Graphs with Nonempty Intersection of Longest Paths

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Abstract

We prove that the intersection of longest paths in a connected graph G is nonempty if and only if for every block B of G the longest paths in G which use at least one edge of B have nonempty intersection. This result is used to show that if every block of a graph G is Hamilton-connected, almost-Hamilton-connected, or a cycle then all longest paths in G intersect. (We call a bipartite graph almost-Hamilton-connected if every pair of vertices is connected by a path containing an entire bipartition set.) We also show that in a split graph all longest paths intersect. (A graph is split if there exists a partition of its vertex set into a stable set and a complete set).

1 Introduction

Throughout the paper, all graphs considered will be finite and *connected*.

In [3] T. Gallai asked if it is true that in every graph G there exists a vertex which is contained in each longest path of G. H. Walther [10] answered this question negatively by exhibiting an example of a planar bipartite graph on 25 vertices, in which every vertex is missed by some longest path. The smallest known graph with empty intersection of longest paths which has 12 vertices is due to T. Zamfirescu [12] (see Fig. 1). Many further examples of such graphs can be found in [12], and also [5].

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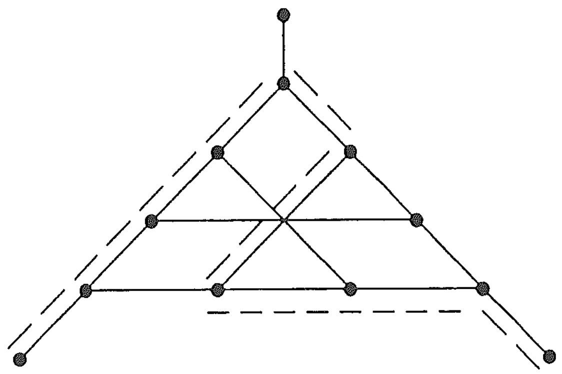


Figure 1: Zamfirescu's graph and a longest path.

In this paper we investigate some sufficient conditions for a graph to have nonempty intersection of longest paths. In Section 2 we present some simple examples of such graphs, including the class of split graphs. Motivated by Zamfirescu's graph which is not 2-connected, we examine in the main part of the paper under what additional conditions on the blocks does the graph itself have nonempty intersection of longest paths. In Section 3, we derive a necessary and sufficient condition which is satisfied, for example, by graphs whose blocks are cycles, cubes, complete bipartite graphs, or Hamilton–connected graphs.

2 Terminology and examples

For a graph G, let $\mathbf{P}(G)$ denote the set of all longest paths in G. If B is a block of G, let $\mathbf{P}_B(G)$ denote the set of all longest paths in G which have at least one edge in B. We shall denote the length of a path P by $|P|$. If P is a path in G and x, y two vertices of P, let P_{xy} denote the section of P between x and y. By intersection of two or more paths we mean the intersection of their vertex sets. Using this convention, we are interested in conditions on G which imply that $\cap\mathbf{P}(G) \neq \emptyset$. We shall often use the following well-known result. For a proof, see Ore [7, p. 31].

Lemma 2.1. *If $P, Q \in \mathbf{P}(G)$ then $P \cap Q \neq \emptyset$.*

A graph G is *traceable* if it possesses a Hamilton path, and *hypotraceable* if it is not traceable but $G\backslash v$ is traceable for every $v \in V(G)$. A graph is *Hamilton–connected* if every pair of its vertices is connected by a Hamilton path. A bipartite graph is *almost-Hamilton-connected* if every pair of its vertices is connected by a path which contains all vertices of one of the bipartition sets; we shall call such a path an *almost-Hamilton path.* A related concept is that of *Hamilton-laceable graphs* defined by G. Simmons for the case when the cardinalities of the two bipartition sets differ by at most one (cf. [8]).

A graph is *split* if there exists a partition of its vertex set into a stable set and a complete set. A *block graph* is a graph which arises as the intersection graph of the blocks of some graph. A *cactus* is a graph in which no two circuits share an edge.

Example 2.2. If G is traceable then every longest path in G is a Hamilton path and hence $\cap\mathbf{P}(G) = V(G)$. Thus every sufficient condition for traceability is also sufficient for nonemptiness of $\cap\mathbf{P}(G)$. If G is vertex-transitive then the converse holds as well: if $\cap\mathbf{P}(G) \neq \emptyset$ then $\cap\mathbf{P}(G) = V(G)$ and hence G is traceable. Therefore the famous question of L. Lovász [6] whether every vertex-transitive graph is traceable can be rephrased in the following way: Is it true that $\cap\mathbf{P}(G) \neq \emptyset$ for every vertex-transitive graph G?

