at least one edge, then $c(U_e, x) \geq 1$ on $[-1, \infty)$ by the induction assumption and the fact that $c(U_e, -1) = 1$. So, we conclude that $c'(G, x) > 0$ on $[-1, \infty)$. ∎

Proposition 3.1 quickly yields a zero-free interval for cube polynomials of graphs from $\mathcal{M}^*$. For if G has at least one edge and $G_1, G_2, \ldots, G_k$ are its connected

components, then

$$c(G, -1) = \sum_{i=1}^{k} c(G_i, -1) = k > 0.$$

If G has no edges, then $c(G, x) = |V(G)| > 0$. This implies the following consequence:

Corollary 3.2. *Let G be a graph from $\mathcal{M}^*$. Then $c(G, x)$ has no zeros in $[-1, \infty)$.*

Let T be a tree on $n \geq 2$ vertices, then $c(T, x) = n + (n - 1)x$. So the only (rational) zero of $c(T, x)$ is $-(n/(n - 1))$. However, as we show in the next result, much more is true—all possible rational zeros are already realized on trees.

Theorem 3.3. *Let G be a median graph. If $c(G, a) = 0$ for a rational number a, then $a = -((t + 1)/t)$ for some integer $t > 0$.*

Proof. By Proposition 3.1, we may assume that $a < -1$. So, let $a = -(s/t)$, where $s, t \in \mathbb{N}$ and $s > t + 1$. Suppose first that a is not an integer. Then $t > 1$, and we may assume that s and t are relatively prime. Since $s - t > 1$ this also implies that $(s - t)$ and t are relatively prime. In what follows we prove:

Claim A   *There exists $p \in \mathbb{N}$ such that $t^p c(G, a) \equiv t \pmod{s - t}$.*

The proof of the claim is by induction on the number of peripheral expansion steps. As $c(K_1, a) = 1$, we have $tc(K_1, a) = t$ and $t \equiv t \pmod{s - t}$ which shows the basis of induction.

Let G be constructed from a median graph G' by a peripheral expansion with respect to G_0. Then

$$c(G, x) = c(G', x) + (x + 1)c(G_0, x),$$

hence

$$c(G, a) = c(G', a) + (a + 1)c(G_0, a),$$

and so

$$t\,c(G, a) = t\,c(G', a) - (s - t)c(G_0, a). \tag{1}$$

By induction, for a median graph G' there exists $p \in \mathbb{N}$ such that $t^p c(G', a) \equiv t \pmod{s - t}$. Also, note that if κ is the size of the largest hypercube of G_0, then for every $k \geq \kappa$ the number $t^k c(G_0, a)$ is an integer. Let $\phi(n)$ denote Euler's function. Set $k \in \mathbb{N}$ large enough so that $k\phi(s - t) > \kappa$, and multiply equation (1)

by $t^{k\phi(s-t)+p-1}$:

$$t^{k\phi(s-t)+p}c(G,a) = t^{k\phi(s-t)+p}c(G',a) - (s-t)t^{k\phi(s-t)+p-1}c(G_0,a).$$

By the choice of k, the number $u := t^{k\phi(s-t)+p-1}c(G_0,a)$ is an integer. Hence by a small rearrangement, we get:

$$t^{k\phi(s-t)+p}c(G,a) = (t^p c(G',a))t^{k\phi(s-t)} - (s-t)u\,.$$

Using $t^p c(G,a) \equiv t \pmod{s-t}$, there exists an integer h such that

$$t^{k\phi(s-t)+p}c(G,a) = (h(s-t)+t)t^{k\phi(s-t)} - (s-t)u,$$

and so

$$t^{k\phi(s-t)+p}c(G,a) = h(s-t)t^{k\phi(s-t)} + t^{k\phi(s-t)+1} - (s-t)u\,.$$

By the well-known fact (Euler's Theorem) that $t^{\phi(n)} \equiv 1 \pmod{n}$, when t and n are relatively prime, we have $t^{k\phi(s-t)+1} = l(s-t)+t$ for some integer l. Finally,

$$t^{k\phi(s-t)+p}c(G,a) = (s-t)(t^{k\phi(s-t)} - u + l) + t\,.$$

This settles the claim for G and hence Claim A is proved by induction.

