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Note

On median graphs and median grid graphs

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Abstract

Let G be a Q_4-free median graph on n vertices and m edges. Let k be the number of equivalence classes of Djoković–Winkler's relation Θ and let h be the number of Q_3's in G. Then we prove that $2n - m - k + h = 2$. We also characterize median grid graphs in several different ways, for instance, they are the grid graphs with $m - n + 1$ squares. To obtain these results we introduce the notion of square-edges, i.e. edges contained in exactly one 4-cycle of a graph.

Keywords: Median graph; Cube-free median graph; Planar graph; Grid graph

1. Introduction

Median graphs have been extensively studied by now, see [4] for a bibliographic survey on their characterizations and related topics. A milestone in the theory of median graphs is Mulder's convex expansion theorem from [7,8]. We will apply this theorem in the sequel, however, to keep this note reasonable short we will not explain it in detail. Since it appeared several time in the literature we believe this is justified.

In this note we first characterize when a graph, obtained from a median graph by adding or removing an edge, is a median graph. Then we give a relation between the number of vertices, the number of edges, the number of Θ-classes and the number of

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induced cubes in a Q_4-free median graph. In this way, we extend the result of [5]. In the last section we characterize median grid graphs in several different ways.

In what follows, we give necessary definitions. A cycle of length four is called a 4-*cycle* or a *square*. The *distance* $d(u,v)$ between vertices u and v of a graph G will be the usual shortest path distance. A subgraph H of a graph G is an *isometric* subgraph if $d_H(u,v) = d_G(u,v)$, for all $u,v \in V(H)$. A subgraph H of a graph G is *convex*, if for any u, $v \in V(H)$ all shortest u,v-paths lie completely in H. The *convex hull* of a subgraph H in a graph G, denoted by $con(H)$, is the intersection of all convex subgraphs of G containing H, i.e., the smallest convex subgraph containing H. A subgraph H of G is 2-*convex* if for any two vertices u and v of H with $d_G(u,v)=2$, every common neighbor of u and v belongs to H.

A vertex x is a *median* of a triple of vertices u, v and w if $d(u,x)+d(x,v)=d(u,v)$, $d(v,x)+d(x,w)=d(v,w)$ and $d(u,x)+d(x,w)=d(u,w)$. A connected graph G is a *median graph* if every triple of its vertices has a unique median. A (median) graph is called *cube-free* it contains no 3-cube Q_3.

For a graph G, Djoković–Winkler's relation Θ [2,9] is defined on $E(G)$ as follows. If $e=xy \in E(G)$ and $f=uv \in E(G)$, then $e\Theta f$ if $d(x,u)+d(y,v) \neq d(x,v)+d(y,u)$. It is well known that Θ defines an equivalence relation on the edge set $E(G)$ of a median graph G.

2. Square-edges in median graphs

Let e be an edge of a connected graph G. If e is a bridge then it is easy to see that G is a median graph if and only if both components of $G \backslash e$ are median graphs. In the following, we may thus assume that e lies on some cycle of G. If G is a median graph then e necessarily lies on some 4-cycle. For edges which lie on more than one 4-cycle we have:

Proposition 1. *Let e be an edge of a graph G which lies in at least two* 4-*cycles. Then not both G and $G\backslash e$ are median graphs.*

Proof. Let $G' = G\backslash e$. Let G be a median graph and let $e = uv$ lie on at least two 4-cycles, say $uvwz$ and $uvz'w'$. Since G is bipartite we have $z \neq z'$ and $w \neq w'$. If $z = w'$ or $w = z'$ then the five vertices induce $K_{2,3}$, which is not possible in a median graph. Hence w, w', z and z' are pairwise different vertices. If z is adjacent to z', or w is adjacent to w', then we again find a $K_{2,3}$. Thus u,z,w,v,z',w' induce a 6-cycle in G'. If G' would be a median graph, then G would contain a vertex r adjacent to v,w',z, i.e. r would be a median of v,w' and z. But then we obtain that r,u,v,z,w' induce a $K_{2,3}$ in G, a contradiction.

