Monophonic visibility in graphs

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Abstract

Let $S \subseteq V(G)$. Two vertices $u, v \in S$ are monophonically S-visible if there exists a (u,v)-path whose internal vertices do not belong to S. The set S is a monophonic visibility set if every pair of vertices in S is monophonically S-visible. The maximum cardinality of such a set in G is denoted by $\mu_{\rm m}(G)$. We show that the difference between the monophonic visibility number and the mutual-visibility number can be arbitrarily large, although the two parameters coincide on distance-hereditary graphs. We also prove that deciding whether $\mu_{\rm m}(G) \geq k$ is NP-complete, even when restricted to bipartite graphs of diameter 4. We establish different upper and lower bounds on the parameter $\mu_{\rm m}$ involving the order, the minimum and maximum degree, and the connected domination number. Moreover, we determine the monophonic visibility number of arbitrary cactus graphs. We also determine $\mu_{\rm m}(P_2 \,\square\, P_n)$ and $\mu_{\rm m}(P_3 \,\square\, P_n)$, and prove that $\mu_{\rm m}(P_m \,\square\, P_n) = \left(\frac{2}{3} + o(1)\right) mn$, as $m, n \to \infty$. Finally, we show that for every pair of integers $4 \leq a \leq b$, there exists a graph G such that $\mu(G) = a$ and $\mu_{\rm m}(G) = b$.

Keywords: mutual visibility; monophonic visibility set; computational complexity; distance-hereditary graph; cograph; cactus graph; grid graph

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1 Introduction

The graph-theoretic mutual-visibility problem was introduced in [14], motivated by applications in mobile robot networks, wireless sensor networks, and distributed computing. In this context, computer scientists had already investigated problems arising in distributed computing by mobile entities that need to communicate efficiently while ensuring that the messages do not pass through other entities, cf. [16, 21]. Although the graph-theoretic mutual-visibility problem dates back only to 2022, the bibliography on the problem is already huge. See [2–4, 6, 12, 19, 24, 25] and references therein.

A closely related topic is the general position problem which was independently introduced in [8,22] about ten years ago. The bibliography on this topic is even more extensive; a review article [7] lists nearly 150 references. From our perspective, we would like to point out that in 2024, the monophonic position problem was introduced and its investigation initiated in [27].

This line of research was continued in [28], where some extremal problems concerning the monophonic position number of a graph were investigated, while in [15], the lower general position number was compared with its monophonic counterpart.

Connections between the mutual-visibility and general position problems, along with the research on the monophonic position problem, inspired us to explore a monophonic version of mutual visibility.

The paper proceeds as follows. In the second part of the introduction, we provide formal definitions of the concepts under consideration, including the monophonic visibility number. In Section 2, we derive basic general results and consider distance-hereditary graphs. In Section 3, we prove that the corresponding decision problem is NP-complete, even over the class of bipartite graphs of diameter 4. In Section 4 we give different upper and lower bounds for the monophonic visibility number involving the order, minimum degree, maximum degree, and connected domination number. In particular, we prove that if G is a graph of order n and with maximum degree $\Delta \geq 2$, then $\mu(G) \leq \mu_{\rm m}(G) \leq \frac{\Delta-2}{\Delta-1}\,n + \frac{\Delta+1}{\Delta-1}$. In Section 5, we determine the monophonic visibility number of arbitrary cactus graphs and characterize the unicyclic graphs G satisfying $\mu_{\rm m}(G) = {\rm mp}(G)$. In the subsequent section, we determine $\mu_{\rm m}(P_2 \,\square\, P_n)$ and $\mu_{\rm m}(P_3 \,\square\, P_n)$, and prove that $\mu_{\rm m}(P_m \,\square\, P_n) = \left(\frac{2}{3} + o(1)\right) mn$, as $m \to \infty$ and $n \to \infty$. In the main result of Section 7 we show that for every $4 \leq a \leq b$ there exists a graph G with $\mu(G) = a$ and $\mu_{\rm m}(G) = b$. We conclude the paper with several open problems and remarks.

1.1 Definitions

Throughout the paper, the notation $[n]$ represents the set $\{1, \ldots, n\}$ for every positive integer n.

Let $G = (V(G), E(G))$ denote a graph, where $V(G)$ and $E(G)$ are its vertex and edge sets, respectively. Unless explicitly stated otherwise, we assume throughout that G is connected, since the problem studied reduces to connected components. Let $n(G)$ be the order of G and $m(G)$ its size. For any $u, v \in V(G)$, let $d_G(u,v)$ denote the length of a shortest (u,v)-path in G. The *diameter* $\mathrm{diam}(G)$ of G is the largest distance attained between two vertices of G. A subgraph H of G is *isometric* if $d_H(u,v) = d_G(u,v)$ for every $u, v \in V(H)$, and is *convex* if every shortest (u,v)-path in G belongs to H for every $u, v \in V(H)$.

For $u \in V(G)$, let $N_G(u)$ be the *open neighborhood* of u, consisting of all vertices adjacent to u, and let $N_G[u] = N_G(u) \cup \{u\}$ be its *closed neighborhood*. The *degree* of u is denoted by $\deg_G(u)$, while the minimum and maximum degrees of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. If $\deg(u) = 1$, then u is a *pendant vertex* alias *leaf*, while an edge is *pendant* if one of its endpoints is a pendant vertex. Let $\ell(G)$ be the number of pendant vertices in G. A *universal vertex* of G is a vertex with $\deg(u) = n(G) - 1$.

For $S \subseteq V(G)$, the subgraph of G induced by S is denoted by $G[S]$. A vertex whose neighborhood induces a complete subgraph is *simplicial*, and $s(G)$ denotes the number of such vertices in G. A *block graph* is a graph if each of its maximal 2-connected subgraphs (called block) forms a clique. The *connected domination number* $\gamma_c(G)$ of G is the cardinality of a minimum connected dominating set, that is, a set D which induces a connected subgraph of G and every vertex from $V(G) \setminus D$ has a neighbor in D.

The *Cartesian product* $G \,\square\, H$ of graphs G and H has the vertex set $V(G) \times V(H)$ and edges $(g,h)(g',h')$, where either $g = g'$ and $hh' \in E(H)$, or $gg' \in E(G)$ and $h = h'$. For each $h \in V(H)$, let G^h be the subgraph induced by $\{(g,h) :\ g \in V(G)\}$. Then G^h is isomorphic to G and is a *G-layer* of $G \,\square\, H$. Analogously, we define *H-layers*, which are denoted by ${}^gH$ for a given $g \in V(G)$. By abuse of language, when we speak of a layer, we may also mean its vertex set.

For a subset of vertices $S \subseteq V(G)$, vertices $u, v \in V(G)$ are called *S-visible* if there exists a shortest (u,v)-path such that no internal vertex of the path belongs to S. The set S is a *mutual-visibility set* if its vertices are pairwise S-visible. The *mutual-visibility number* of G is the cardinality of a largest mutual-visibility set and is denoted by $\mu(G)$.

The set S is a *general position set* if no shortest path in G contains more than two vertices of S. The largest possible cardinality of such a set is referred to as *general position number* of G, denoted by $\mathrm{gp}(G)$. Similarly, the set S is a *monophonic position set* of G if there is no monophonic (alias induced) path in G that contains more than two elements of S. The *monophonic position number* $\mathrm{mp}(G)$ of G is the number of vertices in a largest monophonic position set of G.

Now let's define a new, key concept for our article. First, we use the term *monophonic path* as a synonym for induced path. Let $S \subseteq V(G)$. Two vertices $u, v \in S$ are *monophonically S-visible* if there exists a monophonic (u,v)-path whose internal vertices do not belong to S. The set S is a *monophonic mutual-visibility set* if every pair of vertices in S is monophonically S-visible. The maximum cardinality of a monophonic mutual-visibility set of G is called the *monophonic mutual-visibility number* of G and is denoted by $\mu_{\mathrm{m}}(G)$. To make our writing flow more smoothly, let's adopt the following conventions. Instead of saying "mutual-visibility" we may simply say "visibility". For instance, saying that S is a monophonic visibility set means that S is a monophonic mutual-visibility set. Also, saying that vertices are *S-m-visible* means that these two vertices are monophonically S-visible. Furthermore, a maximum monophonic visibility set of G is referred to as a *$\mu_{\rm m}$-set*.