Since $(s-t)$ and t are relatively prime, Claim A immediately implies that $c(G,a) \neq 0$, where $a = -(s/t)$, $s,t \in \mathbb{N}$, $s > t+1$. Next, if a is an integer, then $a = -s$, and one can use an analogous (but simpler) argument to prove that $c(G,a) \equiv 1 \pmod{s-1}$. We leave the details to the reader. Hence also in this case we conclude that $c(G,-s) \neq 0$. It follows that a rational zero of $c(G,x)$ can only be of the form $a = -(t+1)/t$ for some $t \in \mathbb{N}$. ∎

Recall that the *bipartite wheel BW_k*, $k \geq 3$, is the graph formed by the cycle C_{2k} and a vertex v adjacent to every second vertex of the cycle. Add a pendant vertex to the bipartite wheel BW_k, $k \geq 4$, so that the cube polynomial of the obtained graph is $kx^2 + (3k+1)x + 2k + 2$. Its zeros are -2 and $-(k+1)/k$, thus the possible rational zeros are also realized by median graphs that are not trees.

4. CHARACTERIZING GRAPHS OF ACYCLIC CUBICAL COMPLEXES

In this section, we characterize a well-known class of median graphs in terms of the zeros of the cube polynomial.

A *cubical complex $\mathcal{K}$* is a finite set of cubes of any dimension that is closed under taking subcubes and non-empty intersections. The (underlying) *graph* of $\mathcal{K}$ has as vertices the 0-dimensional cubes of $\mathcal{K}$, two vertices being adjacent if they constitute

a 1-dimensional cube. Graphs of acyclic cubical complexes were introduced and characterized by Bandelt and Chepoi [3, Theorem] as follows.

Theorem 4.1.   *A graph G is the graph of an acyclic cubical complex if and only if G is a median graph not containing any convex bipartite wheel.*

Another characterization of graphs of acyclic cubical complexes (let us call them shortly *ACC graphs*) involves a certain sequence of *cube contractions*, cf. [3]. Using terminology as in Theorem 2.1 we could restate this result as follows: G is an ACC graph if and only if it can be obtained by a sequence of peripheral cube expansions from the one vertex graph (*peripheral cube expansion* of a graph G is an expansion with respect to cover G, Q where Q is an arbitrary hypercube in G).

By Theorem 3.3, the only candidate for an integer zero of the cube polynomial of a median graph is -2. An infinite family of such graphs was presented at the end of the previous section. For another example take $P_m \square P_n$ and add a pendant vertex to each vertex of degree four. Then the cube polynomial of this graph is $(n - 1)(m - 1)x^2 + ((n - 1)m + (m - 1)n + (n - 2)(m - 2))x + nm + (n - 2)(m - 2)$ with one zero, -2. Let $x_2(n, m)$ be the other zero. Then $\lim_{m,n \to \infty} x_2(n, m) = -1$.

These examples show that the class of median graphs having the property that -2 is a zero of their cube polynomials cannot be "nicely" characterized. However, forcing this property to all 2-connected convex subgraphs yields a characterization of the ACC graphs.

Theorem 4.2.   *Let G be a median graph. Then G is a graph of an acyclic cubical complex if and only if for every 2-connected convex subgraph H of G it holds $c(H, -2) = 0$.*

Proof.   Suppose that G is a median graph which is not an ACC graph. Then, by Theorem 4.1, G contains a convex bipartite wheel BW_n. Since

$$c(BW_n, x) = (1 + 2n) + 3nx + nx^2,$$

we conclude that $c(BW_n, -2) = 1$.

Now, we prove the other direction. Suppose that it is false and suppose that G is a counterexample with the smallest number of vertices. Thus, G is an ACC graph having a 2-connected convex subgraph H with $c(H, -2) \neq 0$. By Theorem 4.1, the subgraph H is also an ACC graph. So, by the minimality of $|V(G)|$, we obtain that $G = H$. Hence, G is 2-connected and $c(G, -2) \neq 0$.