If G' is a median graph then we can almost analogously as above argue that G is not a median graph. Indeed, we first see that w, z, w' and z' are pairwise different vertices if G is assumed to be a median graph. But then we find a $K_{2,3}$ in G, a contradiction. □

By Proposition 1, the only possibility for both G and $G\setminus e$ to be median graphs is when e lies in exactly one 4-cycle. Let us call an edge which in contained in exactly one 4-cycle a *square-edge*. Denote by Q_3^- the graph which we obtain from the 3-cube Q_3 by removing a vertex.

Proposition 2. *Let e be a square-edge of a graph G.*

(i) *If G is a median graph then so is* $G\setminus e$.
(ii) *Suppose that* $G\setminus e$ *is a median graph. Then, G is a median graph if and only if e does not lie in some* Q_3^-.

Proof. For any edge ab, let W_{ab} be the set of vertices of G closer to a than to b and let U_{ab} be those vertices of W_{ab} with a neighbor in $G\setminus W_{ab}$. The sets W_{ba} and U_{ba} are defined analogously. For $X \subseteq V(G)$ let $\langle X\rangle$ denote the subgraph of G induced by X. Let $e = uv$ and let $uxyvu$ be the 4-cycle containing e.

(i) As G is median, U_{uv} is convex and so is also $U_{uv}\setminus u$. But then $G\setminus e$ is obtained from median graphs $\langle W_{uv}\rangle$ and $\langle W_{vu}\rangle$ by a convex expansion over $U_{uv}\setminus u$. Thus, $G\setminus e$ is a median graph by Mulder's convex expansion theorem.

(ii) Observe first that if G is a median graph then, since e is a square-edge, it clearly cannot lie in a Q_3^-. For the converse suppose that e does not lie in some Q_3^- and assume that G is not a median graph. Since $G\setminus e$ is a median graph, $\langle W_{ab}\rangle$ and $\langle W_{ba}\rangle$ are median graphs and convex subgraphs of G. Hence, since G is not median, it follows that $\langle U_{uv}\rangle$ is not a convex subgraph of $\langle W_{uv}\rangle$ or $\langle U_{vu}\rangle$ is not a convex subgraph of $\langle W_{vu}\rangle$. Assume the former case. Recall now that Lemma 2.1 of [3] asserts that in a connected bipartite graph in which every triple of vertices has a median, a subgraph is convex if and only if it is 2-convex and isometric. Clearly $\langle U_{uv}\rangle$ is isometric having in mind that x is the only neighbor of u in $\langle U_{uv}\rangle$. Therefore $\langle U_{uv}\rangle$ is not 2-convex in $\langle W_{uv}\rangle$ which implies that there exits a 4-cycle $uxzw$ such that $z \in U_{uv}$ and $w \notin U_{uv}$. Denote by z' the neighbor of z in U_{ba}. Then, vertices u, v, x, y, z, z', w induce a Q_3^-, a contradiction. □

It is not difficult to see that a cube-free median graph G always contains at least one square-edge, provided it is not a tree. If e is such a square-edge then from Lemma 2(i) we infer that $G\setminus e$ is a cube-free median graph. Thus, by a simple induction we obtain:

Corollary 3. *Let G be a cube-free median graph with n vertices and m edges. Then G contains* $m - n + 1$ *squares.*

With this result in hand we can prove the following extension of the main result of [5].