2 Preliminaries and distance-hereditary graphs

We first establish several basic inequalities and provide examples involving $\mu_{\rm m}$. Next, we characterize graphs with very large monophonic visibility numbers, and also those with very small monophonic visibility numbers. Moreover, we show that the difference $\mu_{\rm m}(G) - \mu(G)$ can be arbitrarily large and also establish some useful lemmas. Finally, we turn to distance-hereditary graphs and two of their important subclasses, namely block graphs and cographs.

It is not hard to verify that $\mu_{\rm m}(K_n) = n$ for $n \geq 1$, that $\mu_{\rm m}(P_n) = 2$ for $n \geq 2$, and that $\mu_{\rm m}(C_n) = 3$ for $n \geq 3$.

Every visibility set of a graph G is also a monophonic visibility set. It follows that $\mu(G) \leq \mu_{\rm m}(G)$. Using [27, Lemma 2.11], which asserts that the set of simplicial vertices of G forms a monophonic position set, we have the following inequality chain for every graph G:

$$s(G) \leq {\rm mp}(G) \leq {\rm gp}(G) \leq \mu(G) \leq \mu_{\rm m}(G)\,. \tag{1}$$

As an example, consider the Petersen graph P. Clearly, $s(P) = 0$. It is known that ${\rm mp}(P) = 3$, and that ${\rm gp}(P) = \mu(P) = 6$, cf. [27, Fig. 1]. Hence $\mu_{\rm m}(P) \geq 6$ by (1). By a direct argument, it can be shown that $\mu_{\rm m}(P) \leq 6$, hence we also have $\mu_{\rm m}(P) = 6$.

In [14], it was shown that $\mu(G) \geq \Delta(G)$ holds for all graphs. Since $\mu(G) \leq \mu_{\rm m}(G)$, the following inequality holds for every graph G:

$$\mu_{\rm m}(G) \geq \Delta(G)\,. \tag{2}$$

2.1 Preliminary results

We next characterize graphs with certain large or small values of the monophonic visibility number.

Proposition 2.1 *For every connected graph G, the following statements hold.*

(i) $\mu_{\rm m}(G) = n(G)$ *if and only if G is a complete graph.*

(ii) $\mu_{\rm m}(G) = 2$ *if and only if $G \cong P_k$ for some $k \geq 2$.*

(iii) $\mu_{\rm m}(G) = n(G) - 1$ *if and only if G is not complete and there exists a vertex $v \in V(G)$ that is adjacent to every vertex $u \in V(G) \setminus \{v\}$ satisfying* $\deg_{G-v}(u) < n(G) - 2$.

(iv) $\mu_{\rm m}(G) = m(G)$ *if and only if $G \cong K_3$ or $G \cong K_{1,r}$ for some $r \geq 2$.*

Proof. (i) This is obvious.

(ii) This follows by (2) and because $\mu_{\rm m}(C_n) = 3$.

(iii) Assume first that $\mu_{\rm m}(G) = n(G) - 1$. Then G is not complete by (i). Let S be a $\mu_{\rm m}$-set of G, and let v be the unique vertex in $V(G) \setminus S$. Let $u \in V(G) \setminus \{v\}$ with $\deg_{G-v}(u) < n(G) - 2$. Then there exists $w \in S$ not adjacent to u. Since $u, w \in S$, there exists a (u, w)-path whose internal vertices avoid S. As $V(G) \setminus S = \{v\}$, this path must be uvw, and hence u is adjacent to v.

Conversely, assume that G is not complete and that there exists $v \in V(G)$ adjacent to every vertex $u \in V(G) \setminus \{v\}$ satisfying $\deg_{G-v}(u) < n(G) - 2$. As G is not complete, $\mu_{\rm m}(G) \le n(G) - 1$. Moreover, the set $V(G) \setminus \{v\}$ is a monophonic visibility set, hence we can conclude that $\mu_{\rm m}(G) = n(G) - 1$.

(iv) Let G be an arbitrary graph with $\mu_{\rm m}(G) = m(G)$. Since G is connected, $m(G) \ge n(G) - 1$. Hence $n(G) - 1 \le m(G) = \mu_{\rm m}(G) \le n(G)$, therefore, $n(G) \in \{m(G), m(G) + 1\}$. If $n(G) = m(G) (= \mu_{\rm m}(G))$, then by (i), $G \cong K_3$. The second case is when $n(G) = m(G) + 1$, that is, when G is a tree. It can be checked that $\mu_{\rm m}(T) = \ell(T)$ for a tree T (cf. Corollary 2.6), hence we can conclude that in this case G must be a star graph of order at least three. □

We add that Proposition 2.1 (iii) is parallel to [14, Lemma 4.8] which asserts that $\mu(G) \ge n(G) - 1$ if and only if there exists a vertex $v \in V(G)$ adjacent to every vertex $u \in V(G) \setminus \{v\}$ with $\deg_{G-v}(u) < n(G) - 2$.

For any graph G we thus have $\mu(G) \le \mu_{\rm m}(G)$. In general, however, $\mu_{\rm m}(G)$ can be arbitrarily larger than $\mu(G)$ as the next construction demonstrates. For $r \ge 3$, let G_r be the graph with vertex set

$$V(G_r) = A \cup B \cup C \cup \{x, y\},$$

where $A = \{a_1, \ldots, a_r\}$, $B = \{b_1, \ldots, b_r\}$, and $C = \{c_1, \ldots, c_r\}$ are pairwise disjoint sets. Each of the sets A, B, and C induces K_r in G_r. Every vertex of A is adjacent to every vertex of C. The vertex x is adjacent to all vertices in $A \cup C$, while the vertex y is adjacent to all vertices in $B \cup C$. Moreover, b_i is adjacent to c_i for every $i \in [r]$, and x is adjacent to y. See Fig. 1 where G_4 is drawn. Observe that $\mathrm{diam}(G_r) = 2$.

Proposition 2.2 *If $r \ge 3$, then $\mu_{\rm m}(G_r) - \mu(G_r) \ge r - 2$.*

Proof. It is straightforward to check that $A \cup B \cup C$ is a monophonic visibility set of G_r. Hence, $\mu_{\rm m}(G_r) \ge 3r$.

Let S be an arbitrary visibility set of G_r. For each $i \in [r]$, at most two vertices of $\{a_i, b_i, c_i\}$ belong to S. Indeed, if $a_i, b_i, c_i \in S$, then a_i and b_i would not be S-visible, since every shortest (a_i, b_i)-path contains c_i as an internal vertex. It follows that $\mu(G_r) \le 2r + 2$, hence $\mu_{\rm m}(G_r) - \mu(G_r) \ge 3r - (2r + 2) = r - 2$. □

It is not difficult to show that the equality holds in Proposition 2.2. However, since our purpose is to demonstrate that the difference can be arbitrarily large, we have formulated the result in the weaker form.

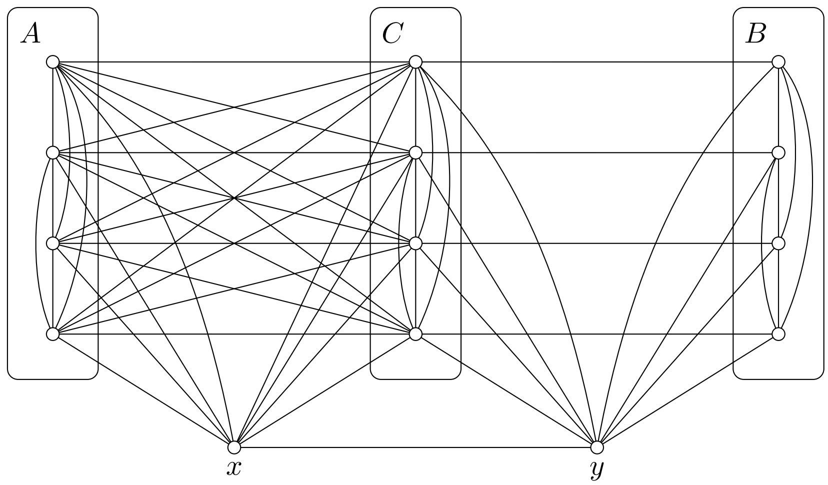


Figure 1: Graph G_4

We now establish two lemmas needed in the sequel.