By the remark above this theorem, G can be constructed from a smaller ACC graph G' by a peripher cube expansion along some hypercube Q_r, which is a subgraph of G'. By the minimality of $|V(G)|$, it follows that $c(G', -2) = 0$. Since G is 2-connected, it follows that $r \geq 1$. Now, Theorem 2.2 (i) implies

$$c(G, x) = c(G', x) + (x + 1)c(Q_r, x) = c(G', x) + (x + 1)(x + 2)^r.$$

With $x = -2$ in the above expression, we obtain that $c(G, -2) = 0$. But this is a contradiction to $c(G, -2) = c(H, -2) \neq 0$. This proves the theorem. ∎

5. REAL ROOTS OF THE CUBE POLYNOMIAL

In this section, we prove that the cube polynomial of every non-trivial median graph has a real zero and, moreover, its largest real zero lies in the interval $[-2, -1)$. On the other hand, we also observe that the absolute values of real zeros of the cube polynomial are not bounded from above in the class of median graphs.

For a median graph G, let $z(G)$ denotes the largest real zero of $c(G, x)$. Then $z(G)$ is well defined, which in particular follows from our next result.

Theorem 5.1. *Let G be a non-trivial median graph. Then $c(G, x)$ has a real zero in the interval $[-2, -1)$. Moreover, for any non-trivial convex subgraph H of G, it holds $z(H) \leq z(G)$.*

Proof. We proceed by induction on the number k of Θ classes of G. If $k = 1$ then $G = K_2$, and $c(K_2, x) = x + 2$, hence $z(K_2) = -2$. Observe that K_2 contains no non-trivial subgraphs, so the second part of the claim is trivial.

Suppose now that G is a median graph with $k \geq 2$. By Theorem 2.1, we may assume that G is constructed from a median graph G' by a peripheral expansion along the convex subgraph G_0 of G'. It is well known that every proper convex (and so median) subgraph of G has at most $k - 1$ Θ classes. So, the assumption $k \geq 2$ implies that G' is non-trivial, and by the induction hypothesis $z' := z(G')$ exists and $z(G') \in [-2, -1)$. In the case of a peripheral expansion the formula of Theorem 2.2 (i) turns into

$$c(G, x) = c(G', x) + (x + 1) c(G_0, x).$$

Now, we consider two cases regarding whether G_0 is trivial. Suppose first that G_0 is the one vertex graph K_1. Notice that $c(G, z') = 0 + z' + 1 < 0$. Since $c(G, -1) = 1$, it follows that $c(G, x)$ has a real zero in the open interval $(z', -1)$.

Suppose now that G_0 is a non-trivial graph. By the induction hypothesis, $z(G_0)$ exists, and moreover $-2 \leq z(G_0) \leq z' < 1$. Since $z(G_0)$ is the largest real zero of $c(G_0, x)$, we conclude that $c(G_0, z') \geq 0$. Hence,

$$c(G, z') = c(G', z') + (1 + z')c(G_0, z') = (1 + z')c(G_0, z') \leq 0.$$

On the other side $c(G, -1) = 1$. This implies that $z(G)$ exists and $z' \leq z(G) < 1$. This establish the first part of the theorem.

In order to prove the second part of the theorem notice that for every convex non-trivial subgraph H of G, there exists a convex subgraph G^* of G such that H is a convex subgraph of G^*, and G can be obtained from G^* by a peripheral expansion. Thus, $z(H) \leq z(G^*) \leq z(G)$, which implies the second part of the claim. ∎

To see that Theorem 5.1 is the best light, consider the following example. Let G be a (median) graph constructed from t vertex disjoint copies of the k-cube Q_k by identifying one vertex of each copy into a single vertex. Then $c(G, x) = t(x + 2)^k - (t - 1)$. Since the number of real zeros of $c(G, x)$ and of the polynomial $x^k - 1$ is the same, $c(G, x)$ has only one real zero if k is odd, and only two real zeros if k is even.

In contrast to Theorem 3.3, real zeros of the cube polynomial in the class $\mathcal{M}$ are not restricted to a bounded interval. To see this, let A be an arbitrary integer greater than 1. Construct a median graph G from a tree on $2A - 2$ vertices by peripherally expanding a copy of K_2 (obtaining a tree or two trees attached to the square). Then $c(G, x) = x^2 + 2Ax + 2A$, and the smaller zero of $c(G, x)$ is $a = -A - \sqrt{A^2 - 2A}$. Since $A > 1$ is an arbitrarily large integer and $a < -A$, we conclude the following:

Proposition 5.2. *There exists a median graph with an arbitrarily small negative real zero of its cube polynomial.*

6. PRODUCT MEDIAN GRAPHS AND MULTIPLE ROOTS

We say that a (median) graph G is a *product* graph if $G = H_1 \square H_2$, where H_1, H_2 are non-trivial (median) graphs.

