*Publ. Math. Debrecen*
**71/3-4** (2007), 267–278

# Edge-counting vectors, Fibonacci cubes, and Fibonacci triangle

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**Abstract.** Edge-counting vectors of subgraphs of Cartesian products are introduced as the counting vectors of the edges that project onto the factors. For several standard constructions their edge-counting vectors are computed. It is proved that the edge-counting vectors of Fibonacci cubes are precisely the rows of the Fibonacci triangle and that the edge-counting vectors of Lucas cubes are $F_{n-1}$-constant vectors. Some problems are listed along the way.

## 1. Introduction

The Cartesian product of graphs is the central graph product [13]. It has numerous appealing algebraic properties and is applicable in a variety of situations. Its fundamental graph property goes back to Sabidussi [24] and Vizing [28]: every connected graph has a unique prime factor decomposition with respect to the Cartesian product. From the algorithmic point of view it took about 20 years of intensive developments to finally prove that the prime factor decomposition can be obtained in linear time [14].

The structure of subgraphs of Cartesian products has been extensively studied as well. There are many classes of graphs that are naturally defined as (metric) subgraphs of Cartesian products, see [1], [4], [6], [17], [26] for a sample of such

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*Mathematics Subject Classification:* 05C75, 11B39.
*Key words and phrases:* Cartesian product of graphs, Fibonacci cubes, Lucas cubes, Fibonacci triangle, partial cubes.
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references. Graphs that are subgraphs of general Cartesian products has been studied as well, see [2], [16], [19], [23] where several characterizations of these graphs are proved.

In this paper we introduce edge-counting vectors for subgraphs of Cartesian products as the vectors that count the edges that project onto the factors. This gives only a partial information about such subgraphs but nevertheless some interesting information can be obtained from these vectors. We demonstrate this fact by Fibonacci cubes and Lucas cubes by considering their natural embedding into hypercubes.

Fibonacci cubes were introduced in [11], [12] as a model for interconnection network and extensively studied afterward, see [15], [18], [20], [22]. An $O(mn)$ algorithm for recognition of Fibonacci cubes is given in [27], while in [25] the complexity has been improved to $O(m \log n)$. (As usual, $n$ stands for the number of vertices and $m$ for the number of edges of a given graph.) A closely related class of graphs is formed by Lucas cubes, see [15], [21].

The paper is organized as follows. In the next section definitions and concepts needed in this paper are given. In the subsequent section we define the edge-counting vectors and give several examples of such vectors. In particular we determine the edge-counting vectors for products of subgraphs, for amalgamations of graphs, and for the canonical metric representation of a graph. In Section 4 we consider the edge-counting vectors of the Fibonacci cubes as subgraphs of hypercubes. This enables us to give a new interpretation to the Fibonacci triangle: the edge-counting vectors of the Fibonacci cubes are just the rows of the triangle. The edge-counting vectors of the extended Fibonacci cubes are also obtained. In the last section we prove that the edge-counting vectors of Lucas cubes are $F_{n-1}$-constant vectors. We also search for other classes of graphs with constant edge-counting vectors and find some more interesting examples.

## 2. Preliminaries

The *Cartesian product* $G \,\square\, H$ of graphs $G$ and $H$ is the graph with the vertex set $V(G) \times V(H)$ where vertices $(g, h)$ and $(g', h')$ are adjacent if $gg' \in E(G)$ and $h = h'$, or $g = g'$ and $hh' \in E(H)$. On Figure 1 the Cartesian product of the complete bipartite graph $K_{1,3}$ with the path on four vertices $P_4$ is shown.

Note that the Cartesian product of two edges (that is, of complete graphs on two vertices) is the 4-cycle $C_4$. Therefore the notation $\square$ has now been adopted

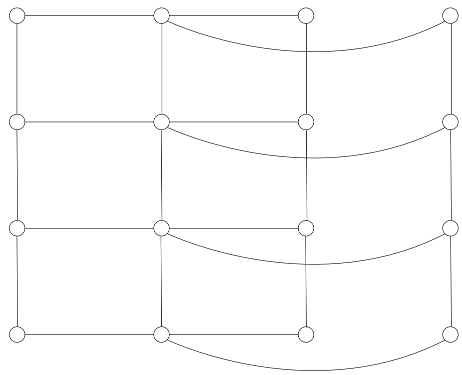

*Figure 1.* Cartesian product $K_{1,3} \,\square\, P_4$.

by most authors in the "product graph community". From the same reason some authors also use the name *box product* for the Cartesian product, see [5].