Lemma 2.3 *If S is a (monophonic) visibility set of G and a vertex $u \in S$ has no neighbor in $V(G) \setminus S$, then $S = N_G[u]$.*

Proof. Since u has no neighbor in $V(G) \setminus S$, we have $N_G[u] \subseteq S$. Suppose now that there exists a vertex $x \in S \setminus N_G[u]$. But then every (u, x)-path contains an internal vertex from $N_G[u] \subseteq S$, a contradiction. $\square$

Lemma 2.4 *Every graph G admits a $\mu_{\rm m}$-set which contains no cut-vertex of G.*

Proof. Let X be the set of cut-vertices of G. Among all $\mu_{\rm m}$-sets of G, let S be selected such that $c = |S \cap X|$ is as small as possible. There is nothing to prove if $c = 0$. Hence, assume that $c \geq 1$ and let $x \in X \cap S$.

Let $H_1, \ldots, H_k$ be the connected components of $G - x$. Since x is a cut-vertex, we have $k \geq 2$.

For the same reason we may without loss of generality assume that all the vertices from $S \setminus \{x\}$ lie in H_1. Let x' be a vertex of H_2 such that $d_G(x, x')$ is as large as possible. Then x' is not a cut-vertex of G. Let now $S' = (S \setminus \{x\}) \cup \{x'\}$. Then we infer that S' is a $\mu_{\rm m}$-set of G. Since $|S' \cap X| < |S \cap X| = c$, this contradicts the choice of S. $\square$

2.2 Distance-hereditary graphs

A graph G is *distance-hereditary* if every connected induced subgraph of G is isometric. Equivalently, distance-hereditary graphs are precisely those in which every induced path is a shortest path. This class of graphs has been continuously studied from its introduction in 1977 [18] to the present day; see [9,26] and the references therein. From our perspective, this class is particularly relevant because of the following observation.

Proposition 2.5 *If G is a distance-hereditary graph, then $\mu_{\rm m}(G) = \mu(G)$.*

Proof. Since induced paths in G coincide with shortest paths, a set of vertices of G is a monophonic visibility set if and only if it is a visibility set. Consequently, $\mu_{\rm m}(G) = \mu(G)$. □

Cicerone and Di Stefano [10] designed a linear-time algorithm for computing the mutual-visibility number of distance-hereditary graphs. By Proposition 2.5, this algorithm also computes their monophonic visibility number in linear time.

It was proved in [14, Theorem 4.2] that $\mu(G) = s(G)$ for every block graph G. Since block graphs are distance-hereditary, Proposition 2.5 yields the following result.

Corollary 2.6 *If G is a block graph, then $\mu_{\rm m}(G) = s(G)$. In particular, if T is a tree, then $\mu_{\rm m}(T) = \ell(T)$.*

A graph is a *cograph* if it contains no induced path isomorphic to P_4; see [13] for numerous characterizations of this graph class. Since cographs are also distance-hereditary, Proposition 2.5, together with [14, Theorem 4.11], yields the following corollary.

Corollary 2.7 *Let G be a connected cograph. Then $\mu_{\rm m}(G) \geq n(G) - 2$ and a $\mu_{\rm m}$-set can be computed in polynomial time.*

3 On the computational complexity

The main finding of this section is as follows.

Theorem 3.1 *It is NP-complete to decide whether $\mu_{\rm m}(G) \geq k$ holds if k is part of the input. The same is true if the problem is considered over the class of bipartite graphs of diameter 4.*

Proof. The problem clearly belongs to NP, as for any graph G and any set $S \subseteq V(G)$, it can be checked efficiently whether S is a monophonic visibility set in G. The NP-hardness will be proved by presenting a polynomial-time reduction from the well-known 3-SAT problem.

Let $F = C_1 \wedge \cdots \wedge C_\ell$ be an instance of the 3-SAT problem over the variables $x_1, \ldots, x_n$. Hence, every clause C_j, for $j \in [\ell]$, is a disjunction of three literals, each of which is either a positive literal x_i or a negative literal $\neg x_i$ for some $i \in [n]$. We define a graph $G(F)$ on the vertex set $Z \cup X \cup C$, where $Z = \{z, z'\}$, X contains three vertices namely x_i^0, x_i^1, and x_i^2 for each variable x_i, while C contains one vertex for each clause such that $C = \{c_1, \ldots, c_\ell\}$. The edge set of $G(F)$ contains the edge zz' and the edges zx_i^0, zx_i^1, $x_i^0x_i^2$, $x_i^2x_i^1$ for every $i \in [n]$. Further, for every clause C_j in F, we add the edge $c_jx_i^0$ if the negative literal $\neg x_i$ is present in C_j; and we add the edge $c_jx_i^1$ if C_j contains the positive literal x_i. Then $G(F)$ contains $2+3n+\ell$ vertices and $1+4n+3\ell$ edges. For an illustration, see Fig. 2. We also observe that $G(F)$ is a bipartite graph of diameter 4, and it can be constructed in linear time from F.

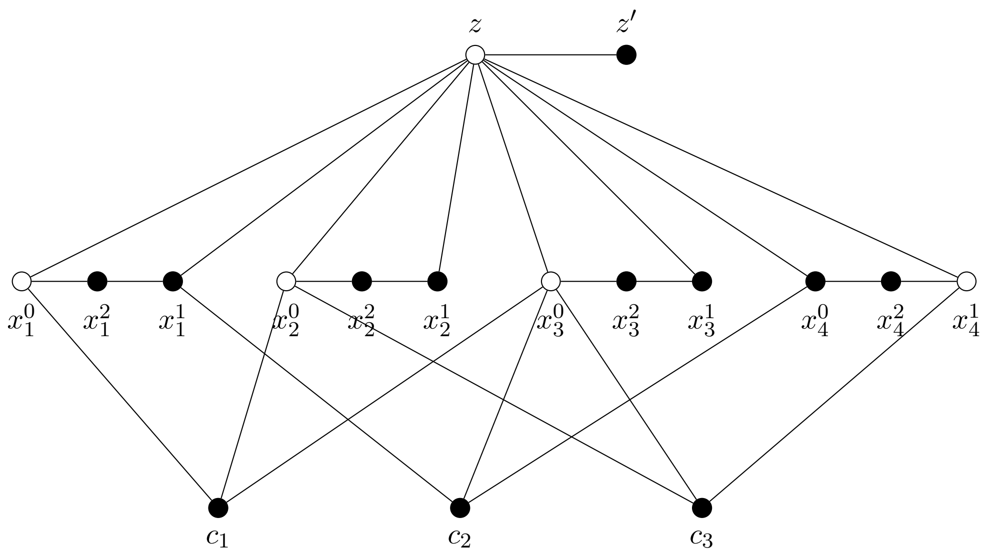


Figure 2: Graph $G(F)$ constructed for the formula $F = (\neg x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_1 \vee \neg x_3 \vee \neg x_4) \wedge (\neg x_2 \vee \neg x_3 \vee x_4)$. Black vertices form a maximum monophonic visibility set in G.

Our goal is to prove that the 3-SAT instance F is satisfiable if and only if $\mu_{\mathrm{m}}(G(F)) = 1+2n+\ell$ holds. Let S be a monophonic visibility set of size at least four in $G(F)$. We first note that $|S \cap \{z, z'\}| \le 1$, since otherwise $z \in S$ blocks all paths between $S \setminus \{z, z'\}$ and z'. Similarly, $|S \cap \{x_i^0, x_i^1, x_i^2\}| \le 2$ for all $i \in [n]$, because otherwise x_i^0 and x_i^1 block all paths starting in x_i^2. We may also assume without loss of generality that $z' \in S$ and $x_i^2 \in S$ for every $i \in [n]$. As a consequence, $|S| \le 1 + 2n + \ell$ holds for every monophonic visibility set S in $G(F)$, and therefore $\mu_{\mathrm{m}}(G(F)) \ge 1 + 2n + \ell$ is equivalent to $\mu_{\mathrm{m}}(G(F)) = 1 + 2n + \ell$.