Example 2.3. If G is hypotraceable then obviously $\cap\mathbf{P}(G) = \emptyset$. An infinite family of hypotraceable graphs was constructed by C. Thomassen in [9].

Example 2.4. A *geodesic* is a path which is shortest between its endpoints. In a tree, longest paths are the same as longest geodesics, which all contain the center of the tree (see, for example, Ore [7, p. 64]). Hence $\cap\mathbf{P}(T) \neq \emptyset$, for every tree T. For an alternative proof of this fact, recall that a family of subtrees of a tree satisfies the Helly property: if any two subtrees in the family intersect, then all subtrees in the family intersect (see, for example, Golumbic [4, p. 92]. As $\mathbf{P}(T)$ is a family of subtrees of T any two of which intersect, by Lemma 2.1, the Helly property implies that $\cap\mathbf{P}(T) \neq \emptyset$. We shall use this technique in the proof of our main result, Theorem 3.3.

Proposition 2.5. *If G is a split graph, then* $\cap\mathbf{P}(G) \neq \emptyset$.

Proof. Let $V(G) = K \cup S$ be a partition of the vertices of G into a clique K and a stable set S. Let S' be a maximal stable set containing S, and $K' = K - S'$. $V(G) = K' \cup S'$ is again a partition of the vertices of G into a clique K' and a stable set S'. Note that $|S' \setminus S| = |K \setminus K'| \leq 1$.

If $K' = \emptyset$ then $V(G) = S'$. Since G is connected it follows that $G \cong K_1$ for which the assertion is obvious. Otherwise let P be a longest path in G. Suppose that for some $x \in K'$, $x \notin V(P)$. Both endpoints of P belong to S', for otherwise Px or xP would be a longer path. Therefore $P = P'uv$, where $u \in K'$ and $v \in S'$ (see Fig. 2.a).

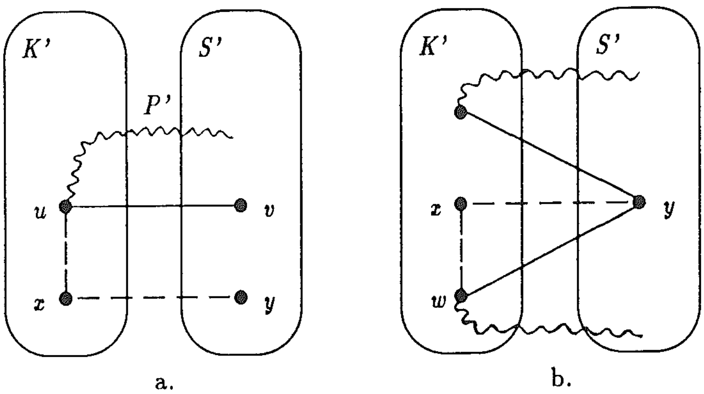


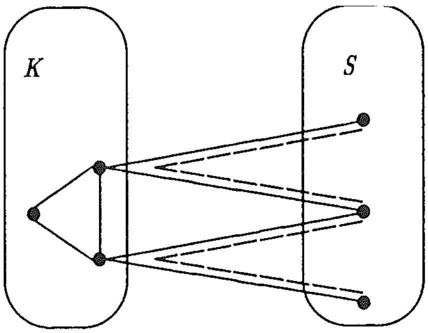
Figure 2: Longest paths in split graphs.

As S' is a maximal stable set, there exists a vertex $y \in S'$ such that $xy \in E(G)$.

If $y \notin V(P)$ then $Q := P'uxy$ is a longer path. If $y \in V(P)$ then let $w \in K'$ be a neighbor of y on P. In this case it is possible to insert x between w and y on P and thus to prolong P (see Fig. 2.b). In both cases, the assumption $x \notin V(P)$ leads to contradiction. It follows that $K' \subseteq \cap \mathbf{P}(G)$ which is therefore not empty. □

That $K \subseteq \cap \mathbf{P}(G)$ need not hold is demonstrated by the graph in Fig. 3.

Figure 3: A split graph and a longest path.



3 Longest Paths in Graphs with Connectivity 1

It may happen that $\cap\mathbf{P}(G) = \emptyset$ although $\cap\mathbf{P}(B) \neq \emptyset$, for every block B of G. An example is furnished by Zamfirescu's graph (see Fig. 1). The purpose of this section is to show that the condition $\cap\mathbf{P}_B(G) \neq \emptyset$, for every block B of G, suffices for $\cap\mathbf{P}(G) \neq \emptyset$.

If e is an edge of G, let $B(e)$ denote the unique block of G containing e.