The Cartesian product graph operation is associative, commutative, and the one vertex graph $K_1$ is the unit. By the associativity we can write $\square_{i=1}^{k} G_i$ for the Cartesian product of factors $G_1, G_2, \ldots, G_k$. Let $v = (v_1, v_2, \ldots, v_k)$ be a vertex of $G = \square_{i=1}^{k} G_i$. A subgraph of $G$ in which we fixed all coordinates except $v_i$ of vertex $v$ is isomorphic to $G_i$ and is called $G_i$-*fiber*.

The simplest Cartesian product graphs are hypercubes. The *k-cube* or a *hypercube* $Q_k$ is the Cartesian product of $k$ factors $K_2$. Hence the vertices of $Q_k$ can be identified with all binary strings of length $k$, two vertices being adjacent if they differ in precisely one position.

A graph $H$ is an *isometric subgraph* of $G$ if $d_H(u, v) = d_G(u, v)$ for any vertices $u, v \in H$, where $d$ is the distance function between vertices. Isometric subgraphs of hypercubes are called *partial cubes*, see [3], [7], [8], [13]. In this paper we always assume that a partial cube $G$ is embedded in the smallest possible hypercube $Q_n$, that is, $n$ is the so-called isometric dimension of $G$. It is well-known that such an embedding is unique. Hence all the edge-counting vectors of partial cubes considered will be unique (modulo permutations of coordinates).

A *Fibonacci string* is a binary string $a_1 a_2 \ldots a_n$ such that $a_i \cdot a_{i+1} = 0$ holds for $i = 1, 2, \ldots, n-1$. In other words, a Fibonacci string is a binary string of length $n$ with no two consecutive ones. The *Fibonacci cube* $\Gamma_n$, has the Fibonacci strings as vertices, two vertices being adjacent whenever they differ in exactly one coordinate. The *Lucas cube* $\Lambda_n$, is the graph with those Fibonacci strings of length $n$ as vertices in which the first and the last bit are not both 1, where two vertices are again adjacent if they differ in exactly one bit. Note that $\Lambda_1 = K_1$,

$\Lambda_2 = P_3$, and $\Lambda_3 = K_{1,3}$. On Figure 2 the Fibonacci cube $\Gamma_4$ and the Lucas cube $\Lambda_5$ are given together with the corresponding labelings of their vertices with Fibonacci strings.

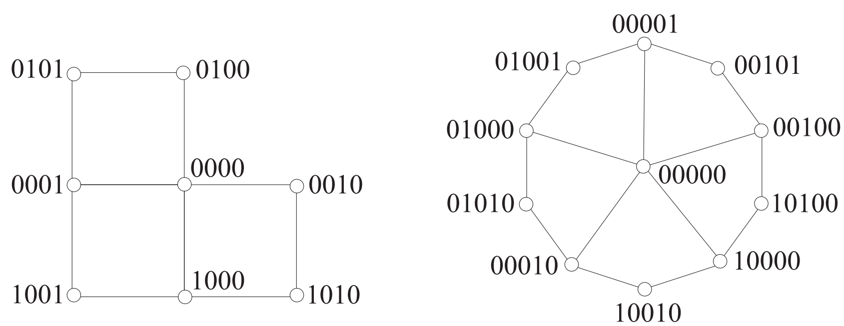


*Figure 2.* Fibonacci cube $\Gamma_4$ and Lucas cube $\Lambda_5$.

## 3. Edge-counting vectors

We now introduce our central concept, the edge-counting vectors, and give several examples of such vectors.

