

# IMPROVED BOUNDS FOR THE CHROMATIC NUMBER OF THE LEXICOGRAPHIC PRODUCT OF GRAPHS

ROLAND KASCHEK

*University of Klagenfurt, Department of Computer Science,  
9020 Klagenfurt, Austria*

AND

SANDI KLAVŽAR\*

*University of Maribor, PF, Koroška cesta 160, 62000 Maribor, Slovenia*

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An upper bound for the chromatic number of the lexicographic product of graphs is proved. It unifies and generalizes several known results and is in particular applied to characterize the graphs that have a complete core. An improved lower bound is also given.

## 1. INTRODUCTION

Among various NP-complete graph problems there are several classical ones, which turned out to be interesting on graph products. Let us mention here the Hamiltonicity<sup>8, 9</sup> and colouring problems. In this paper we continue the investigation of the chromatic number of the strong and the lexicographic product of graphs. Geller, Stahl and Vesztergomby worked on the problem in the seventies<sup>2,3,18,19,20</sup>. Recently, several new motivations for studying these chromatic numbers appeared. The most important recent result is due to Feigenbaum and Schäffer<sup>1</sup>. They proved that the strong product admits a polynomial algorithm for decomposing a given connected graph into its factors. In Linial and Vazirani<sup>13</sup> bounds for the chromatic number of the lexicographic product of graphs are used to study approximation algorithms. Results about the chromatic number of strong products turned out to be important in understanding retracts of strong products<sup>7,10,12</sup>.

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All graphs considered in this paper will be undirected, finite and simple graphs, i.e. graphs without loops or multiple edges. If  $G$  is a graph and  $X \subseteq V(G)$ , then the subgraph of  $G$  induced by  $X$  will be denoted by  $\langle X \rangle$ . An  $n$ -colouring of a graph  $G$  is a function  $f: V(G) \rightarrow \{1, 2, \dots, n\}$ , such that  $xy \in E(G)$  implies  $f(x) \neq f(y)$ . The smallest number  $n$  for which an  $n$ -colouring exists is the 'chromatic number'  $\chi(G)$  of  $G$ .

The order of a largest complete subgraph of a graph  $G$  will be denoted by  $\omega(G)$  and the order of a largest independent set by  $\alpha(G)$ . Clearly,  $\omega(G) \leq \chi(G)$ .

The 'strong product'  $G \boxtimes H$  of graphs  $G$  and  $H$  is the graph with vertex set  $V(G) \times V(H)$  and  $(a, x)(b, y) \in E(G \boxtimes H)$  whenever  $ab \in E(G)$  and  $x = y$ , or  $a = b$  and  $xy \in E(H)$ , or  $ab \in E(G)$  and  $xy \in E(H)$ . The lexicographic product  $G[H]$  of graphs  $G$  and  $H$  is the graph with vertex set  $V(G) \times V(H)$  and  $(a, x)(b, y) \in E(G[H])$  whenever  $ab \in E(G)$ , or  $a = b$  and  $xy \in E(H)$ . Whenever possible we shall denote the vertices of the first factor by  $a, b, c, \dots$  and the vertices of the other factor by  $x, y, z$ .

The main contribution of this paper is an upper bound for the chromatic number of the lexicographic product of graphs (and hence of the strong product) which unifies and generalizes upper bounds from Geller and Stahl<sup>3</sup> and Kalvžar<sup>10, 11</sup>. We prove this upper bound in the next section. In section 3 we give several consequences of the bound. It is shown in particular that if  $G$  is a  $\chi$ -critical graph, then for any graph  $H$ ,  $\chi(G[H]) \leq \chi(H)(\chi(G) - 1) + \lceil \frac{\chi(H)}{\alpha(G)} \rceil$ . We also characterize graphs that have a complete core. In the last section we extend a lower bound form<sup>3,12,18</sup>.

## 2. AN UPPER BOUND

Note first that  $G[K_n]$  is isomorphic to  $G \boxtimes K_n$ . Since furthermore,  $\chi(G[H]) = \chi(G[K_n])$ , if  $\chi(H) = n$  (see Theorem 3 in Geller and Stahl<sup>3</sup>), the results we obtain can also be applied to the strong product of graphs.

This is our main result.