Lemma 3.1. *Let $e_1e_2\ldots e_k$ be the sequence of edges of a path P in G. If $B(e_i) = B(e_j)$ and $i < j$ then $B(e_i) = B(e_{i+1}) = \ldots = B(e_j)$.*

Proof. Otherwise there exists a path in G which connects two vertices of $B(e_i)$ without using any edge of $B(e_i)$. Since any two vertices of $B(e_i)$ are also connected by a path in $B(e_i)$ one gets a circuit which passes through more than one block of G, a contradiction. □

Let P be a path in G, B a block of G, and c a cutpoint of G. We shall call B an *essential block for P* if P uses at least one edge of B, and c an *essential cutpoint for P* if c is a common cutpoint of two essential blocks for P.

Let $T(G)$ denote the block-cutpoint tree of G. If P is a path in G, and $e_1e_2\ldots e_k$ the sequence of edges of P, then let $f(P)$ denote the unique path connecting the vertices corresponding to $B(e_1)$ and $B(e_k)$ in $T(G)$.

Lemma 3.2. *Let P be a path in G. Then a vertex of $T(G)$ lies on $f(P)$ if and only if it corresponds to an essential block or to an essential cutpoint for P.*

Proof. Let $e_1e_2\ldots e_k$ be the sequence of edges of P. The sequence

$$B(e_1)B(e_2)\ldots B(e_k) \tag{1}$$

contains all the blocks which are essential for P. Let

$$B_1B_2\ldots B_m \tag{2}$$

be a maximal subsequence of (1) with the property that $B_i \neq B_{i+1}$, for $i = 1, 2, \ldots, m-1$. (To obtain (2), replace each group of consecutive identical blocks in (1) with a single block from the group.) By Lemma 3.1, all blocks appearing in (2) are distinct. Let c_i denote the common cutpoint of B_i and B_{i+1} in (2), for $i = 1, 2, \ldots, m-1$. Then the cutpoints $c_1, c_2, \ldots, c_{m-1}$ are essential for P, and there are no others, or else there would exist a circuit not contained in any single block. The sequence of vertices of $T(G)$ corresponding to the sequence

$$B_1 c_1 B_2 c_2 \ldots B_{m-1} c_{m-1} B_m$$

is a path connecting the vertices corresponding to $B(e_1)$ and $B(e_k)$ in $T(G)$. As this path is unique, it coincides with $f(P)$, and the lemma is proved. □

Theorem 3.3. *$\cap \mathbf{P}(G) \neq \emptyset$, if and only if $\cap \mathbf{P}_B(G) \neq \emptyset$, for every block B of G.*

Proof. That this condition is necessary follows from the fact that $\mathbf{P}_B(G) \subseteq \mathbf{P}(G)$. To prove sufficiency, we distinguish two cases.

Case 1. For every pair of paths $P, Q \in \mathbf{P}(G)$, there exists a block B which is essential for both P and Q.

Let $T(G)$ be the block-cutpoint tree of G, and $f(P)$ the path corresponding to P in $T(G)$. We claim that in this case all pairs of paths in the family $\{f(P) \mid P \in \mathbf{P}(G)\}$ intersect. By Lemma 2.1, $P \cap Q \neq \emptyset$. Let B be an essential block for both P and Q. Then by Lemma 3.2, both $f(P)$ and $f(Q)$ contain the vertex corresponding to B in $T(G)$, proving the claim.

By Helly property, there exists a vertex $v \in V(T(G))$ which is contained in $f(P)$, for every $P \in \mathbf{P}(G)$. If v corresponds to a block B of G then by Lemma 3.2, $\mathbf{P}(G) = \mathbf{P}_B(G)$, and therefore $\cap \mathbf{P}(G) \neq \emptyset$. If v corresponds to a cutpoint c of G then by Lemma 3.2, c lies on every $P \in \mathbf{P}(G)$ which again implies that $\cap \mathbf{P}(G) \neq \emptyset$.

Case 2. There exists a pair of paths $P, Q \in \mathbf{P}(G)$ such that no block of G is essential for both P and Q.

By Lemma 2.1, $P \cap Q \neq \emptyset$. If $P \cap Q$ contains more than one vertex then either P and Q share an edge or there exists a circuit in G formed by edges of P and Q. In both cases, P and Q have a common essential block, contrary to our assumption. Thus, let x be the unique vertex in $P \cap Q$.

We claim that $x \in \cap \mathbf{P}(G)$. Assume that $R \in \mathbf{P}(G)$ does not contain x. By Lemma 2.1, $P \cap R \neq \emptyset$ and $Q \cap R \neq \emptyset$. Let $y \in P \cap R$ be such that P_{xy} contains no other vertex of R, and let $z \in Q \cap R$ be such that Q_{xz} contains no other vertex of R. The assumption that R does not contain x implies that $x \neq y$, $x \neq z$ and $y \neq z$ (see Fig. 4).