Let $H$ be a subgraph of the Cartesian product $G = \Box_{i=1}^{k} G_i$, $k \geq 1$. Let $e = hh'$ be an edge of $H$, where $h = (h_1, h_2, \ldots, h_k)$ and $h' = (h'_1, h'_2, \ldots, h'_k)$. Then there exists exactly one $i$ such that $h_i h'_i \in E(G_i)$, while $h_j = h'_j$ for $j \neq i$. We will say that the edge $e$ is of *type* $i$. For $i = 1, 2, \ldots, k$ let

$$E_i(H; G) = \{e \in E(H) \mid e \text{ is of type } i\},$$

set $e_i(H; G) = |E_i(H; G)|$, and let

$$v(H; G) = (e_1(H; G), e_2(H; G), \ldots, e_k(H; G))$$

be the *edge-counting vector* of the subgraph $H$ of the Cartesian product $G$.

Note that $v(H; G)$ is well-defined since $H$ is a fixed subgraph of a given Cartesian product $G$. In general, however, a graph $H$ can have different embeddings into a Cartesian product $G$, and a given graph $G$ can have different representations as Cartesian product. For instance, let $G = P_3 \,\Box\, P_3$. Then $v(P_3; G)$ can be any of the vectors $(2, 0)$, $(1, 1)$, and $(0, 2)$, depending which subgraph $P_3$ of

$G$ we select. Also, let the $n$-cube $Q_n$ be represented as usual: $G = \Box_{i=1}^{n} K_2$. Then the edge-counting vector $v(Q_n; G) = (2^{n-1}, \ldots, 2^{n-1})$ is a vector of length $n$. However, if we set $G = K_2 \,\Box\, Q_{n-1}$ then the edge-counting vector becomes $v(Q_n; G) = (2^{n-1}, (n-1)2^{n-1})$.

Unless stated otherwise, for a connected Cartesian product $G$ we will assume that its representation as a Cartesian product is the unique prime factor decomposition of $G$. In particular, the $n$-cube $Q_n$ will be always represented as $\Box_{i=1}^{n} K_2$.

Let $H$ be a subgraph of $G = \Box_{i=1}^{k} G_i$ and $H'$ a subgraph of $G' = \Box_{i=1}^{\ell} G'_i$. Then the *natural product embedding* of $H \,\Box\, H'$ as a subgraph into $G \,\Box\, G'$ is defined as follows. Let $h \in V(H)$ and $h' \in V(H')$ correspond to $(g_1, \ldots, g_k) \in V(G)$ and $(g'_1, \ldots, g'_\ell) \in V(G')$, respectively. Map the vertex $(h, h')$ of $H \,\Box\, H'$ into the vertex $(g_1, \ldots, g_k, g'_1, \ldots, g'_\ell)$ of $G \,\Box\, G'$.

**Proposition 3.1.** *Let $H$ be a subgraph of $G = \Box_{i=1}^{k} G_i$ and $H'$ a subgraph of $G' = \Box_{i=1}^{\ell} G'_i$. Then for the natural product embedding of $H \,\Box\, H'$ into $G \,\Box\, G'$, the edge-counting vector $v(H \,\Box\, H'; G \,\Box\, G')$ equals to*

$$(n' e_1(H; G), \ldots, n' e_k(H; G), n e_1(H'; G'), \ldots, n e_\ell(H'; G')),$$

*where $n = |V(G)|$ and $n' = |V(G')|$.*

Proof. Consider $e_i(H \,\Box\, H'; G \,\Box\, G')$, where $1 \le i \le k + \ell$. Let $e$ be an edge of $H \,\Box\, H'$ of type $i$ and assume for simplicity that $i = 1$. Then

$$e = (g_1, g_2, \ldots, g_k, g'_1, \ldots, g'_\ell)(x, g_2, \ldots, g_k, g'_1, \ldots, g'_\ell),$$

where $g_1 x \in E(G_1)$. Now, for any edge $(g_1, g_2, \ldots, g_k)(x, g_2, \ldots, g_k)$ from $E_1(H; G)$, the last $\ell$ coordinates $g'_1, \ldots, g'_\ell$ can be arbitrarily selected. In other words,

$$e_1(H \,\Box\, H'; G \,\Box\, G') = e_1(H; G)|V(G')|.$$

The same argument applies to the other coordinates, hence the result. $\Box$

Let $H$ and $H'$ be isomorphic subgraphs of graphs $G$ and $G'$, respectively. Then the amalgamation of $G$ and $G'$ along $H$ and $H'$ is the graph obtained from the disjoint union of $G$ and $G'$ by identifying (in view of an isomorphism $H \to H'$) the subgraphs $H$ and $H'$. For our purposes, the following special amalgamations will be useful.