In this section, we first use the cube polynomial in order to characterize product graphs among median graphs. Then we apply this result as well as Theorems 3.3 and 5.1 to show that the cube polynomial of degree p has a p-multiple zero precisely when it is a Cartesian product of p trees of the same order. We start with a lemma from the metric graph theory.

Lemma 6.1. *Let H and K be convex, non-disjoint subgraphs of a median graph G. Then $H \cup K$ is an isometric subgraph of G.*

Proof. First note that $H \cup K$ is a connected subgraph. Suppose that $H \cup K$ is not isometric. Then, let $u \in H \setminus K$ and $v \in K \setminus H$ be closest vertices such that in $H \cup K$ there is no u, v-path of length $d_G(u, v)$. Let $k = d_G(u, v)$. Hence, by the minimality of k, internal vertices of every shortest path between u and v lie in $G \setminus (H \cup K)$. Let $u = u_0, u_1, \ldots, u_k = v$ be such a shortest path, and let $u = w_0, w_1, \ldots, w_k, \ldots, w_{k+s} = v$ be a shortest u, v-path in $H \cup K$. Again by the minimality of k, it follows that $u = w_0, w_1, \ldots, w_k$ is a shortest path also in G, that is, $d_G(u, w_k) = k$. Let z be the median of the triple u, v, w_k. Since z is on a shortest path between u and v, we infer that $z \in G \setminus (H \cup K)$. Since w_k is a vertex of H or K (or both), this is a contradiction to convexity of one of these (or even both) subgraphs. ∎

We are now ready to characterize product median graphs.

Theorem 6.2. *Let G be a median graph. Then G is a product graph with $G = H \square K$, if and only if G contains convex subgraphs H and K such that $|V(H) \cap V(K)| = 1$ and $c(G, x) = c(H, x)c(K, x)$.*

Proof. Let G be a product graph. Then it is clear that it contains convex subgraphs H and K with $|V(H) \cap V(K)| = 1$ and with $c(G, x) = c(H, x)c(K, x)$, recall Theorem 2.2 (iii).

For the converse, let G be a median graph with convex subgraphs H and K such that $|V(H) \cap V(K)| = 1$ and $c(G, x) = c(H, x)c(K, x)$. Since

$$c'(G, x) = c'(H, x)c(K, x) + c(H, x)c'(K, x),$$

we infer that

$$c'(G, -1) = c'(H, -1) + c'(K, -1).$$

Hence by Theorem 2.2 (v), the number of Θ-classes of G is the sum of the numbers of Θ classes of H and K.

We claim that H and K do not contain edges of the same Θ-class. Let $e = uv$ be an edge of H and $f = wz$ an edge of K, and let a be the unique vertex of $H \cap K$. Since G is bipartite, we may choose the notation so that $d(u, a) = d(v, a) - 1 = s$, and $d(w, a) = d(z, a) - 1 = t$. Using Lemma 6.1 we deduce

$$d(u, w) + d(v, z) = 2(s + t + 1) = d(u, z) + d(v, w).$$

By the definition of Θ, the edges uv and wz are not from the same Θ-class of G, and the claim is proved.

From the above two paragraphs we conclude that every edge of G is in relation Θ either with an edge of H or with an edge of K (but not with both). This implies that G is isomorphic to a subgraph of $H \square K$. Indeed, for a vertex $x \in V(G)$ set $p_H(x)$ (resp. $p_K(x)$) to be the closest vertex to x in H (resp. K). Then it is easy to check that $p(x) = (p_H(x), p_K(x))$ defines the appropriate embedding p of G into the Cartesian product $H \square K$.

Note that the number of edges of G is $c'(G, 0)$. Since

$$c'(G, 0) = c'(H, 0)c(K, 0) + c(H, 0)c'(K, 0),$$

we obtain

$$|E(G)| = |E(H)||V(K)| + |V(H)||E(K)|,$$

which is possible only if G is isomorphic to $H \square K$. ∎

Now everything is ready for an algebraic characterization of Cartesian products of trees of the same order. Notice that Cartesian products of trees are metrically characterized in [2].