Theorem 4. *Let G be a Q_4-free, median graph on n vertices and m edges. Let k be the number of equivalence classes of the relation Θ and let h be the number of subgraphs of G isomorphic to Q_3. Then*

$$2n - m - k + h = 2.$$

Proof. The proof is by induction on the number of vertices. The claim is obviously true for $G = K_1$. So, we may assume that G is the convex expansion of the median graph G' with respect to the subgraphs G_1' and G_2' with $G_0' = G_1' \cap G_2'$. By induction, we have $2n' - m' - k' + h' = 2$ for G', where k', n', m' and h' are the corresponding parameters of G'. Let $t_v = |V(G_0')|$ and $t_e = |E(G_0')|$. Observe that $n = n' + t_v$, $m = m' + t_v + t + t_e$ and $k = k' + 1$. Since G is Q_4-free if follows that G_0' is cube-free. Note also that $h - h'$ is actually the number of 4-cycles of G_0'. So, by Corollary 3, we obtain that $h - h' = t_e - t_v + 1$, and so we can compute:

$$\begin{aligned} 2n - m - k + h &= 2(n' + t_v) - (m' + t_v + t_e) - (k' + 1) + (h' + t_e - t_v + 1) \\ &= 2n' - m' - k' + h' \\ &= 2. \quad \square \end{aligned}$$

By the above theorem, we can explicitly determine the number of faces of a plane median graph. Note that such a graph has no subgraph isomorphic to Q_4.

Corollary 5. *Let G be a median plane graph with n vertices, f faces, k Θ-classes and h subgraphs isomorphic to Q_3. Then $f = n - k + h$.*

Proof. Plug the Euler's formula $n - m + f = 2$ into Theorem 4. $\square$

3. Median grid graphs

A *grid graph* is a subgraph of a complete grid, where by a complete grid we mean the Cartesian product of two paths $P_n \Box P_m$. Complete grids are median graphs. In this section we address the question which grid graphs are median graphs. For instance, these graphs can be characterized as those grid graphs which contain no isometric cycles of length at least 6. (This result follows immediately from the fact that median graphs can be characterized as the graphs in which convex hulls of isometric cycles are hypercubes.) In order to present several additional characterizations of median grid graphs we need some more definitions.

A graph G is a *square-edge graph* if the following holds. There exists a sequence of connected graphs $G_0, G_1, \ldots, G_r = G$ and a sequence of edges $e_0, e_1, \ldots, e_{r-1}$ such that G_0 is a tree, e_i is a square-edge in G_i and $G_{i+1} = G_i \backslash e_i$. This class of graphs was introduced and studied in [6].

Recall that $\oplus$ denotes the symmetric sum of cycles, where we consider cycles as set of edges. Recall in addition that a *plane graph* is a planar graph together with an embedding in the plane. An edge of a plane graph is called an *outer edge* if it lies on the outer face of its embedding. In the rest of this note, whenever we will consider grid graphs as planar graphs, we will assume that they are embedded in the plane in the natural way — as a subgraph of a complete grid.

Let H be a subgraph of a plane graph G. Then we call H *inner-convex*, if $con(H)\backslash H$ contains only vertices and edges of the outer face of H. For grid graphs we have the following straightforward interpretation of this concept.

Lemma 6. *A plane grid graph is inner-convex if and only if all its inner faces are 4-cycles.*

Proof. Let H be a subgraph of a complete grid G. If every inner face of H is a 4-cycle then $con(H)\backslash H$ clearly contains only vertices of the outer face of H. Conversely, if an inner face would be of length at least 6, then $con(H)\backslash H$ would contain all the vertices and edges of G lying in this face. □

We are now ready for the main theorem of this section.

Theorem 7. *For a connected grid graph G, the following statements are equivalent.*

(i) *G is a median graph.*
(ii) *G is a square-edge graph.*
(iii) *For every cycle C of G, there exist 4-cycles $C_1, C_2, \ldots, C_k$, such that $C = C_1 \oplus C_2 \oplus \cdots \oplus C_k$.*
(iv) *G is inner-convex.*
(v) *G contains $m - n + 1$ squares.*

Proof. (i) ⇒ (ii) Let G be a median grid graph. Then it is cube-free and, as we have observed earlier, it contains at least one square-edge (provided it is not a tree). Then Proposition 2 and induction complete the argument.

(ii) ⇒ (iii) This application is proved (for any square-edge graph) in [6].