*Theorem 1* — Let  $G$  and  $H$  be any graphs,  $\chi(H) = m$ . Let  $\{X_i\}_{i \in \{1, 2, \dots, k\}}$  be a partition of a set  $X \subseteq V(G)$ . Let  $\chi(G - X_i) = n_i$ ,  $i \in \{1, 2, \dots, k\}$  and  $\chi(\langle X \rangle) = s$ . Then

$$\begin{aligned} \chi(G[H]) &\leq (n_1 + n_2 + \dots + n_r) \lceil \frac{m}{k} \rceil \\ &\quad + (n_{r+1} + n_{r+2} + \dots + n_k + s) \lceil \frac{m}{k} \rceil \\ &\quad + \chi(\langle X_1 \cup X_2 \cup \dots \cup X_r \rangle), \end{aligned}$$

where  $m = pk + r$ ,  $0 \leq r < k$ .

PROOF : Let  $h$  be an  $m$ -colouring of  $H$  and let  $f_i$  be an  $n_i$ -colouring of  $G - X_i$ ,  $i \in \{1, 2, \dots, k\}$ . Furthermore, for  $j \in \{1, 2, \dots, m\}$  write  $j = pk + r_j$ , where



$$V(L) = \{(a, x) \mid a \in V(G) \text{ and } x \in V(H_a)\}$$

$$E(L) = \{(a, x) (b, y) \mid ab \in V(G) \text{ or } a = b \text{ and } xy \in E(H_a)\}.$$

Clearly, if for every  $a \in V(G)$ ,  $H_a$  is isomorphic to a graph  $H$ , then  $L$  is isomorphic to the lexicographic product  $G[H]$ . We can extend Theorem 1 a little more :

*Corollary 2* — Let  $G$  be a graph and let  $\mathcal{H} = \{H_a\}_{a \in V(G)}$  be a family of graphs. Let  $\{X_i\}_{i \in \{1, 2, \dots, k\}}$  be a partition of a set  $X \subseteq V(G)$  and let  $\chi(G - X_i) = n_i$ ,  $i \in \{1, 2, \dots, k\}$ . If  $\max \{\chi(H_a) \mid a \in V(G)\} = m$  and  $\chi(\langle X \rangle) = s$ , then

$$\begin{aligned} \chi(G[\mathcal{H}]) \leq (n_1 + n_2 + \dots + n_r) \left\lceil \frac{m}{k} \right\rceil + (n_{r+1} + n_{r+2} + \dots + n_k + s) \left\lfloor \frac{m}{k} \right\rfloor \\ + \chi(\langle X_1 \cup X_2 \cup \dots \cup X_r \rangle), \end{aligned}$$

where  $m = pk + r$ ,  $p \leq r < k$ .

PROOF : Follows from the result that if  $\chi(H) = m$  then  $\chi(G[H]) = \chi(G[K_m])$  (see Geller and Stahl<sup>3</sup>), and an observation that  $\chi(G[\mathcal{H}]) \leq \chi(G[K_m])$ . □

### 3. APPLICATIONS OF THE UPPER BOUND

A graph  $G$  is called vertex-critical or  $\chi$ -critical if  $\chi(G - x) < \chi(G)$  for every  $x \in V(G)$ . We have :

*Corollary 3*<sup>11</sup> — If  $G$  is a  $\chi$ -critical graph, then for any graph  $H$ ,

$$\chi(G[H]) \leq \chi(H) (\chi(G) - 1) + \left\lceil \frac{\chi(H)}{\alpha(G)} \right\rceil.$$

PROOF : Let  $\chi(G) = n$  and let  $\chi(H) = m$ . Let  $\alpha(G) = k$  and let  $X = \{x_1, x_2, \dots, x_k\}$  be an independent set of  $G$ . Let  $m = pk + r$ ,  $0 \leq r < k$ . For  $i \in \{1, 2, \dots, k\}$  set  $X_i = \{x_i\}$ . Clearly,  $\chi(\langle X \rangle) = 1$  and  $\chi(G - X_i) = n - 1$ ,  $i \in \{1, 2, \dots, k\}$ .