Assume first that F is satisfiable and the assignment $\varphi\colon \{x_1, \ldots, x_n\} \to \{\text{true, false}\}$

satisfies each clause in F. We define a set of size $1+2n+\ell$ as

$$S = V(G(F)) \setminus (\{z\} \cup \{x_i^0 \colon \varphi(x_i) = \text{false}\} \cup \{x_i^1 \colon \varphi(x_i) = \text{true}\}).$$

We state that a (u,v)-path without internal vertices from S exists for every vertex pair u, v from $V(G(F))$. First let $v = z$. The property clearly holds for u and z if u is the neighbor of z. If $u = x_i^2$ then, according to the value of $\varphi(x_i)$, one of the paths $x_i^2 x_i^0 z$ and $x_i^2 x_i^1 z$ satisfies the requirement. Since φ satisfies F, every clause vertex c_j in $G(F)$ has a neighbor, a variable vertex x_i^0 or x_i^1, which does not belong to S. Hence, a (c_j, z)-path exists such that the internal vertex is not in S. Finally, if u and v are vertices different from z, then from the (u,z)-path and (v,z)-path satisfying the property, a (u,v)-path without internal vertices from S can be constructed. This implies that also every two vertices from S are S-m-visible. Therefore, S is a monophonic visibility set in $G(F)$ and in turn, $\mu_{\rm m}(G(F)) = 1+2n+\ell$.

To prove the other direction of the equivalence, we assume that $\mu_{\rm m}(G(F)) = 1+2n+\ell$. Let S be a $\mu_{\rm m}$-set in $G(F)$. As discussed at the beginning, we may assume that the complement set of S contains z and, for each $i \in [n]$, exactly one vertex from $\{x_i^0, x_i^1\}$. We may therefore define the following truth assignment over $\{x_1, \ldots, x_n\}$:

$$\phi(x_i) = \begin{cases} \text{true}, & x_i^0 \in S; \\ \text{false}, & x_i^1 \in S. \end{cases}$$

Recall that $z' \in S$ and $C \subseteq S$. It follows that $N[c_j] \not\subseteq S$ and therefore, c_j has a neighbor, say x_i^0 such that $x_i^0 \notin S$. Then, $x_i^1 \in S$ and $\phi(x_i) = \text{false}$. Consequently, the negative literal $\neg x_i$ satisfies clause C_j by ϕ. The argument is analogous if $x_i^1 \notin S$ holds for a neighbor x_i^1 of c_j. We conclude that the formula F is satisfiable.

3-SAT is a well-known NP-complete problem. Since we proved that it can be reduced (in polynomial time) to the decision problem of $\mu_{\rm m}(G) \geq k$ over the class of bipartite graphs of diameter 4, the NP-hardness of the latter problem follows. This finishes the proof of the theorem. □

4 General bounds

In this section, we establish general upper and lower bounds on the monophonic visibility number.

We begin with an upper bound in terms of the order and maximum degree of a graph. The following notation will be used in its proof. If S_1 and S_2 are disjoint subsets of $V(G)$, then $E(S_1, S_2)$ denotes the set of edges with one endpoint in S_1 and the other in S_2.

Theorem 4.1 *If G is a connected graph on n vertices and $\Delta(G) = \Delta \geq 2$, then*

$$\mu_{\rm m}(G) \leq \frac{\Delta - 2}{\Delta - 1}\, n + \frac{\Delta + 1}{\Delta - 1}\,. \tag{3}$$

Proof. First note that if $G \cong K_n$, then $\mu_{\rm m}(G) = n$ and (3) holds with equality. In the continuation, we may assume that G is not a complete graph and therefore $\mu_{\rm m}(G) < n$ by Proposition 2.1(i).

Let S be a $\mu_{\rm m}$-set of G and let $Y = V(G) \setminus S$. Since G is connected, there is a vertex $v \in S$ that has a neighbor in Y. Let $N_G(v) \cap Y = \{u_1, \dots, u_s\}$ where $1 \le s \le \Delta$. Note that $|N_G(v) \cap S| \le \Delta - s$. Since S is a monophonic visibility set, v and z are S-m-visible for every $z \in S$. Thus, for every $z \in S$ which is not adjacent to v, we may fix a path $vu_i \dots z$, with an appropriate $i \in [s]$, such that all its internal vertices belong to Y. Define now R as the set of the internal vertices of these paths. Then $R \subseteq Y$, and since every vertex in R belongs to a path starting with $u_i \in N_G(v) \cap Y$, the induced subgraph $G[R]$ consists of at most s components, say $G[R_1], \dots, G[R_k]$, where $k \le s$ and $R = R_1 \cup \dots \cup R_k$. We also introduce the notation $r_i = |R_i|$ for $i \in [k]$.

A component $G[R_i]$ contains at least $r_i - 1$ edges and each $w \in R_i$ satisfies $\deg_G(w) \le \Delta$. Therefore,

$$|E(S, R_i)| \le \Delta\, r_i - 2(r_i - 1) = (\Delta - 2)\, r_i + 2.$$

The number of edges between R and S can then be estimated as

$$\begin{aligned} |E(S,R)| &\le \sum_{i=1}^{k} ((\Delta - 2)\, r_i + 2) = (\Delta - 2)|R| + 2k \\ &\le (\Delta - 2)(n - |S|) + 2s. \end{aligned} \tag{4}$$

By the definition of R, every vertex in $S \setminus (N_G(v) \cup \{v\})$ is incident to at least one edge from $E(S, R)$, and v is incident to s further edges from this set. Since $|N_G(v) \cap S| \le \Delta - s$, this implies

$$|E(S,R)| \ge (|S| - (\Delta - s) - 1) + s = |S| - \Delta + 2s - 1. \tag{5}$$

From (4) and (5) we derive

$$|S| - \Delta - 1 \le (\Delta - 2)(n - |S|).$$

As $\Delta \ge 2$ and $\mu_{\rm m}(G) = |S|$, this directly implies the upper bound (3). $\square$

If $\Delta(G) = 2$, Theorem 4.1 yields $\mu_{\rm m}(G) \le 3$, and this bound is tight for all cycles. If $\Delta(G) = 3$, we derive $\mu_{\rm m}(G) \le \frac{n}{2} + 2$. If $\Delta(G) = 4$, then $\mu_{\rm m}(G) \le \frac{2}{3}n + \frac{5}{3}$.

Together with the inequality chain (1), Theorem 4.1 directly yields the following new result.

Corollary 4.2 *If G is a connected graph on n vertices and $\Delta(G) = \Delta \ge 2$, then*

$$\mu(G) \le \frac{\Delta - 2}{\Delta - 1}\, n + \frac{\Delta + 1}{\Delta - 1}\,. \tag{6}$$

Let G be a graph of order n containing a universal vertex, that is, $\Delta(G) = n-1$. Then the upper bound of (3) evaluates to exactly n. This means that if $\Delta(G) = n-1$, then the equality in (3) holds if and only if $G \cong K_n$. We next consider graphs with $\Delta(G) = n-2$. To characterize the equality cases in (3) under this condition, we introduce the following family of graphs.

We define the family $\mathcal{H}$ of graphs $G(H,k)$ constructed as follows. Let n and k be given integers, where $n \geq 4$ and $k \in [n-3]$. Let H be an arbitrary graph of order $n-k-1$ with no universal vertex. Then the graph $G(H,k)$ is constructed from the disjoint union of H, a complete graph K_k, and a *core vertex* u as follows. Add all possible edges between H and K_k, and add all edges between the core vertex u and all the vertices of H, see Fig. 3. Note that $G(\overline{K}_{n-2},1) \cong K_{2,n-2}$.

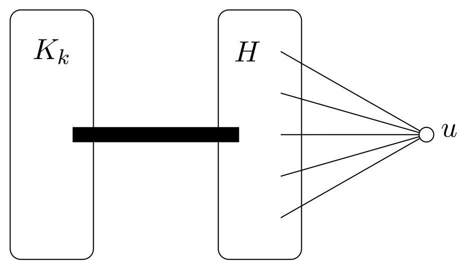


Figure 3: Graph $G(H,k)$.