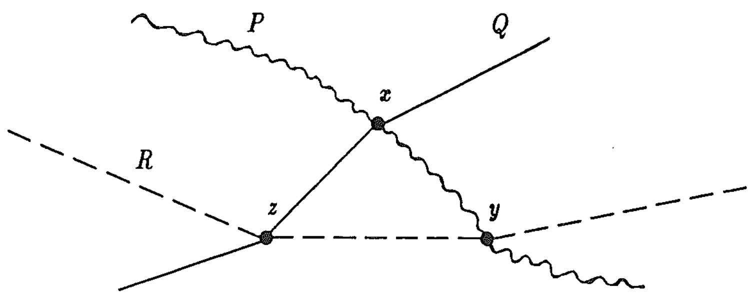


Figure 4: P, Q and R form a circuit.

Let W be the closed walk obtained by concatenating P_{xy}, R_{yz}, and Q_{zx}, in this order. We are going to show that W is a circuit. Let u be a vertex of P_{xy}. If $u \in Q_{zx}$ then $u \in P \cap Q$ and hence $u = x$. If $u \in R_{yz}$ then $u \in P \cap R$. By our choice of y, no inner vertex of P_{xy} belongs to R. Therefore $u = y$. This proves that $P_{xy} \cap Q_{zx} = \{x\}$ and $P_{xy} \cap R_{yz} = \{y\}$. Interchanging the roles of P_{xy} and Q_{zx}, one sees likewise that $Q_{zx} \cap R_{yz} = \{z\}$. It follows that W is a circuit.

Let B be the block of G containing W. As $x \neq y$ and $x \neq z$, the lengths of P_{xy} and Q_{zx} are positive. This implies that B is essential for both P and Q, contrary to our assumption. Therefore R must contain x. Since R was arbitrary, it follows that $x \in \cap \mathbf{P}(G)$, and the proof is complete. □

In the rest of the section we present several types of blocks which satisfy the condition of Theorem 3.3.

Lemma 3.4. *If a path P is longest between its endpoints then it is longest between any two of its essential cutpoints.*

Proof. Let P be a longest path connecting x and y, and let u and v be two essential cutpoints for P. Suppose that there exists a path Q connecting u and v such that $|Q| > |P_{uv}|$. We claim that Q contains no vertex of P_{xu} or P_{vy} other than u, v. If not, consider the case when Q and P_{xu} have an inner vertex in common. Let w be the first such vertex on Q. Observe that $f(P_{uv}) = f(Q)$ in $T(G)$. Therefore by Lemma 3.2, the sequences of essential blocks for P_{uv} and for Q coincide. Together with the fact that u is an essential cutpoint for P, this implies that $P_{wu}Q_{uw}$ is a circuit which contains edges from at least two different blocks, which is not possible. Similarly, one can show that no inner vertex of Q lies on P_{vy}, thus proving the claim. Hence $P_{xu}QP_{vy}$ is a path longer than P, a contradiction. □

Proposition 3.5. *Let B be a block of a graph G. If B is Hamilton-connected, then $\cap\mathbf{P}_B(G) \neq \emptyset$.*

Proof. If $P \in \mathbf{P}_B(G)$, then P enters B at two distinct vertices. As B contains a Hamilton path between these two vertices and P is a longest path, P passes through every vertex of B, by Lemma 3.4. □

Example 3.6. Wheels, complete graphs on $n \geq 4$ vertices with at most $n-4$ edges removed, and graphs on n vertices with minimal degree $\delta \geq (n+1)/2$ are all Hamilton-connected (see, for example, Capobianco and Molluzzo [2, p. 182].

It is easy to see that a Hamilton-connected block is either hamiltonian or K_2. That the condition "Hamilton-connected" in Proposition 3.5 cannot be weakened to "hamiltonian or K_2" is again demonstrated by Zamfirescu's graph in Fig. 1, which consists of one hamiltonian block and three K_2's.

Proposition 3.7. *Let B be a bipartite block of a graph G. If B is almost-Hamilton-connected, then $\cap\mathbf{P}_B(G) \neq \emptyset$.*

Proof. Let $V(B) = V_1 \cup V_2$ be a bipartition of B, and $P \in \mathbf{P}_B(G)$. First we prove that

$$V_1 \subseteq V(P) \quad \text{or} \quad V_2 \subseteq V(P). \tag{3}$$

If not, let u and v be the vertices in which P enters B, and R an almost-Hamilton path between u and v. By definition, R contains all vertices of V_1 or V_2, and hence $|R| > |P_{uv}|$. But this is in contradiction with Lemma 3.4.