Theorem 6.3.   *Let G be a median graph with the cube polynomial $c(G, x)$ of degree p. Then, $c(G, x)$ has a p-multiple zero if and only if G is a Cartesian product of p trees all of the same order.*

Proof.   If $p = 0$, then the claim is trivial assuming that for every graph G its zero power is $G^0 = K_1$. And, if $p = 1$ then G is a tree, and so the claim is also trivial. Now, we may assume that $p \geq 2$.

If G is a product of p trees of order $\ell + 1$, then by Theorem 2.2 (iii), $c(G, x) = (\ell x + \ell + 1)^p$. Thus, $-((\ell + 1)/\ell)$ is a p-multiple zero of $c(G, x)$.

Suppose that the other direction is false and let G be a counterexample with $|V(G)|$ as small as possible. Then $c(G, x)$ is a polynomial of degree p with a p-multiple zero x_0, but G is not a Cartesian product of p trees all of the same order. By Viète rule, $p \cdot x_0 = -(\alpha_{p-1}(G)/\alpha_p(G))$, hence x_0 is a rational number. Therefore, Theorem 3.3 implies that $x_0 = -((\ell + 1)/\ell)$ for some integer $\ell \geq 1$. Since $c(G, -1) = 1$, it follows that $\alpha_p(G) = \ell^p$. So, we can rewrite the cube polynomial of G as

$$c(G, x) = (\ell x + \ell + 1)^p.$$

We now distinguish two cases:

Case 1.   Every component of ∂G is a product of $p - 1$ trees on $\ell + 1$ vertices.

Let uv be an edge of G, and U_{uv} the component of ∂G. Denote by $T_1, \ldots, T_{p-1}$ the (convex) trees of order $\ell + 1$ in U_{uv} for which U_{uv} is isomorphic to $T_1 \square T_2 \square \cdots \square T_{p-1}$. We may assume that u is a pendant vertex of the tree T_1 (otherwise one could choose another vertex with this property), and let z be the unique neighbor of u in T_1.

We claim that $V(T_1) \cap V(U_{uz}) = \{u\}$. Since $V(T_1) \cap V(U_{uz})$ is convex (it is the intersection of two convex sets) we infer that it induces a subtree of T_1. On the other hand, since it lies in $V(U_{uz})$ it consists of vertices closer to u than to z. Combining both observations we derive that the subtree $T_1 \cap U_{uz}$ of T_1 can only consist of the pendant vertex u and the claim follows.

Now, since $c(G, x) = c(T_1, x)c(U_{uz}, x)$ and $|V(T_1) \cap V(U_{uz})| = 1$ we derive by Theorem 6.2 that $G = T_1 \square U_{uz}$. In other words, G is the Cartesian product of p trees on $\ell + 1$ vertices.

Case 2.   There is a component U^* of ∂G which is not a product of $p - 1$ trees of order $\ell + 1$.

Note that for each integer $r \in \{0, \ldots, p\}$, each component U' of $\partial^r G$ is isomorphic to a convex subgraph of G. If U' is trivial, then $c(U', x_0) = 1$. And, if U' is non-trivial, then by Theorem 5.1, it follows $z(U') \leq x_0$. Hence, $c(U', x_0) \geq 0$ always holds.

By the minimality of G, it follows that x_0 is not a $(p - 1)$-multiple zero of U^*. So one can conclude that for some integer $s \in \{0, \ldots, p - 2\}$, the s-th

derivative $c^{(s)}(U^*, x)$ does not have x_0 as a zero, i.e. $c^{(s)}(U^*, x_0) \neq 0$. Since by Theorem 2.2 (iv), $c^{(s)}(U^*, x) = c(\partial^s U^*, x)$, we conclude from the above paragraph that $c^{(s)}(U^*, x_0) > 0$.

Using Theorem 2.2 (iv) again, we can write

$$c^{(s+1)}(G, x) = c(\partial^{s+1} G, x).\tag{2}$$

Note now that $\partial^s U^* \subseteq \partial^{s+1} G$. Since for each connected component U' of $\partial^{s+1} G$, we have $c(U', x_0) \geq 0$ and, in addition, $c^{(s)}(U^*, x_0) = c(\partial^s U^*, x_0) > 0$, it follows that the right side of (2) is positive at x_0. On the other hand, since $s + 1 < p$ and x_0 is a p-multiple zero of G, the left side of (2) at x_0 is 0, a contradiction. ∎

REFERENCES

[1] S. P. Avann, Metric ternary distributive semi-lattices, Proc Amer Math Soc 12 (1961), 407–414.