Let $H$ and $H'$ be subgraphs of Cartesian products $G$ and $G'$, respectively. Let $A(H, H')$ be the graph that is obtained by amalgamating an arbitrary vertex of $H$

with an arbitrary vertex of $H'$. (Sometimes this is called a vertex-amalgamation.) A *natural amalgamation embedding* of $A(H,H')$ as a subgraph into $G \,\square\, G'$ is defined as follows. Embed $H$ in any $G$-fiber and let $u$ be the vertex of $G \,\square\, G'$ into which the amalgamated vertex of $H$ is mapped. Clearly, the amalgamated vertex of $H'$ is also mapped into $u$. Then embed $H'$ in the unique $G'$-fiber that intersects $u$. The following result, stated for further reference, follows easily.

**Proposition 3.2.** *Let $H$ and $H'$ be subgraphs of the Cartesian products $G$ and $G'$, respectively. Let $v(H;G) = (a_1,\ldots,a_k)$ and $v(H';G') = (a'_1,\ldots,a'_\ell)$. Then*

$$v(A(H,H'); G \,\square\, G') = (a_1,\ldots,a_k,a'_1,\ldots,a'_\ell)$$

*for the natural amalgamation embedding of $A(H,H')$ into $G \,\square\, G'$.*

For the final example in this section consider the Graham–Winkler's canonical metric representation from [9]. So let

$$\alpha : G \to G/E_1 \,\square\, \cdots \,\square\, G/E_k$$

be the canonical metric representation of the graph $G$, see [9] or [13] for its definition. Then by the definition of the embedding,

$$v(G; G/G_1 \,\square\, \cdots \,\square\, G/G_k) = (|E_1|,\ldots,|E_k|)\,.$$

We note that with a similar method induced subgraphs of Hamming graphs in particular [17] and induced subgraphs of Cartesian graphs in general [23] can be treated.

## 4. Fibonacci cubes and Fibonacci triangle

In this section we consider the edge-counting vectors of Fibonacci cubes as subgraphs of hypercubes. Clearly, $\Gamma_n$ is a subgraph of $Q_n$, just identify the vertices of $\Gamma_n$ with the corresponding vertices of $Q_n$. Call this embedding the *natural embedding* of $\Gamma_n$ into $Q_n$.

In the rest we will often use the well-known fact that $\Gamma_n$ contains $F_{n+2}$ vertices, cf. [12].

**Theorem 4.1.** *Let $n \geq 1$. Then for the natural embedding of $\Gamma_n$ into $Q_n$,*

$$v(\Gamma_n; Q_n) = (F_1F_n, F_2F_{n-1}, \ldots, F_nF_1).$$

Proof. Observe that $e_1(\Gamma_n; Q_n) = |\{10b_3 \dots b_n\}|$, where $b_3 \dots b_n$ is an arbitrary Fibonacci string. Therefore, $e_1(\Gamma_n; Q_n) = F_n = F_1 F_n$. We similarly get that $e_n(\Gamma_n; Q_n) = |\{b_1 \dots b_{n-2}01\}|$, hence $e_n(\Gamma_n; Q_n) = F_n = F_n F_1$. Let $2 \le i \le n-1$, then

$$e_i(\Gamma_n; Q_n) = |\{b_1 \dots b_{i-2}010b_{i+2} \dots b_n\}|.$$

Since $b_1 \dots b_{i-2}$ is an arbitrary Fibonacci string of length $i-2$ and $b_{i+2} \dots b_n$ is an arbitrary Fibonacci string of length $n-i-1$, we conclude that for $2 \le i \le n-1$, $e_i(\Gamma_n; Q_n) = F_i F_{n-i+1}$ and the proof is complete. □