Substituting into Theorem 1 we obtain in the case  $r = 0$  that

$$\chi(G[H]) \leq (k(n - 1) + 1) p = m(n - 1) + p.$$

In the case  $r > 0$  we have :

$$\begin{aligned} \chi(G[H]) &\leq r(n - 1) \left\lceil \frac{m}{k} \right\rceil + ((k - r)(n - 1) + 1) \left\lfloor \frac{m}{k} \right\rfloor + 1 \\ &= (n - 1) (rp + 1) + (k - r)p + (p + 1) \\ &= m(n - 1) + \left\lceil \frac{m}{k} \right\rceil, \end{aligned}$$

and the proof is complete. □

Let  $G$  be a graph and let  $V(G) = \{a_1, a_2, \dots, a_p\}$ . We construct the graph  $M(G)$  from  $G$  by adding  $p + 1$  new vertices  $b_1, b_2, \dots, b_p, b$ . The vertex  $b$  is joined to each

vertex  $b_i$  and the vertex  $b_i$  is joined to every vertex to which  $a_i$  is adjacent in  $G$ . The graph  $M(P_5 \cup P_3)$  is shown in Fig. 1.

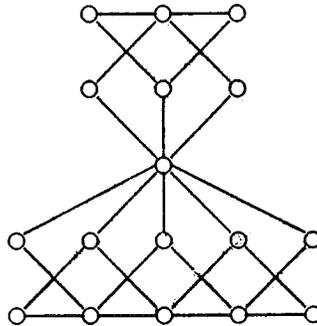


FIG. 1. The graph  $M(P_5 \cup P_3)$ .

The construction is due to Mycielski<sup>14</sup> (and hence our notation). From now on we write  $MG$  instead of  $M(G)$ . It is not hard to see that if  $G$  is  $n$ -chromatic,  $\chi$ -critical and triangle-free, then  $MG$  is  $(n + 1)$ -chromatic,  $\chi$ -critical and triangle-free (see Halin<sup>4</sup>, p. 277, Exercise 2). If the Mycielski procedure is applied  $n$  times consecutively, the final graph will be denoted by  $M^n G$ . Also, let  $M^n G[H]$  denote  $(M^n G)[H]$ .

It is shown in Klavžar<sup>10</sup> that for a bipartite graph  $H$ ,  $\chi(M^n C_5[H]) \leq 2n + 4$ , for any  $n \geq 1$ . However, this upper bound does not follow from Corollary 3, although the graphs  $M^n C_5$  are  $\chi$ -critical. In fact, Corollary 3 gives us  $\chi(M^n C_5[H]) \leq 2n + 5$ . We shall prove in Theorem 4 below that the bound  $2n + 4$  can be derived from Theorem 1. Moreover, Theorem 4 generalizes the result of Klavžar<sup>10</sup> from bipartite  $H$  to  $m$ -chromatic  $H$  and from  $C_5$  to  $C_{2l+1}$ .

*Theorem 4* — Let  $H$  be a graph,  $\chi(H) = m$ . Then for  $n \geq 1$  and  $l \geq 2$ ,

$$\chi(M^n C_{2l+1}[H]) \leq \begin{cases} (n + 2)m; & m \text{ is even,} \\ (n + 2)m + 1; & m \text{ is odd.} \end{cases}$$

PROOF : The proof is by induction on  $n$ . Throughout the proof let  $C$  denote the cycle  $C_{2l+1}$ . We first prove the theorem for  $n = 1$ . Let  $a_0, a_1, \dots, a_{2l}$  be consecutive vertices of  $C$  and let  $V(MC) = \{a_0, a_1, \dots, a_{2l}, b_0, b_1, \dots, b_{2l}, b\}$ . Define

$$X_1 = \{a_0, a_2, \dots, a_{2l-2}, a_{2l-1}\}$$

and

$$X_2 = \{a_1, a_3, \dots, a_{2l-3}, b_1, b\}.$$

We claim that

$$\chi(MC - X_1) = \chi(MC - X_2) = \chi((X_1 \cup X_2)) = 2.$$

It is easy to check that

$$\{a_1, a_3, \dots, a_{2l-3}, a_{2l}, b\} \cup \{b_0, b_1, \dots, b_{2l}\}$$

is a bipartition of the graph  $MC - X_1$  and

$$\{a_0, a_{2l-1}, b_0, b_3, b_5, \dots, b_{2l-1}\} \cup \{a_2, a_4, \dots, a_{2l}, b_2, b_4, \dots, b_{2l}\}$$

a bipartition of the graph  $MC - X_2$ . Finally,

$$\{a_0, a_2, \dots, a_{2l-2}, b\} \cup \{a_1, a_3, \dots, a_{2l-1}, b_1\}$$

is a bipartition of the graph  $\langle X_1 \cup X_2 \rangle$ . The claim is proved.

