Discrete Mathematics, Algorithms and Applications

A note on Frame-Stewart Conjecture*

The article [2] by Roberto Demontis deals with the so-called *Frame-Stewart Conjecture* which has been open since 1941 and makes a statement about the number of moves necessary (and sufficient) to solve the Tower of Hanoi problem in the presence of $k \geq 4$ pegs; see [3, Chapter 5] for a comprehensive description of the problem. The conjecture was confirmed for the special case $k = 4$ by T. Bousch in [1], but the general case remained open. In [2] Demontis now claims to have solved this almost 80 years old question. Unfortunately, his article contains fatal flaws that negate his main result. In order to keep research on the Frame-Stewart Conjecture alive, the authors of the present letter feel obliged to point out some of the most serious deficiencies of [2].

Our main concern is with the set X, defined in Definition 5 of [2], that plays a vital role in the paper. It consists of all *costs* of discs, i.e. the numbers of moves made by them, that occur in an *ideal sequence* of moves for n discs, labelled 1 to n from small to large, and k pegs. In particular, any minimal *demolishing sequence*, leading from the start configuration with all n discs united on one peg up to and including the only move made by the largest disc n, is ideal according to Definition 4, part (1) of [2]. Demontis summarizes his allegations at the top of the last page with the claim (his point (1)) that X is made up of all powers of 2, a “fact” on which the argument for his Theorem 1 heavily depends. The following observation shows that this claim, and consequently Demontis’ proof, are false.

Proposition 1. *There is a symmetric (with respect to the central move of the largest disc) minimal solution for the Tower of Hanoi problem with* 4 *pegs and* 11 *discs where the number of moves made by disc 1 is not a power of 2.*

Proof. For moves we will use the notation of [2], but instead of writing ∞ when a disc is put on some empty peg, we will be more specific and denote (the bottoms of) the pegs by A, B, C, and D and assume that we start with all discs on peg A. So $(1, 2, D)$ means that disc 1, lying on disc 2 (on peg A) is moved to the empty peg D. We define the sequence of moves

$$\begin{aligned} S_1 = {} & (1,2,D)(2,3,B)(3,4,C)(2,B,3)(4,5,B) \\ & (2,3,5)(3,C,4)(2,5,3)(1,D,2)(5,6,D)(6,7,C)(5,D,6)(7,8,D)\,. \end{aligned}$$

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This is a demolishing sequence for discs 1 to 7 ending with discs 1 to 4 on peg B, 5 and 6 on C, and 7 on D; all larger discs are still on their starting peg A. We continue with the following sequence obtained by reflection from the previous one:

$$\begin{aligned} S_2 = &(5,6,8)(6,C,7)(5,8,6)(1,2,8)(2,3,5)(3,4,C)(2,5,3) \\ &(4,B,5)(2,3,B)(3,C,4)(2,B,3)(1,8,2)\,. \end{aligned}$$

All discs from 1 to 7 are united on peg D. The so far untouched discs larger than 7 are now transferred, avoiding peg D, according to the sequence

$$S_3 = (8,9,C)(9,10,B)(8,C,9)(10,11,C)(8,9,11)(9,B,10)(8,11,9)(11,A,B)\,.$$

This results in disc 11 on peg B, discs 8 to 10 on C and the smaller ones on D. So the sequence $S = S_1S_2S_3$ is a demolishing sequence and known to be minimal according to [1] because the total length is 33. The number of moves made by discs 1 through 11 in this sequence is 4, 8, 4, 2, 4, 2, 1, 4, 2, 1, and 1, respectively.

Similarly, if we replace S_2 with

$$\begin{aligned} S_2' = &(5,6,8)(6,C,7)(5,8,6)(1,2,5)(2,3,C)(3,4,8)(1,5,3) \\ &(4,B,5)(1,3,B)(3,8,4)(2,C,3)(1,B,2)\,, \end{aligned}$$

we find that $S' = S_1S_2'S_3$ is a minimal demolishing sequence where the number of moves made by discs 1 through 11 is 6, 6, 4, 2, 4, 2, 1, 4, 2, 1, and 1, respectively. If we extend this sequence symmetrically with the 32 moves obtained from the first 32 moves performed in inverse order and with pegs A and B interchanged, we obtain a minimal solution to transfer all discs from peg A to peg B with the number of moves made by discs 1 through 11 being 12, 12, 8, 4, 8, 4, 2, 8, 4, 2, and 1, respectively. $\square$

Remark. The proof shows that $\{1,2,4,6,8\} \subset X$ and if we write M for the M_4 of [2, Definition 6], we see from our sequences S and S' that $M(1) \geq 3$, $M(2) \geq 3$, $M(4) \geq 4$, $M(6) \geq 2$, and $M(8) \geq 1$.