[2] H.-J. Bandelt, G. Burosch, and J. M. Laborde, Cartesian products of trees and paths, J Graph Theory 22 (1996), 347–356.

[3] H.-J. Bandelt and V. Chepoi, Graphs of acyclic cubical complexes, Eur J Combin 17 (1996), 113–120.

[4] H.-J. Bandelt and J. Hedlíková, Median algebras, Discrete Math 45 (1983), 1–30.

[5] H.-J. Bandelt, L. Quintana-Murci, A. Salas, and V. Macaulay, The fingerprint of phantom mutations in mitochondrial DNA data, Am J Human Genet 71 (2002), 1150–1160.

[6] F. Brenti, G. F. Royle, and D. G. Wagner, Location of zeros of chromatic and related polynomials of graphs, Canad J Math 46 (1994), 55–80.

[7] B. Brešar, W. Imrich, S. Klavžar, H. M. Mulder, and R. Škrekovski, Tiled partial cubes, J Graph Theory 40 (2002), 91–103.

[8] B. Brešar, S. Klavžar, and R. Škrekovski, The cube polynomial and its derivatives: The case of median graphs, Electron J Combin 10 (2003), #R3, 11.

[9] I. J. Brown, K. Dilcher, and R. J. Nowakowski, Roots of independence polynomials of well covered graphs, J Algebraic Combin 11 (2000), 197–210.

[10] D. Djoković, Distance preserving subgraphs of hypercubes, J Combin Theory Ser B 14 (1973), 263–267.

[11] C. D. Godsil, Algebraic Combinatorics, Chapman & Hall, New York, 1993.

[12] C. D. Godsil and I. Gutman, On the theory of the matching polynomial, J Graph Theory 5 (1981), 137–144.

[13] J. Haglund, K. Ono, and D. Wagner, Theorems and conjectures involving rook polynomials with only real zeros, In Topics in Number Theory, S. D. Ahlgren,

G. E. Andrews, and K. Ono (Eds), Kluwer Academic Publishers, Dordrecht, 1999, pp. 207–221.

[14] W. Imrich and S. Klavžar, Product Graphs: Structure and Recognition, John Wiley & Sons, New York, 2000.

[15] W. Imrich, S. Klavžar, and H. M. Mulder, Median graphs and triangle-free graphs, SIAM J Discrete Math 12 (1999), 111–118.

[16] B. Jackson, A zero-free interval for chromatic polynomials of graphs, Combin Probab Comput 2 (1993), 325–336.

[17] S. Klavžar and H. M. Mulder, Median graphs: Characterizations, location theory and related structures, J Combin Math Combin Comp 30 (1999), 103–127.

[18] S. Klavžar, H. M. Mulder, and R. Škrekovski, An Euler-type formula for median graphs, Discrete Math 187 (1998), 255–258.

[19] H. M. Mulder, The structure of median graphs, Discrete Math 24 (1978), 197–204.

[20] H. M. Mulder, The interval function of a graph, Math Centre Tracts 132, Math Centre, Amsterdam, 1980.

[21] H. M. Mulder, The expansion procedure for graphs, In Contemporary Methods in Graph Theory, R. Bodendiek (Ed), Bibliographisches Institut, Mannheim, 1990, pp. 459–477.

[22] L. Nebeský, Median graphs, Comment Math Univ Carolinae 12 (1971), 317–325.

[23] R. Škrekovski, Two relations for median graphs, Discrete Math 226 (2001), 351–353.

[24] P. S. Soltan and V. D. Chepoi, Solution of the Weber problem for discrete median metric spaces (Russian), Trudy Tbiliss Mat Inst Razmadze Akad Nauk Gruzin SSR 85 (1987), 52–76.

[25] P. Winkler, Isometric embeddings in products of complete graphs, Discrete Appl Math 7 (1984), 221–225.

[26] D. R. Woodall, The largest real zero of the chromatic polynomial, Discrete Math 172 (1997), 141–153.