We now apply Theorem 1 letting  $X = X_1 \cup X_2$ . For  $r = 0$  we have

$$\chi(MC[H]) \leq (2 + 2 + 2) \left\lfloor \frac{m}{2} \right\rfloor = 3m,$$

and for  $r = 1$  we obtain

$$\begin{aligned} \chi(MC[H]) &\leq 2 \left\lceil \frac{m}{2} \right\rceil + 4 \left\lfloor \frac{m}{2} \right\rfloor + 2 \\ &= 2(p + 1) + 4p + 2 \\ &= 6p + 4 = 3(2p + 1) + 1 = 3m + 1. \end{aligned}$$

This proves the Theorem for  $n = 1$ . Let  $n \geq 1$  and suppose that  $m$  is even. Let  $f$  be an  $((n + 2)m)$ -colouring of  $M^n C[H]$ . Let  $V(M^n C) = \{a_1, a_2, \dots, a_p\}$  and let  $b_1, b_2, \dots, b_p, b$  be the corresponding vertices in the graph  $M^{n+1} C$ . We extend  $f$  to a colouring of  $M^{n+1} C[H]$  in the following way. Set  $f(b_i, x) = f(a_i, x)$ ,  $i \in \{1, 2, \dots, p\}$ , and  $x \in V(H)$ . In addition, colour vertices of the subgraph  $\langle \{b\} \times V(H) \rangle$  with  $m$  new colours. It is straightforward to verify that we have a proper colouring with  $(n + 3)m$  colours. If  $m$  is odd we proceed in the same way to get an  $((n + 3)m + 1)$ -colouring.  $\square$

*Corollary 5<sup>10</sup>* — For  $n \geq 1$ ,  $\chi(M^n C_5[H]) \leq 2n + 4$ , for any bipartite graph  $H$ .

A subgraph  $R$  of a graph  $G$  is a retract of  $G$  if there is a map  $r : V(G) \rightarrow V(R)$  satisfying  $xy \in E(G) \Rightarrow r(x)r(y) \in E(R)$  and  $r(x) = x$ , for all  $x \in V(R)$ . The map  $r$  is called a retraction,  $r$  is an edge preserving map (a homomorphism) which fixes  $R$ . It is well-known that if  $R$  is a retract of  $G$  then  $\chi(R) = \chi(G)$ .

A subgraph  $H$  of  $G$  is called a core of  $G$  if there is a homomorphism  $G \rightarrow H$  but no homomorphism  $G \rightarrow H'$  for any proper subgraph  $H'$  of  $H$ . Principal properties of graph cores are summarised in Hell and Nešetřil<sup>5</sup>. In particular, the core of a graph is unique up to isomorphism and the core of  $G$  is a retract of  $G$  (minimal with respect to inclusion).

As an application of Theorem 4 to graph retracts one can give a new infinite sequence of pairs of graphs  $G$  and  $G'$  such that  $G'$  is not a retract of  $G$  while  $G'[K_n]$  is a retract of  $G[K_n]$ . The construction is similar to those in Klavžar<sup>10,12</sup> hence we will not give it here. Here is another application. For a graph  $G$ , let  $X_G$  be the core of  $G$ .

*Theorem 6* — For a graph  $G$ , the following conditions are equivalent :

- (i)  $X_G$  is complete.
- (ii)  $\chi(G) = \omega(G)$ .
- (iii)  $X_G$  is  $\chi$ -critical and  $\chi(G[K_2]) = 2\chi(G)$ .

PROOF : Since  $\chi(G) = \chi(X_G) = n$ , (i) is clearly equivalent to (ii) and furthermore, (i) (and (ii)) implies (iii). Hence it remains to show that (iii) implies (i) (or (ii)).