Theorem 4.1 immediately implies the following result, cf. [15]:

**Corollary 4.2.** *For any* $n \ge 1$, $|E(\Gamma_n)| = \sum_{i=1}^{n} F_i F_{n-i+1}$.

The *Fibonacci triangle* is defined with

$$F_{n,m} = F_m F_{n-m+1}, \quad 1 \le m \le n,$$

where $n$ denotes the row and $m$ the position in the $n$-th row of the entry $F_{n,m}$ [10]. It follows immediately from the definition that it is centrally symmetric, that is, $F_{n,m} = F_{n,n-m+1}$. The first several rows of the Fibonacci triangle are shown in Table 1.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | | | | | | | |
| 1 | 1 | | | | | | |
| 2 | 1 | 2 | | | | | |
| 3 | 2 | 2 | 3 | | | | |
| 5 | 3 | 4 | 3 | 5 | | | |
| 8 | 5 | 6 | 6 | 5 | 8 | | |
| 13 | 8 | 10 | 9 | 10 | 8 | 13 | |
| 21 | 13 | 16 | 15 | 15 | 16 | 13 | 21 |

*Table 1.* The first few rows of the Fibonacci triangle

Theorem 4.1 gives the following reinterpretation of the Fibonacci triangle.

**Corollary 4.3.** *For any* $n \ge 1$, *the vector* $v(\Gamma_n; Q_n)$ *coincides with the* $n$*-th row of the Fibonacci triangle.*

Let us write $V_i$ for $V(\Gamma_i)$. Then it is clear that $V_{i+2} = 0V_{i+1} \cup 10V_i$. With this property it is natural to define the *extended Fibonacci cube of order* $n$, $\Gamma_n^i$, $0 \le i \le n$, as follows. The vertex set $V_n^i$ of $\Gamma_n^i$ is defined recursively by $V_{n+2}^i =$

$0V^i_{n+1} \cup 10V^i_n$, where $V^i_i$ is the set of all binary strings of length $i$ and $V^i_{i+1}$ the set of all binary strings of length $i+1$. Note that $\Gamma^i_i = Q_i$, $\Gamma^i_{i+1} = Q_{i+1}$, and $\Gamma^0_n = \Gamma_n$.

Extended Fibonacci cubes where introduced in [30]. In [29] Whitehead and Zagaglia Salvi showed that extended Fibonacci cubes are Cartesian products of Fibonacci cubes and hypercubes, more precisely:

$$\Gamma^i_n = \Gamma^0_{n-1} \,\Box\, Q_i = \Gamma_{n-1} \,\Box\, Q_i.$$

As $\Gamma_{n-1}$ embeds into $Q_{n-1}$, it follows that $\Gamma^i_n$ naturally embeds into $Q_{n-1+i}$. Combining this fact with Proposition 3.1 and Theorem 4.1 we thus have:

**Corollary 4.4.** *For any $n \geq i \geq 0$,*

$$v(\Gamma^i_n; Q_{n-1+i}) = (2^i F_1 F_{n-1}, \ldots, 2^i F_{n-1} F_1, 2^{i-1} F_{n+1}, \ldots, 2^{i-1} F_{n+1}),$$

*where the term $2^{i-1}F_{n+1}$ appears $i$ times.*

We close the section with the following problem.

**Problem 4.5.** *Which partial cubes are uniquely (modulo its permutations) determined by its edge-counting vector?*

All hypercubes have this property as well as $C_6$ and $P_3$. This can also be checked to be true for Fibonacci cubes for small $n$'s. In general we pose a question whether Fibonacci cubes can be characterized among partial cubes by this property. More precisely, is a partial cube $G$ isomorphic to $\Gamma_n$ provided that $v(G; Q_n)$ is the $n$-th row of the Fibonacci triangle? Note that one can easily find graphs that are not partial cubes but have the same edge-counting vectors as $\Gamma_n$, $n \geq 4$.