Theorem 4.3 *Let G be a connected graph on $n \geq 4$ vertices with $\Delta(G) = \Delta = n-2 \geq 2$. Then the following conditions are equivalent.*

(i) $\mu_{\rm m}(G) = \left\lfloor \dfrac{\Delta-2}{\Delta-1}n + \dfrac{\Delta+1}{\Delta-1} \right\rfloor$;

(ii) $\mu_{\rm m}(G) = n-1$;

(iii) $G \in \mathcal{H}$;

(iv) $\mu(G) = n-1$.

Proof. Since $n \geq 4$ and $\Delta = n-2$, we have

$$\left\lfloor \frac{\Delta-2}{\Delta-1}n + \frac{\Delta+1}{\Delta-1} \right\rfloor = \left\lfloor \frac{n^2-3n-1}{n-3} \right\rfloor = \left\lfloor n - \frac{1}{n-3} \right\rfloor = n-1\,.$$

This implies that (i) and (ii) are equivalent statements.

(ii) $\Rightarrow$ (iii): Let G be a graph with $\Delta(G) = \Delta = n-2$ and $\mu_{\rm m}(G) = n-1$. Let S be a $\mu_{\rm m}$-set of G and let u be the unique vertex not in S. Let $H = G[N_G(u)]$ and let

$R = S \setminus N_G(u)$. Note first that $R \neq \emptyset$, for otherwise u would be a universal vertex of G and hence $\Delta(G) = n-1$. If $x \in R$, then we get that $N_G[x] = R \cup N_G(u) = S$ by Lemma 2.3. This implies that $G[R]$ is a complete graph and that each vertex from R is adjacent to each vertex of H. Moreover, H has no universal vertex, since any universal vertex of H would also be universal in G, contradicting $\Delta(G) = n-2$. In particular, $|V(H)| \geq 2$. Setting $k = |R|$, we therefore have $1 \leq k \leq n-3$. It now follows from the construction that $G = G(H,k)$, and hence $G \in \mathcal{H}$.

(iii) $\Rightarrow$ (ii): Let $G = G(H,k)$ be an arbitrary graph from $\mathcal{H}$, and let u be the core vertex of it. Since H has no universal vertex, G is not complete and thus $\mu_{\rm m}(G) \leq n-1$ by Proposition 2.1(i). On the other hand, by the construction of G we get that $V(G)\setminus\{u\}$ is a monophonic visibility set of G and we can conclude that $\mu_{\rm m}(G) = n-1$.

The equivalence between (iii) and (iv) is proved along parallel lines as the one between (ii) and (iii). $\square$

Note that Theorem 4.3 provided infinitely many sharpness examples for the bound of Theorem 4.1, as well as for the bound of Corollary 4.2.

In the second part of this section, we shift our focus to lower bounds for monophonic visibility number and prove the following result.

Theorem 4.4 *If G is a connected graph, then $\mu_{\rm m}(G) \geq n(G) - \gamma_c(G)$.*

Proof. Let D be a connected dominating set of G with $|D| = \gamma_c(G)$. We claim that $\overline{D} = V(G) \setminus D$ is a monophonic visibility set of G. Let u and v be arbitrary vertices of $\overline{D}$. Since D is a dominating set, u has a neighbor $u' \in D$, and v has a neighbor $v' \in D$. As D induces a connected subgraph of G, there exists a monophonic (u',v')-path whose all vertices are from D. Hence we get that u and v are $\overline{D}$-m-visible.

We have thus proved that $\overline{D} = V(G)\setminus D$ is a monophonic visibility set of G. Therefore, $\mu_{\rm m}(G) \geq |\overline{D}| = n(G) - \gamma_c(G)$. $\square$

Corollary 4.5 *If G is a graph of diameter two, then $\mu_{\rm m}(G) \geq n(G) - \delta(G) - 1$.*

Proof. Let u be a vertex of G with $\deg_G(u) = \delta(G)$. Then $N_G[u]$ forms a connected dominating set of G. As $|N_G[u]| = \delta(G) + 1$, the conclusion follows by Theorem 4.4. $\square$

As already mentioned, $\mu_{\rm m}(P) = 6$, where P is the Petersen graph. Since from [29, Theorem 4.1] we get that $\gamma_c(P) = 4$, the Petersen graph is a sporadic example for the sharpness of the bound of Theorem 4.4 as well as of Corollary 4.5. We next construct an infinite family of sharp examples.

For $p \geq q \geq 3$, let $G_{p,q}$ be the graph constructed as follows. Start with the Hamming graph $K_p \,\Box\, K_q$. Then add new vertices u_i, $i \in [q]$, and for each i, add edges between u_i and all the vertices in the $i^{\rm th}$ K_p-layer in $K_p \,\Box\, K_q$. In this way, this layer together with the vertex u_i induces a complete graph K_{p+1}, which we denote by H_i. See Fig. 4, where $p = 5$ and $q = 4$, so that $H_1 \cong \cdots \cong H_4 \cong K_6$. Finally, add one more vertex w and add the edges wu_i, $i \in [q]$, see Fig. 4 again.

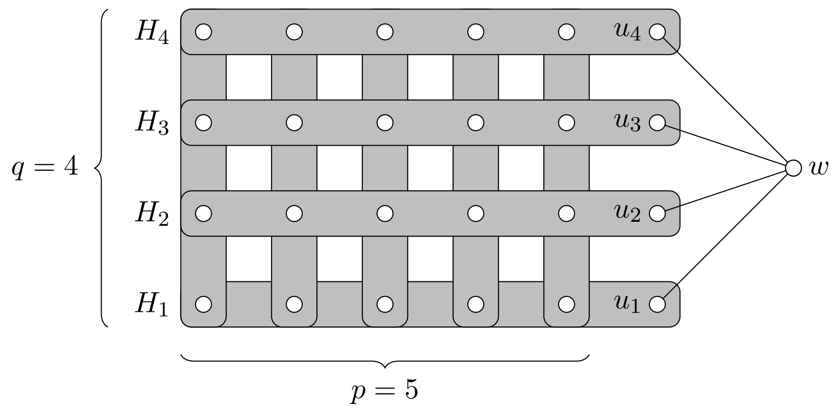


Figure 4: Graph $G_{5,4}$, where gray areas represent complete subgraphs

Proposition 4.6 *If $p \geq q \geq 3$, then $\gamma_c(G_{p,q}) = q + 1$ and $\mu_{\mathrm{m}}(G_{p,q}) = pq$.*

Proof. The set $\{w\} \cup \{u_i : \ i \in [q]\}$ is a connected dominating set of $G_{p,q}$, hence $\gamma_c(G_{p,q}) \leq q + 1$. To prove the reverse inequality, suppose for a contradiction that $G_{p,q}$ contains a connected dominating set D with $|D| = q$. If for some $i \in [q]$, $D \cap V(H_i) = \emptyset$, then the vertices of H_i must be respectively dominated by pairwise different vertices of D, which means that $|D| \geq p + 1 \geq q + 1$. Hence we have $|D \cap V(H_i)| \geq 1$ for every $i \in [q]$, and, as we have assume that $|D| = q$, we must actually have $|D \cap V(H_i)| = 1$, $i \in [q]$. Since D induces a connected subgraph, all the vertices of D must lie in the same K_q-layer of $K_p \,\square\, K_q$. But then w is not dominated by D, the final contradiction which proves that $\gamma_c(G_{p,q}) \geq q + 1$. We can conclude that $\gamma_c(G_{p,q}) = q + 1$.

As we have proved that $\gamma_c(G_{p,q}) = q + 1$, Theorem 4.4 implies that $\mu_{\mathrm{m}}(G_{p,q}) \geq pq$. To prove that also $\mu_{\mathrm{m}}(G_{p,q}) \leq pq$ holds, suppose that $G_{p,q}$ admits a monophonic visibility set S with $|S| = pq + 1$. Denote the vertices of $K_p \,\square\, K_q$ by $x_{i,j}$, $i \in [p]$, $j \in [q]$, where the indices reflect natural affiliation with layers. Setting $X = \{w\} \cup \{u_i : \ i \in [q]\}$, we have $r = |S \cap X| \geq 1$. Note that $r \leq q$ because if $r = q + 1$ would hold, then all the neighbors of w are in S, hence Lemma 2.3 gives $|S| = \left|N_{G_{p,q}}[w]\right| = q + 1 < pq + 1$, a contradiction. We now consider two cases.