Let  $X_G$  be  $\chi$ -critical and let  $\chi(G[K_2]) = 2\chi(G)$ . Since  $X_G$  is a retract of  $G$ , it easily follows that  $X_G[K_2]$  is a retract of  $G[K_2]$ . Hence

$$\chi(X_G[K_2]) = \chi(G[K_2]) = 2\chi(G) = 2\chi(X_G).$$

Now Corollary 3 implies that  $\chi(X_G[K_2]) \leq 2(\chi(X_G) - 1) + \lceil 2/\alpha(X_G) \rceil$ . It follows that  $\alpha(X_G) = 1$ . Thus  $X_G$  is complete. □

In connection with the last theorem we pose two problems. Let  $k \geq 2$ . Call a graph  $G(\chi, k)$ -simple, if  $\chi(G[K_k]) = k\chi(G)$ .

*Problem 1* — Characterize  $(\chi, k)$ -simple graphs. In particular, characterize  $(\chi, 2)$ -simple graphs.

Clearly, a graph  $G$  with  $\omega(G) = \chi(G)$  is  $(\chi, k)$ -simple for every  $k$ . However, there are infinite series of 3-chromatic graphs without triangles that are  $(\chi, 2)$ -simple and  $(\chi, 3)$ -simple, respectively<sup>11</sup>. By Corollary 3,  $\chi$ -critical, incomplete graphs are not  $(\chi, k)$ -simple for any  $k$ .

*Problem 2* — Characterize the graphs  $G$  that have  $\chi$ -critical core.

For example, perfect graphs have complete cores. On the other hand, every retract-rigid graph (i.e. a graph without proper retracts) is a proper core of some graph.

#### 4. A LOWER BOUND

In this section we slightly generalize a lower bound given in Geller and Stahl<sup>3</sup>, Klavžar and Milutinović<sup>10</sup> and Stahl<sup>18</sup> by the following theorem. The proof idea is essentially the same as in Klavžar and Milutinović<sup>12</sup>. For  $a, b \in V(G)$  we will write  $ab[H_a, H_b]$  instead of  $\langle \{a, b\} \rangle [H_a, H_b]$  and  $a[H_a]$  instead of  $\langle \{a\} \rangle [H_a]$ .

*Theorem 7* — Let  $G$  be a graph with at least one edge and  $\mathcal{H} = \{H_a\}_{a \in V(G)}$  be a family of graphs. Then

$$\chi(G[\mathcal{H}]) \geq \chi(G) + \min\{\chi(ab[H_a, H_b]) \mid ab \in E(G)\} - 2.$$

PROOF : Let  $L = G[\mathcal{H}]$  and let  $c$  be a colouring of  $L$  with colours  $1, 2, \dots, \chi(L)$ . Denote by  $m$  the number  $\min\{\chi(H_a) \mid a \in V(G)\}$ . Let for  $a \in V(G)$ ,  $m_a = \min\{c(z) \mid z \in a[H_a]\}$ . Finally let  $m_G = \min\{\chi(ab[H_a, H_b]) \mid ab \in E(G)\}$ . Define the mapping  $\gamma : V(G) \rightarrow \{1, 2, \dots, \chi(L) - m_G + 2\}$ , in the following way :

$$\gamma(a) = \begin{cases} m_a; & m_a < \chi(L) - m_G + 2, \\ \chi(L) - m_G + 2; & \text{otherwise.} \end{cases}$$

To show that  $\gamma$  is a colouring let  $ab \in E(G)$  and assume  $\gamma(a) = \gamma(b)$ . We claim that  $\gamma(a) \neq m_a$ . Suppose not. Then  $\gamma(a) = m_a = m_b = \gamma(b)$ . But because every vertex from  $a[H_a]$  is adjacent to every vertex from  $b[H_b]$ , we have a contradiction. Now  $\gamma(a) \neq m_a$  implies  $m_a \geq \chi(L) - m_G + 2$ . As  $\gamma(a) = \gamma(b)$  it follows also  $m_b \geq \chi(L) - m_G + 2$ . Therefore, for to colour  $ab[H_a, H_b]$  only  $\chi(L) - (\chi(L) - m_G - 1) = m_G - 1$  colours are available, a contradiction. Thus  $\gamma$  is a colouring of  $G$ . It follows  $\chi(G) \geq \chi(L) - m_G + 2$  and hence the result.  $\square$

*Corollary* 8<sup>3,12,18</sup> — Let  $G$  and  $H$  be graphs and let  $E(G) \neq \emptyset$ . Then

$$\chi(G[H]) \geq \chi(G) + 2\chi(H) - 2.$$

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