## 5. Lucas cubes and constant edge-counting vectors

Let $H$ be a subgraph of a Cartesian product $G$ with $v(H; G) = (\ell, \ldots, \ell)$. Then we say that $v(H; G)$ is a *constant* edge-counting vector, more precisely $\ell$-*constant*. In this section we first prove that the edge-counting vector of the Lucas cube $\Lambda_n$ (as a subgraph of $Q_n$) is $F_{n-1}$-constant. After that several more graphs with constant edge-counting vectors are constructed.

Vertices of the Lucas cube $\Lambda_n$ can be obtain from the vertices of the Fibonacci cubes $\Gamma_{n-1}$ and $\Gamma_{n-3}$ as follows: $V(\Lambda_n) = 0V(\Gamma_{n-1}) \cup 10V(\Gamma_{n-3})0$. For the proof that $v(\Lambda_n; Q_n)$ is $F_{n-1}$-constant we need the following easy lemma.

**Lemma 5.1.** *Let $n \geq 4$. Then for any $i$ with $3 \leq i \leq n-3$,*

$$F_{i-1}F_{n-i+1} + F_{i-2}F_{n-i} = F_{n-1}.$$

Proof. For $i = 3$ we have $F_2F_{n-2} + F_1F_{n-3} = F_{n-1}$. For the induction step we can compute in the following way: $F_iF_{n-i} + F_{i-1}F_{n-i-1} = (F_{i-1} + F_{i-2})F_{n-i} + F_{i-1}F_{n-i-1} = F_{i-1}(F_{n-i} + F_{n-i-1}) + F_{i-2}F_{n-i} = F_{i-1}F_{n-i+1} + F_{i-2}F_{n-i} = F_{n-1}$. □

**Theorem 5.2.** *Let $n \geq 2$. Then for the natural embedding of $\Lambda_n$ into $Q_n$,*

$$v(\Lambda_n; Q_n) = (F_{n-1}, F_{n-1}, \ldots, F_{n-1}).$$

Proof. Since $\Lambda_2 = P_3$ and $\Lambda_3 = K_{1,3}$ we have $v(\Lambda_2; Q_2) = (1,1) = (F_1, F_1)$ and $v(\Lambda_3; Q_3) = (1,1,1) = (F_2, F_2, F_2)$. Assume in the rest that $n \geq 4$.

Observe first that $e_1(\Lambda_n; Q_n) = |\{10b_3 \ldots b_{n-1}0\}|$, where $b_3 \ldots b_{n-1}$ is an arbitrary Fibonacci string. Therefore $e_1(\Lambda_n; Q_n) = F_{n-1}$. By symmetry we have $e_n(\Lambda_n; Q_n) = |\{0b_2 \ldots b_{n-2}01\}|$, hence $e_n(\Lambda_n; Q_n) = F_{n-1}$. Similarly we have $e_2(\Lambda_n; Q_n) = |\{010b_4 \ldots b_n\}| = F_{n-1}$ and again by symmetry $e_{n-1}(\Lambda_n; Q_n) = F_{n-1}$.

For $3 \leq i \leq n-3$ use the fact that $V(\Lambda_n) = 0V(\Gamma_{n-1}) \cup 10V(\Gamma_{n-3})0$. Then

$$e_i(\Lambda_n; Q_n) = |\{0b_2 \ldots b_{i-2}010b_{i+2} \ldots b_n\}| + |\{10b_3 \ldots b_{i-2}010b_{i+2} \ldots b_{n-1}0\}|,$$

where $b_2 \ldots b_{i-2}$, $b_{i+2} \ldots b_n$, $b_3 \ldots b_{i-2}$, and $b_{i+2} \ldots b_{n-1}$ are arbitrary Fibonacci strings of length $i-3$, $n-i-1$, $i-4$, and $n-i-2$, respectively. Therefore for $3 \leq i \leq n-3$, $e_i(\Lambda_n; Q_n) = F_{i-1}F_{n-i+1} + F_{i-2}F_{n-i}$. The proof is complete by Lemma 5.1. □

Theorem 5.2 and the Proposition 7 of [15] immediately imply:

**Corollary 5.3.** *For any $n \geq 2$,*

$$F_{n-1} = \frac{1}{n} \sum_{i=1}^{n-1} F_i L_{n-1-i}.$$

Propositions 3.1 and 3.2 suggest how to obtain many subgraphs with constant edge-counting vectors. This is done in the next two corollaries, respectively.

