

Note

## Strong products of Kneser graphs<sup>☆</sup>

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### Abstract

Let  $G \boxtimes H$  be the strong product of graphs  $G$  and  $H$ . We give a short proof that  $\chi(G \boxtimes H) \geq \chi(G) + 2\omega(H) - 2$ . Kneser graphs are then used to demonstrate that this lower bound is sharp. We also prove that for every  $n \geq 2$  there is an infinite sequence of pairs of graphs  $G$  and  $G'$  such that  $G'$  is not a retract of  $G$  while  $G' \boxtimes K_n$  is a retract of  $G \boxtimes K_n$ .

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### 1. Introduction and definitions

Finding a (prime factor) decomposition of a given graph with respect to a graph product is one of the basic problems in studying graph products from the algorithmic point of view. Among the four most interesting graph products (the lexicographic, the direct, the Cartesian and the strong product) the Cartesian product [1, 2, 11] and the strong product [3] are known to have polynomial algorithms for finding prime factor decompositions of connected graphs. An overview of complexity results for other products can be found in [3]. Because of Feigenbaum and Schäffer's polynomial result, it seems to be of vital interest to study those parameters of strong products of graphs whose determination is in general NP-complete.

In this note we give a lower bound for the chromatic number of the strong product of graphs, thus answering questions of Vesztergombi [9] and Jha [5]. Kneser graphs are then used to demonstrate that this lower bound is sharp. In Section 3 we give some more insight into the structure of retracts of strong products of graphs.

All graphs considered in this note will be undirected, simple graphs, i.e., graphs without loops or multiple edges. An  $n$ -coloring of a graph  $G$  is a function  $f$  from  $V(G)$

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to  $\{1, 2, \dots, n\}$ , such that  $xy \in E(G)$  implies  $f(x) \neq f(y)$ . The smallest number  $n$  for which an  $n$ -coloring exists is the *chromatic number*  $\chi(G)$  of  $G$ . The size of a largest complete subgraph of a graph  $G$  will be denoted by  $\omega(G)$ . A subgraph  $R$  of a graph  $G$  is a *retract* of  $G$  if there is an edge-preserving map  $r: V(G) \rightarrow V(R)$  with  $r(x) = x$ , for all  $x \in V(R)$ . The map  $r$  is called a *retraction*.

The *strong product*  $G \boxtimes H$  of graphs  $G$  and  $H$  is the graph with vertex set  $V(G) \times V(H)$  and  $(a, x)(b, y) \in E(G \boxtimes H)$  whenever  $ab \in E(G)$  and  $x = y$ , or  $a = b$  and  $xy \in E(H)$ , or  $ab \in E(G)$  and  $xy \in E(H)$ . The *lexicographic product*  $G[H]$  of graphs  $G$  and  $H$  is the graph with vertex set  $V(G) \times V(H)$  and  $(a, x)(b, y) \in E(G[H])$  whenever  $ab \in E(G)$ , or  $a = b$  and  $xy \in E(H)$ .

## 2. A lower bound and Kneser graphs

Vesztergombi [9] shows that if both  $G$  and  $H$  have at least one edge then  $\chi(G \boxtimes H) \geq \max\{\chi(G), \chi(H)\} + 2$ . In [5] Jha generalizes this lower bound to  $\chi(G \boxtimes H) \geq \chi(G) + n$ , where  $n = \omega(H)$ . Both authors ask for a better lower bound.

In [8] Stahl introduces the  *$n$ -tuple coloring* of a graph  $G$  as an assignment of  $n$  distinct colors to each vertex of  $G$ , such that no two adjacent vertices share a color. Further,  $\chi_n(G)$  is the smallest number of colors needed to give  $G$  an  $n$ -tuple coloring. It is straightforward to verify that

$$\chi_n(G) = \chi(G \boxtimes K_n).$$

One can also derive this as follows. Stahl [8] observes that  $\chi_n(G) = \chi(G[K_n])$ . Since the graphs  $G[K_n]$  and  $G \boxtimes K_n$  are isomorphic we conclude  $\chi_n(G) = \chi(G \boxtimes K_n)$ .

It follows from the previous remark that the case where one of the factors in the strong product is a complete graph has already been addressed in [8]. This is of special interest, because if  $\chi(H) = \omega(H) = n$  then it is not hard to see that  $\chi(G \boxtimes H) = \chi(G \boxtimes K_n)$ . In [8] a lower bound on  $\chi_n(G)$  is given. Here we present an alternative short proof of this lower bound. Our proof is a generalization of the proofs given in [9, 5].

**Theorem 2.1.** *If  $G$  has at least one edge then*

$$\chi(G \boxtimes K_n) \geq \chi(G) + 2n - 2.$$

**Proof.** We may assume  $n \geq 2$ . Let  $\chi(G \boxtimes K_n) = n + s$ . Since  $G$  has at least one edge,  $s \geq n$ . Let  $f$  be an  $(n + s)$ -coloring of  $G \boxtimes K_n$ . Let  $V(K_n) = \{v_1, v_2, \dots, v_n\}$ .