Case 1: $w \in S$.
In this case, we may without loss of generality, assume that $S \cap X = \{w, u_1, \ldots, u_{r-1}\}$. As $|S| = pq + 1$ and $r = |S \cap X|$, there are exactly $r - 1$ vertices of $K_p \,\square\, K_q$ which do not belong to S. If $r = 1$, this means that all the vertices of $K_p \,\square\, K_q$ belong to S.

But then no two vertices of $K_p \,\square\, K_q$ at distance two are S-m-visible. Hence assume in the rest that $r \geq 2$. Since $r-1 < p$, there exists a K_q-layer, say the one with vertices $x_{1,1}, \ldots, x_{1,q}$, whose all vertices belong to S. We now distinguish two subcases. In the first assume that for each $i \in [r-1]$ we have $|S \cap V(H_i)| = p$, that is, exactly one vertex from H_i is not from S. Since $S \cap V(H_j) = V(H_j) \setminus \{u_j\}$ holds for each $j \in \{r, \ldots, q\}$, we infer that w and the vertices from $S \cap V(H_i)$, $i \in [q-1]$ are not S-m-visible. Indeed, the only neighbors of w from $V(G) \setminus S$ are $u_r, \ldots, u_q$, but all the neighbors of each of these vertices are from S. In the second subcase we assume that there exists $i \in [r-1]$ such that $|S \cap V(H_i)| = p+1$. But then all the neighbors of $x_{1,i}$ are in S, hence Lemma 2.3 gives $|S| = \left|N_{G_{p,q}}[x_{1,i}]\right| = p+q+1 < pq+1$, a contradiction.

Case 2: $w \notin S$.
In this case we may without loss of generality, assume that $S \cap X = \{u_1, \ldots, u_r\}$. As in the first case, there are exactly $r-1$ vertices of $K_p \,\square\, K_q$ which do not belong to S. Hence we may assume without loss of generality that $|S \cap V(H_1)| = p+1$. We may again, without loss of generality, assume that all the vertices $x_{1,1}, x_{1,2}, \ldots, x_{1,q}$ belong to S. But then all the neighbors of $x_{1,1}$ lie in S and Lemma 2.3 yields a contradiction $|S| = \left|N_{G_{p,q}}[x_{1,1}]\right| = p+q+1 < pq+1$. □

5 Cactus graphs

A *cactus* is a (connected) graph in which every edge belongs to at most one cycle. In this section, we determine the monophonic visibility number of an arbitrary cactus and characterize the unicyclic graphs G satisfying $\mu_{\rm m}(G) = {\rm mp}(G)$. Recall that $\ell(G)$ denotes the number of pendant vertices in G.

Theorem 5.1 *Let G be a cactus graph that is not isomorphic to a cycle. Let s_1 be the number of cycles whose set of vertices of degree two is nonempty and independent, and let s_2 be the number of cycles containing two adjacent vertices of degree two. Then $\mu_{\rm m}(G) = \ell(G) + s_1 + 2s_2$.*

Proof. Let C be the set of all vertices lying on cycles of G. We call a vertex $u \in C$ a branch vertex if $d_G(u) > 2$, and a non-branch vertex otherwise. Let $C_1, \ldots, C_k$ denote the cycles of G. For a branch vertex $u \in C$, let u_1 and u_2 be the neighbors of u on the cycle containing u, and let R_u denote the component of $G - \{u_1, u_2\}$ that contains u.

By Lemma 2.4, we may choose a $\mu_{\rm m}$-set S of G that contains no cut-vertex. Hence, no branch vertex belongs to S, and every vertex of $S \setminus C$ is a leaf of G.

We may further assume that $|V(C_i) \cap S| \leq 2$ for every $i \in [k]$. Indeed, suppose that $|V(C_i) \cap S| = 3$ for some $i \in [k]$. Since G is not isomorphic to a cycle, C_i contains a branch vertex; let u be such a vertex. Since S contains no cut-vertex, we have $u \notin S$. Moreover, no vertex of $R_u - \{u\}$ belongs to S, otherwise a vertex of $R_u - \{u\}$ and a vertex of $V(C_i) \cap S$ would not be S-m-visible. This allows us to replace any vertex of $V(C_i) \cap S$

by any vertex of R_u that is not a cut-vertex of G, obtaining another $\mu_{\rm m}$-set with only two vertices on C_i. Repeating this replacement whenever necessary shows that such a $\mu_{\rm m}$-set can be chosen.

First, consider a cycle C_i containing no non-branch vertices. Then every vertex of the cycle is a cut-vertex in G, and therefore $V(C_i) \cap S = \emptyset$.

Next, consider a cycle C_i whose set of non-branch vertices is nonempty and independent. If C_i contains exactly one non-branch vertex, then clearly $|V(C_i) \cap S| \leq 1$. Assume that C_i contains at least two non-branch vertices and suppose that $w_1, w_2 \in V(C_i) \cap S$. Since w_1 and w_2 are not adjacent and their neighbors are all branch vertices, there exist branch vertices $v_1, v_2 \in V(C_i)$ such that both internally disjoint (v_1, v_2) paths on C_i contain a vertex from $\{w_1, w_2\}$. It follows that at least one of R_{v_1} and R_{v_2} contains no vertex of S; without loss of generality, $V(R_{v_1}) \cap S = \emptyset$. Let x be a vertex of R_{v_1} that is not a cut-vertex of G. This allows us to replace w_2 by x, obtaining another $\mu_{\rm m}$-set with only one vertex on C_i. Repeating this replacement whenever necessary shows that we may assume that each of the s_1 cycles contributes at most one vertex to a $\mu_{\rm m}$-set.

Finally, consider a cycle containing two adjacent non-branch vertices. By our choice of S, each of the s_2 cycles contributes at most two vertices. Combining the three cases, we conclude that $\mu_{\rm m}(G) \leq \ell(G) + s_1 + 2s_2$.

On the other hand, choosing all leaves of G, together with one non-branch vertex from each of the s_1 cycles and two adjacent non-branch vertices from each of the s_2 cycles, yields a monophonic visibility set of G. Therefore, $\mu_{\rm m}(G) = \ell(G) + s_1 + 2s_2$. $\square$

Theorem 5.1 immediately yields the following result for unicyclic graphs.

Corollary 5.2 *Let G be a unicyclic graph with cycle C. Then*

$$\mu_{\rm m}(G) = \begin{cases} \ell(G), & \text{if } C \text{ contains no vertex of degree } 2; \\ \ell(G) + 2, & \text{if } C \text{ contains two adjacent vertices of degree } 2; \\ \ell(G) + 1, & \text{otherwise.} \end{cases}$$

Combining Corollary 5.2 with [27, Theorem 3.5], the following result can be deduced.

Corollary 5.3 *Let G be a unicyclic graph that is not a cycle and let C be its cycle of length n. Then $\mu_{\rm m}(G) = {\rm mp}(G)$ if and only if at least one of the following conditions holds.*

(i) $n = 3$.

(ii) *Every vertex of C has degree at least 3.*

(iii) *C has exactly one vertex of degree at least 3.*

(iv) *$n = 4$ and C has exactly two vertices of degree at least 3, where these two vertices are not adjacent and at least one of the respective trees attached to them is a path.*

6 Monophonic visibility in grids

In this section, we first determine $\mu_{\rm m}(P_2 \,\Box\, P_n)$ and $\mu_{\rm m}(P_3 \,\Box\, P_n)$. We then consider the general case and establish lower and upper bounds on $\mu_{\rm m}(P_m \,\Box\, P_n)$ that imply $\mu_{\rm m}(P_m \,\Box\, P_n) = \left(\frac{2}{3} + o(1)\right) mn$, as $m \to \infty$ and $n \to \infty$.

Recall that $\mu_{\rm m}(P_1 \,\Box\, P_n) = \mu_{\rm m}(P_n) = 2$ for every $n \geq 2$ and that $\mu_{\rm m}(P_2 \,\Box\, P_2) = \mu_{\rm m}(C_4) = 3$. Moreover, it can be checked directly that $\mu_{\rm m}(P_2 \,\Box\, P_3) = 4$. The remaining cases with $\min\{m, n\} \leq 3$ are covered by the following theorem.