(iii) ⇒ (iv) Let G be a subgraph of $P_n \Box P_m$ and suppose that G is not inner-convex. By Lemma 6 there exists at least one inner face F which is not a 4-cycle. Let

$$t = \min\{i \mid (i,j) \in V(P_n \Box P_m) \cap F\}.$$

We may assume that t is as small as possible among all inner faces which are not 4-cycles. We claim that F cannot be obtained by a symmetric sum of 4-cycles.

Let $e=uv$ be an edge of F with $u=(t,j_1)$ and $v=(t,j_2)$ for some j_1 and j_2. As G is a plane grid graph, such an edge clearly exists. If e is not a square-edge, then the claim clearly holds. Otherwise let $e_1, e_2, \ldots, e_k$, $k \geqslant 1$, be the sequence of edges where e_1 is the edge opposite to e in the square containing e, $e_2 \neq e$ the edge opposite to e_1, … Note now that any symmetric sum of 4-cycles which contains e necessarily contains also e_k, which proves the claim.

(iv) $\Rightarrow$ (v) Let G be an inner-convex grid graph. Then it follows from Lemma 6 that there exists a sequence of connected graphs: $P_q \Box P_r = G_0, G_1, \ldots, G_s = G$, and a sequence of edges $e_0, e_1, \ldots, e_{s-1}$, such that e_i is an outer edge in G_i and $G_{i+1} = G_i \backslash e_i$. Note that any edge e_i is either a square-edge or incident to a pendant vertex (i.e. a vertex of degree 1). Clearly, $P_q \Box P_r$ has $(q-1)(p-1)$ squares which is equal to

$$|E(P_q \Box P_r)| - |V(P_q \Box P_r)| + 1 = (q(r-1) + r(q-1)) - rq + 1.$$

In addition, when we remove an edge e_i, we either remove a pendant vertex or we decrease the number of squares by one, thus any G_i has $|E(G_i)| - |V(G_i)| + 1$ squares and so has $G_s = G$.

(v) $\Rightarrow$ (i) Let G be a grid graph with $m - n + 1$ squares. We proceed by induction on the number of squares. If $m - n + 1 = 0$, then G is a tree and thus a median graph. Suppose now that G has at least one square. If e is not a square-edge it is contained in another square and let e' be the opposite edge to e in it. Repeating this procedure, having in mind that G can be represented as a plane grid graph, we find a square-edge f of G. Let $G' = G - f$ and note that G' has one edge less and one square less than G. Thus, by induction, G' is a median graph. Proposition 2(ii) completes the proof. □

We wish to add that (i) $\Rightarrow$ (iii) is proved in [1] for any median graph. We end this note with the following observation.

Proposition 8. *Let G be a plane median grid graph. Then the length of its outer face is $4n - 2m - 4$, which is twice the number of its Θ-classes.*

Proof. If G is a tree, then G has $n - 1 = 2n - m - 2$ Θ-classes. By induction, using Theorem 7(ii) we conclude that any plane median graph G has $2n - m - 2$ Θ-classes. It thus remains to show that the length of the outer face of G is twice the number of its Θ-classes.

The proof is by induction on the number k of 4-cycles of G. If $k = 0$, i.e. if G is a tree, then each Θ-equivalence class consists of a single edge and each edge is counted twice on the outer face. Hence the proposition for $k = 0$.

Suppose now that $k \geqslant 1$ and let $e = uv$ be an outer edge of G which lies in a 4-cycle. Such an edge exists because $k \geqslant 1$ and moreover, e is a square-edge. By Lemma 2 the graph $G' = G \backslash e$ is a median grid graph. Since G' has one 4-cycle less than G, the induction assumption holds for G'.

Let $uvwz$ be the 4-cycle of G containing e. Then in G the edges uz and vw belong to the same Θ-equivalence class, while in G' they are in different equivalence classes. It follows that in G' we have one more Θ-equivalence class than in G. Moreover, the length of the outer face of G' is by 2 greater than the length of the outer face of G. Since the proposition holds for G' it holds for G as well. □

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