For  $u \in V(G)$  set  $m_u = \min\{f(u, v_1), f(u, v_2), \dots, f(u, v_n)\}$ . Note that  $m_u \leq s + 1$ . Define a mapping  $g: G \rightarrow \{1, 2, \dots, s + 2 - n\}$  by

$$g(u) = \begin{cases} m_u, & m_u \leq s + 1 - n; \\ s + 2 - n, & m_u \geq s + 2 - n. \end{cases}$$

We claim that  $g$  is a coloring of  $G$ . Suppose that  $uv \in E(G)$  and  $g(u) = g(v)$ . If  $g(u) = g(v) \leq s + 1 - n$ , then  $m_u = m_v$ , which is impossible since  $uv \in E(G)$ . Suppose that  $g(u) = g(v) = s + 2 - n$ . As  $uv \in E(G)$  the vertices  $\{u, v\} \times V(K_n)$  induce a complete  $K_{2n}$  in  $G \boxtimes K_n$ . Hence, these  $2n$  vertices should be colored with different colors from the set  $\{s + 2 - n, s + 3 - n, \dots, s + n\}$  which contains only  $2n - 1$  elements. This contradiction proves the claim.

It follows that  $\chi(G) \leq s + 2 - n$ . Since  $s = \chi(G \boxtimes K_n) - n$  we get

$$\chi(G) \leq \chi(G \boxtimes K_n) - 2n + 2,$$

which completes the proof.  $\square$

**Corollary 2.2.** *If  $G$  has at least one edge then*

$$\chi(G \boxtimes H) \geq \chi(G) + 2\omega(H) - 2,$$

*for any graph  $H$ .*

The lower bound of Theorem 2.1 is sharp in the sense that for every  $n$  there exists a graph for which this bound is attained. We illustrate this using Kneser graphs. The vertices of Kneser graph  $KG_{n,k}$  are the  $n$ -subsets of the set  $\{1, 2, \dots, 2n + k\}$  and two vertices are adjacent if and only if they are disjoint. Lovász proves in [7] that  $\chi(KG_{n,k}) = k + 2$ , thus setting a conjecture of Kneser. Vesztegombi observes in [10] that  $\chi(KG_{n,k} \boxtimes K_n) \leq 2n + k$ . On the other hand, combining Lovász's result with Theorem 2.1 we get  $\chi(KG_{n,k} \boxtimes K_n) \geq k + 2 + 2n - 2 = 2n + k$ . Thus

**Corollary 2.3.** *For  $n \geq 1$  and  $k \geq 0$*

$$\chi(KG_{n,k} \boxtimes K_n) = 2n + k.$$

### 3. On retracts of strong products

For the strong product  $G \boxtimes H$  of connected graphs  $G$  and  $H$ , it is shown in [4] that every retract  $R$  of  $G \boxtimes H$  is of the form  $R = G' \boxtimes H'$ , where  $G'$  is an isometric subgraph of  $G$  and  $H'$  is an isometric subgraph of  $H$ . It is also conjectured in [4] that every retract of strong products of a large class of graphs is a product of retracts of the factors. The conjecture is true for triangle-free graph.

It is demonstrated in [6] that there exist graphs  $G, H, G'$  and  $H'$  such that  $G' \boxtimes H'$  is a retract of  $G \boxtimes H$  yet  $G'$  is not a retract  $G$ . The examples are based on the Mycielski graphs. Using Corollary 2.3 we are able to construct another series of counterexamples. These examples involve arbitrary complete graphs while the examples with the Mycielski graphs admit only  $K_2$  as a second factor.

**Theorem 3.1.** *For every  $n \geq 2$  there is an infinite sequence of pairs of graphs  $G$  and  $G'$  such that  $G'$  is not a retract of  $G$  while  $G' \boxtimes K_n$  is a retract of  $G \boxtimes K_n$ .*

**Proof.** Let  $k \geq 2$  and let  $H_{n,k}$  be a graph which we get from a copy of the graph  $KG_{n,k}$  and a copy of the complete graph  $K_{k+1}$  by joining a vertex  $u$  of  $KG_{n,k}$  with a vertex  $v$  of  $K_{k+1}$ . We claim that  $\chi(H_{n,k} \boxtimes K_n) = n(k+1)$ . By Corollary 2.3 we have a  $(2n+k)$ -coloring  $f$  of  $KG_{n,k} \boxtimes K_n$ . Next we color the layer  $\{v\} \boxtimes K_n$  by any  $n$  colours from  $f$  which were not used in the layer  $\{u\} \boxtimes K_n$ . Extend the coloring in the obvious way to get a  $n(k+1)$ -coloring of  $H_{n,k}$  and the claim is proved.

It follows that we have a retraction from  $H_{n,k} \boxtimes K_n$  onto the subgraph  $K_{k+1} \boxtimes K_n$ . Note finally that since  $\chi(H_{n,k}) = k+2$  and retracts are isochromatic subgraphs, there is no retraction  $V(H_{n,k}) \rightarrow V(K_{k+1})$ .  $\square$

We finally remark that the question of characterising graphs  $G$ ,  $H$ ,  $G'$  and  $H'$ , for which  $G' \boxtimes H'$  is a retract of  $G \boxtimes H$  where  $G'$  is not a retract  $G$  remains open.

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