Theorem 6.1 *The following statements hold.*

(i) $\mu_{\rm m}(P_2 \,\Box\, P_n) = n$ *holds for every* $n \geq 4$.

(ii) $\mu_{\rm m}(P_3 \,\Box\, P_n) = 2n$ *holds for every* $n \geq 3$.

Proof. We set $V(P_k) = [k]$ for every path discussed in the proof.

(i) It is straightforward to check that the set $S_1 = \{(1, i) \colon i \in [n]\}$ is a monophonic visibility set in $P_2 \,\Box\, P_n$ for every $n \geq 4$. Hence, $\mu_{\rm m}(P_2 \,\Box\, P_n) \geq n$.

For the other direction, suppose for a contradiction that $\mu_{\rm m}(P_2 \,\Box\, P_n) \geq n + 1$ and let S be a $\mu_{\rm m}$-set in the grid. Since $|S| \geq n + 1$, there is a P_2-layer $P_2^i = \{(1, i), (2, i)\}$ such that $P_2^i \subseteq S$. If $2 \leq i \leq n - 1$, this layer cuts the grid into two parts, namely a $(2 \times (i - 1))$-grid Y_1 and a $(2 \times (n - i))$-grid Y_2 such that all paths between the two parts contain a vertex from S. Therefore, S meets at most one of Y_1 and Y_2. Furthermore, by Lemma 2.3 the set S cannot contain the closed neighborhood of a corner vertex if $|S| \geq 4$. Therefore, $P_2^1 \subseteq S$ implies $P_2^2 \cap S = \emptyset$ and similarly, $P_2^n \subseteq S$ implies $P_2^{n-1} \cap S = \emptyset$. These observations imply the following properties.

- S cannot contain three or more different P_2-layers.
- If P_2^i and P_2^j are two different layers included in S, it is enough to consider the following three cases: either $i = 1$ and $j = n$, or $2 \leq i < j \leq n - 1$, or $i = 1$ and $3 \leq j \leq n - 1$. Based on the observations, we infer that $|S| \leq n$ in all three cases, contradicting our supposition.
- If S contains only one P_2-layer, namely P_2^i, then $i \in \{1, n\}$ implies $|S| \leq 2 + (n - 2) = n$, while $2 \leq i \leq n - 1$ implies $|S| \leq 2 + \max\{i - 1, n - i\} \leq n$, a contradiction.

These contradictions imply $|S| = \mu_{\rm m}(P_2 \,\Box\, P_n) \leq n$ for all $n \geq 4$ that finishes the proof for (i).

(ii) Observe that $S_2 = \{(\ell, i) \colon \ell \in \{1, 3\},\ i \in [n]\}$ is a monophonic visibility set in $P_3 \,\Box\, P_n$ and therefore, $\mu_{\rm m}(P_3 \,\Box\, P_n) \geq 2n$. The argument for the reverse inequality is analogous to that in case (i). That is we suppose for a contradiction that $\mu_{\rm m}(P_3 \,\Box\, P_n) \geq 2n + 1$. It follows that every $\mu_{\rm m}$-set S contains a P_3-layer. If S contains three different P_3-layers, we directly get a contradiction with the m-visibility property. If S includes two or one P_3-layers, the basic observations from (i) are still true and restrict the number of vertices in S. We can derive a contradiction for all cases and conclude that $\mu_{\rm m}(P_3 \,\square\, P_n) = 2n$ holds. $\square$

The determination of $\mu_{\rm m}(P_m \,\square\, P_n)$ for every m and n seems to be a problem that is hard to solve. We establish here a lower and an upper bound on $\mu_{\rm m}(P_m \,\square\, P_n)$ such that the difference of the bounds is linear in terms of m and n. Before stating the result, we introduce the following notation for every pair (a, b) of positive integers:

$$\sigma(a,b) = \begin{cases} \frac{2}{3}ab - \frac{1}{3}a, & \text{if } a \equiv 0 \pmod 3; \\ \frac{2}{3}ab - \frac{1}{3}a - \frac{2}{3}b + \frac{7}{3}, & \text{if } a \equiv 1 \pmod 3; \\ \frac{2}{3}ab - \frac{1}{3}a - \frac{1}{3}b + \frac{2}{3}, & \text{if } a \equiv 2 \pmod 3). \end{cases}$$

Theorem 6.2 *Let $P_m \,\square\, P_n$ be a grid such that $m \geq 4$ and $n \geq 4$. Then it holds that*

$$\max\{\sigma(m,n), \sigma(n,m)\} \leq \mu_{\rm m}(P_m \,\square\, P_n) \leq \frac{2}{3}mn + \frac{5}{3}. \tag{7}$$

Proof. The upper bound in (7) directly follows from the general upper bound in Theorem 4.1 applied to graphs with maximum degree 4.

The lower bound is proved by construction. Consider the grid $P_m \,\square\, P_n$. Set now

$$\begin{aligned} Y_0 &= \emptyset, \\ Y_1 &= \{(m,1),(m,n)\}, \\ Y_2 &= \{(m,i)\colon i \in [n]\}, \end{aligned}$$

and, for every $0 \leq j \leq \lfloor \frac{m}{3} \rfloor - 1$ let

$$\begin{aligned} X_j = \{(\ell,i)\colon \ell \in \{3j+1, 3j+3\},\ i \in \{3,\ldots,n\}\} \\ \cup \{(3j+1,1),(3j+2,1),(3j+3,1)\}. \end{aligned}$$

A monophonic visibility set of the grid $P_m \,\square\, P_n$ can be obtained as

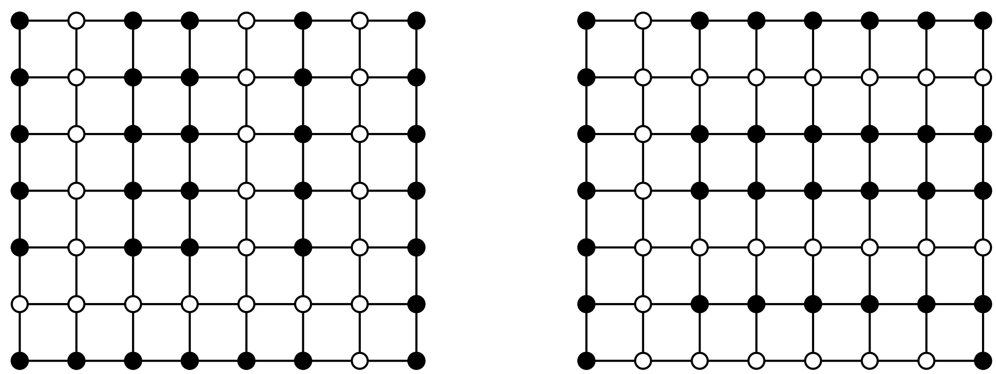
$$S = Y_r \cup \bigcup_{j=0}^{\lfloor \frac{m}{3} \rfloor - 1} X_j,$$

where $r = m \bmod 3$. It is straightforward to check that S is a monophonic visibility set of the grid. In fact, every vertex $v \in S$ has a neighbor outside of S and the subgraph induced by the complement of S is connected. Hence, $|S| \leq \mu_{\rm m}(P_m \,\square\, P_n)$. Since every X_j contains $3 + 2(n-2) = 2n - 1$ vertices, we obtain

$$\mu_{\rm m}(P_m \,\square\, P_n) \geq |S| = |Y_r| + \left\lfloor \frac{m}{3} \right\rfloor (2n-1). \tag{8}$$

Further calculation according to the value of the remainder r shows that the lower bound in (8) equals $\sigma(m,n)$. Another monophonic visibility set of size $\sigma(n,m)$ can be constructed if we consider the same grid as $P_n \,\square\, P_m$. Then $\mu_{\rm m}(P_m \,\square\, P_n) \geq \sigma(n,m)$ also follows and the proof is completed. $\square$

Remark 6.3 *The upper bound $\frac{2}{3}mn + \frac{5}{3}$ in (7) can be reduced to $\frac{2}{3}mn$ if the proof of Theorem 4.1 is rewritten for grids. We may consider two cases in this new version: either there is a corner vertex in X, or all corner vertices belong to the complement.*



(a) (b)

Figure 5: Monophonic visibility sets in $P_8 \,\square\, P_7$ according to the construction in the proof of Theorem 6.2. The black vertices in (a) and (b) show monophonic visibility sets of size $\sigma(8,7)$ and $\sigma(7,8)$, respectively.