**Corollary 5.4.** *Let $H \subseteq G$ and $H' \subseteq G'$ be as in Proposition 3.1. Then $v(H \,\square\, H'; G \,\square\, G')$ is a constant edge-counting vector if and only if $v(H; G)$ is an $i$-constant edge-counting vector, $v(H'; G')$ is a $j$-constant edge-counting vector, and $j|H| = i|H'|$. Moreover, in this case we have $v(H \,\square\, H'; G \,\square\, G') = (j|H|, \ldots, j|H|)$.*

**Corollary 5.5.** *Let $A(H, H')$ be an amalgam of $H$ and $H'$ where both $H$ and $H'$ have $\ell$-constant edge-counting vectors. Then $A(H, H')$ has an $\ell$-constant edge-counting vector as well.*

In the rest we give some more partial cubes with constant edge-counting vectors. First two trivial examples: the edge-counting vector of an arbitrary tree is 1-constant, and the edge-counting vector of an even cycle is 2-constant.

A nice class of partial cubes is formed by *bipartite wheels* $BW_n$, $n \geq 3$. $BW_n$ is a graph obtained from the cycle $C_{2n}$ and the *central vertex* $v$ by joining every second vertex of the cycle with $v$. Note that $\Lambda_5 = BW_5$. It is straightforward to verify that $v(BW_n; Q_n)$ is 3-constant.

We define *extended bipartite wheels*, $EBW_n^\ell$, $n \geq 3$, $0 \leq \ell \leq \lceil \frac{n}{2} \rceil - 2$, as follows. For $\ell = 0$ we set $EBW_n^0 = BW_n$. For $\ell > 0$ connect on the $i$-th step, $i = 1, \ldots, \ell$, vertices $x$ and $y$ by path of length 2, if $d(x, y) = 2$ and $x$ and $y$ are on maximum distance from $v$ in $EBW_n^{i-1}$. Note that $EBW_n^\ell$ is not a partial cube anymore if $\ell > \lceil \frac{n}{2} \rceil - 2$. See Figure 3 where $EBW_7^1$ and $EBW_7^2$ are shown. It is not difficult to verify that $v(EBW_n^\ell; Q_n) = (3 + 2\ell, \ldots, 3 + 2\ell)$.

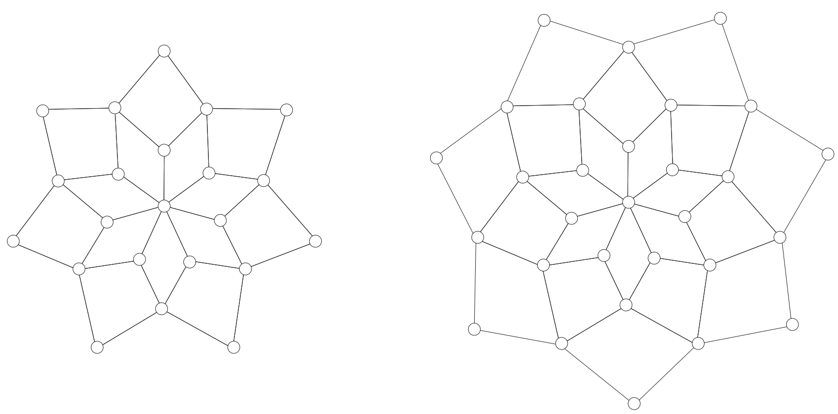

*Figure 3.* Extended wheels $EBW_7^1$ and $EBW_7^2$

It seems an interesting project to classify all partial cubes (or all median graphs) with constant edge-counting vectors.

ACKNOWLEDGEMENT. We thank the referees for several useful remarks on the presentation of the paper.

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*(Received November 10, 2005; revised November 16, 2006)*