WLOG assume that $|V_1| \leq |V_2|$. First we consider the case $|V_1| < |V_2|$. As the cardinalities of $V(P) \cap V_1$ and $V(P) \cap V_2$ may differ by at most one, it follows from (3) that every $P \in \mathbf{P}_B(G)$ contains all vertices of V_1.

In the case $|V_1| = |V_2|$ there are two subcases. If every $P \in \mathbf{P}_B(G)$ enters B in different bipartition sets then by (3), $V(B) \subseteq \cap\mathbf{P}_B(G)$. In the opposite case, let $P \in \mathbf{P}_B(G)$ be a path which enters B in the same bipartition set, say V_1. We claim that then $V_1 \subseteq \cap\mathbf{P}_B(G)$. If not, there exists a path $Q \in \mathbf{P}_B(G)$ which enters B at $x, y \in V_2$. Let $P = P_1P_2P_3$ and $Q = Q_1Q_2Q_3$, where P_2 and Q_2 are the sections of P and Q within B. WLOG assume that $|P_1| \geq |P_3|$ and $|Q_1| \geq |Q_3|$. Let R be an almost-Hamilton path in B connecting the common vertex of P_1 and P_2 with the common vertex of Q_1 and Q_2. It follows from (3) that $|R| = |P_2| + 1 = |Q_2| + 1$. Therefore $|Q_1| \geq |P_3|$. Now the concatenation of P_1, R and Q_1 is longer than P. □

Example 3.8. Complete bipartite graphs $K_{m,n}$ are almost–Hamilton–connected. If $m \neq n$ then any longest path between two vertices contains the smaller set, and if $m = n$ then a longest path between two vertices avoids at most one vertex.

Example 3.9. In the n-cube Q_n, two vertices are connected by a Hamilton path if they are at odd distance, and by a path which contains all but one vertex if they are at positive even distance. This can be easily proved by induction on n. Hence n-cubes are almost–Hamilton–connected.

Proposition 3.10. *Let B be a block of G with the following property: If P is a path in B and u a vertex of B not on P then there exists a path Q in B connecting one endpoint of P with u, such that $|Q| > |P|$. Then $\cap \mathbf{P}_B(G) \neq \emptyset$.*

Proof. For every vertex $v \in V(B)$, let P_v be a longest path starting at v and avoiding all other vertices of B, and let $l(v) := |P_v|$. Let u be a vertex of B such that $l(u) = \max\{l(v) \mid v \in V(B)\}$. We claim that $u \in \cap \mathbf{P}_B(G)$.

To see this, suppose that $P \in \mathbf{P}_B(G)$ and $u \notin V(P)$. Let x, y be the vertices in which P enters B. By hypothesis of the proposition there exists a path Q in B connecting u with one endpoint – say x – of P_{xy}, and longer than P_{xy} (see Fig. 5). Then the path $P_x Q P_u$ is longer than P, proving the claim. □

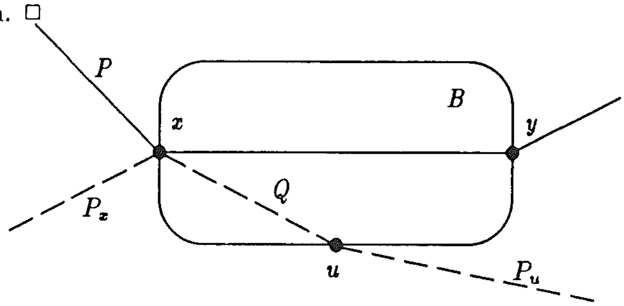


Figure 5: Longest paths passing through a block.

Example 3.11. Cycles and Hamilton–connected graphs satisfy the condition of Proposition 3.10. In a cycle, any path P not containing u can be extended on either side to u. In a Hamilton–connected graph, any Hamilton path between an endpoint of P and u can play the role of Q.

Corollary 3.12. *If every block of G is Hamilton–connected, almost–Hamilton–connected or a cycle then $\cap \mathbf{P}(G) \neq \emptyset$.*

Proof. This follows from Theorem 3.3, using Propositions 3.5, 3.7 and 3.10. □

Corollary 3.13. *If G is a block graph or a cactus, then* $\cap P(G) \neq \emptyset$.

Proof. It is well known that a graph is a block graph if and only if its blocks are complete, and that it is a cactus if and only if its blocks are cycles or K_2's. Thus the assertion follows from Corollary 3.12. □

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