Let us assume for the moment that $x_1 = 1$, $x_2 = 2$, $x_3 = 4$, and $x_4 = 6$ (i.e. that 3 and 5 are not in X), and that $M(x_1) = 3 = M(x_2)$, $M(x_3) = 4$, and $M(x_4) = 2$. Considering Lemma 2 of [2] for $n = 11$, we have $M(x_1) + M(x_2) + M(x_3) = 10$, so $i = 4$ and $T = 1$. The lower bound then becomes

$$H_4(11) \geq 1 + (2 \times 2) + (3 \times 2 \times 2) + (4 \times 2 \times 4) + (1 \times 2 \times 6) = 61\,.$$

But the Frame-Stewart methods make 65 moves for the 4-peg, 11-disc problem, so this is not a sharp bound on the Frame-Stewart cost! If any of the assumptions made earlier in this paragraph are false, the bound becomes still lower because then the cost of a disc of low cost will replace the cost of a disc of higher cost in the estimate in Lemma 2, yielding a decreased lower bound. The problem here is that while there exist ideal sequences, e.g. S, in which 4 discs each move 4 times, and there exist ideal sequences, e.g. S', in which 2 discs each move 6 times, Lemma 2 comes up short of the Frame-Stewart cost by allowing for the possible existence of an ideal sequence with both properties.

In fact, Lemma 2 cannot be used to prove that the 4-peg Frame-Stewart sequences are minimal for any $n \geq 11$. It is well known that the Frame-Stewart methods can generate demolishing sequences in which $i+1$ discs each make 2^{i-1} moves for all $i \geq 2$. If these demolishing sequences are in fact minimal, they are ideal sequences, and we have $M(2^{i-1}) \geq i+1$ for $i \geq 2$. Under the assumptions that $x_i = 2^{i-1}$ and $M(x_i) = i+1$, the bound in Lemma 2 is equal to the Frame-Stewart cost. The existence of 6 in set X lowers this bound by replacing the cost of the most costly disc with a cost of 6.

The proof of Theorem 1 in [2] depends on Lemma 4, which depends in turn on Lemma 2; any attempt to prove the minimality of the Frame-Stewart sequences along this route is doomed to failure.

There are other fatal flaws in [2]. Corollary 3 can *not* be proved as easily as Demontis wants to make us believe: it is true that disc x can be put on a peg which is never visited by larger discs, but the smaller ones may be, after the (first) move of x, in a distribution disadvantageous for the moves of the larger discs. So Demontis' Corollary 3 still has the status of a conjecture only which is probably, if at all, not easy to verify. Moreover, a logical blunder occurs immediately after the proof of Lemma 3: $I_k(x_i) \leq L_k(x_{i-1})$ does *not* imply that $m_k(x_i) \leq M_k(x_{i-1})$. The hypothesis of Lemma 4 has not been verified, so the conclusion is not valid. A serious logical mishap is following Lemma 5: the claims that $I_k(x_i) \leq \binom{k-3+i}{k-2}$ and $I_k(x_{i-1}) \leq \binom{k-4+i}{k-2}$ imply almost nothing about $I_k(x_i) - I_k(x_{i-1})$. In fact, the data strongly suggest that $x_3 = 4$, $x_4 = 6$, and $I_3(x_3) = 6 = I_3(x_4)$, which would make $m_3(x_4) = 0$, contradicting the claim just before Lemma 6. With $m_3(x_4) = 0$, the hypothesis of Lemma 6 (whose meaning has to be guessed because i is used simultaneously as a free and as a bound variable) is false, making its conclusion invalid; in fact, we have seen that the conclusion is false.

Demontis definitely hasn't answered the question he posed in the title of his note. What P. K. Stockmeyer wrote in 1994 [4, p. 4] is valid today, in 2019:

"But the optimality of the Frame-Stewart algorithm remains a conjecture."

References

[1] T. Bousch, La quatrième tour de Hanoï, *Bull. Belg. Math. Soc. Simon Stevin* **21** (2014) 895–912.

[2] R. Demontis, What is the least number of moves needed to solve the k-peg Towers of Hanoi problem?, *this journal* **11**(1) (2019) 1930001.

[3] A. M. Hinz, S. Klavžar, U. Milutinović and C. Petr, *The Tower of Hanoi—Myths and Maths* (Springer/Birkhäuser, Basel, 2013).[a]

[4] P. K. Stockmeyer, Variations on the Four-Post Tower of Hanoi Puzzle, *Congr. Numer.* **102** (1994) 3–12.

[a] Although there is a second edition of this book (Springer/Birkhäuser, Cham, 2018), we nevertheless cite the first edition because it was available at the time of submission of [2].

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