7 Realizability results

We say that a pair (a,b) of positive integers is $(\mu,\mu_{\rm m})$-*realizable* if there exists a graph G with $\mu(G) = a$ and $\mu_{\rm m}(G) = b$. In this section, we study the $(\mu,\mu_{\rm m})$-realizability of such pairs. The main result shows that a pair (a,b) with $4 \leq a \leq b$ is always $(\mu,\mu_{\rm m})$-realizable.

By (1), $\mu(G) \leq \mu_{\rm m}(G)$ holds for every graph and therefore, we consider only pairs with $a \leq b$. If $\mu(G) = 1$, then $G \cong K_1$ and $\mu_{\rm m}(G) = 1$ follows. If $\mu(G) = 2$, then G is a path of order at least two [14, Lemma 2.8] and $\mu_{\rm m}(G) = 2$ also holds. Consequently, the pairs $(1,b)$ with $b \geq 2$ and the pairs $(2,b)$ with $b \geq 3$ are not $(\mu,\mu_{\rm m})$-realizable.

The following lemma will help us prove the main result of this section.

Lemma 7.1 *If G' is a graph obtained from a graph G by adding a pendant vertex to it, then $\mu_{\rm m}(G) \leq \mu_{\rm m}(G') \leq \mu_{\rm m}(G) + 1$.*

Proof. Since G is an induced subgraph of G', every monophonic visibility set of G is also a monophonic visibility set of G'. Hence, $\mu_{\rm m}(G) \le \mu_{\rm m}(G')$.

Let u be the pendant vertex in G' added to G, and let v be its neighbor in G. Let S' be a $\mu_{\rm m}$-set of G'. If $u \notin S'$, then $S' \subseteq V(G)$. Thus, S' is also a monophonic visibility set of G, so that $|S'| = \mu_{\rm m}(G') \le \mu_{\rm m}(G)$. If $u \in S'$, then $S' \setminus \{u\}$ is a monophonic visibility set of G, which implies $|S' \setminus \{u\}| \le \mu_{\rm m}(G)$. Therefore, $\mu_{\rm m}(G') \le \mu_{\rm m}(G) + 1$. $\square$

The main result of the section now reads as follows.

Theorem 7.2 *Let (a, b) be a pair of positive integers with $3 \le a \le b$.*

(i) The pairs $(3, 3)$ and $(3, 4)$ are $(\mu, \mu_{\rm m})$-realizable.

(ii) Every pair (a, b) with $4 \le a \le b$ is $(\mu, \mu_{\rm m})$-realizable. Moreover, all these pairs are $(\mu, \mu_{\rm m})$-realizable over the class of bipartite graphs.

Proof. (i) For the pair $(3, 3)$, it is enough to consider a cycle C_n. Indeed, we have $\mu(C_n) = 3$ and $\mu_{\rm m}(C_n) = 3$. For the pair $(3, 4)$, consider the graph H drawn in Fig. 6.

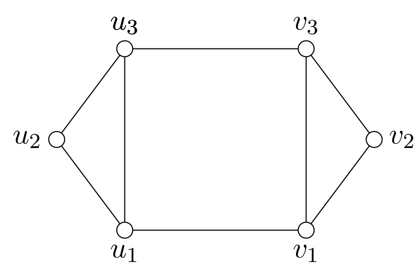


Figure 6: Graph H

The set $\{u_1, u_2, v_1, v_2\}$ is a monophonic visibility set of H, hence $\mu_{\rm m}(H) \ge 4$. Suppose now that S is a monophonic visibility set of H with $|S| = 5$. Assuming without loss of generality that $u_1, u_2, u_3 \in S$, we get a contradiction having Lemma 2.3 in mind. We can conclude that $\mu_{\rm m}(H) = 4$.

The set $\{u_1, u_3, v_2\}$ is a visibility set, hence $\mu(H) \ge 3$. Suppose now that S' is a visibility set of H with $|S'| = 4$. Using Lemma 2.3 again we infer that then $|S' \cap \{u_1, u_2, u_3\}| = 2$ and $|S' \cap \{v_1, v_2, v_3\}| = 2$. If $S' \cap \{u_1, u_2, u_3\} = \{u_1, u_2\}$, then $v_1 \notin S'$. So $v_2, v_3 \in S'$ but then u_2 and v_2 are not visible. By the symmetry, the only remaining case is when $S' = \{u_1, u_3, v_1, v_3\}$, but then u_1 and v_3 are not visible. We can conclude that $\mu(H) = 3$.

(ii) For this assertion we distinguish four cases depending on the value of a.

Case 1: $a = 4$.
Consider the graph $G = P_b \,\Box\, P_2$. Since each of the two P_b-layers contains at most two

visible vertices, we infer that $\mu(G) = 4$. On the other hand, Theorem 6.1 (i) yields $\mu_{\rm m}(G) = b$. Hence, every pair $(4, b)$ with $b \geq 4$ is $(\mu, \mu_{\rm m})$-realizable.

Case 2: $a \geq 6$ is even.
Write $a = 2k$, where $k \geq 3$, and let $s = b - a$. We construct a graph $G(a, b)$ as follows.

For each $i \in [k] \setminus \{2\}$, let

$$R_i : v_1^i v_2^i u_2^i u_1^i v_1^i$$

be a 4-cycle. Let R_2 be the grid $P_{s+2} \,\square\, P_2$ with the vertex set

$$\{v_1^2, \ldots, v_{s+2}^2, u_1^2, \ldots, u_{s+2}^2\}.$$

Finally, add the edges $v_2^1 v_1^2$, $v_{s+2}^2 v_1^3$, and $v_2^i v_1^{i+1}$ for every $i \in \{3, \ldots, k-1\}$. See Fig. 7 where $G(8, 11)$ is drawn. Clearly, $G(a, b)$ is bipartite.

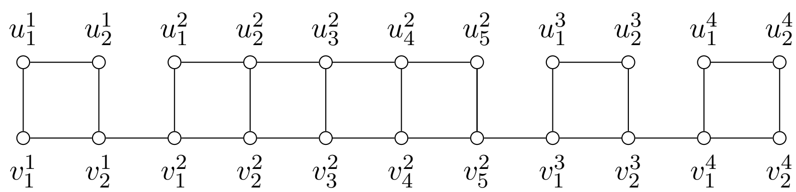


Figure 7: Graph $G(a, b)$ defined in the proof of Theorem 7.2 for $a = 8$ and $b = 11$

By Lemma 2.5 of Di Stefano [14], there exists a μ-set of $G(a, b)$ containing no cut-vertex. Let S be such a μ-set of $G(a, b)$.

Let Q be the path induced by

$$\bigcup_{i \in [k] \setminus \{2\}} \{v_1^i, v_2^i\} \cup \{v_1^2, \ldots, v_{s+2}^2\}.$$

Since Q is convex in $G(a, b)$, the set $S \cap V(Q)$ is a visibility set of Q. Therefore, $\mu(P_n) = 2$ yields $|S \cap V(Q)| \leq 2$.

We next focus on the end-cycles R_1 and R_k. Since X contains no cut-vertex, $v_2^1 \notin S$. Moreover, $|S \cap V(R_1)| \neq 3$, since otherwise $\{v_1^1, u_1^1, u_2^1\} \subseteq S$, and then u_1^1 would not be visible with any vertex of $S \setminus V(R_1)$. Consequently, $|S \cap V(R_1)| \leq 2$. On the other hand, $S \cap V(R_1) \neq \emptyset$, since otherwise $S \cup \{u_2^1\}$ would also be a visibility set of $G(a, b)$, contradicting the maximality of X. Therefore, $1 \leq |S \cap V(R_1)| \leq 2$. By symmetry, $1 \leq |S \cap V(R_k)| \leq 2$.

Now observe that

$$S \cap V\big(Q \setminus (R_1 \cup R_k)\big) = \emptyset.$$

Indeed, every shortest path joining a vertex of R_1 to a vertex of R_k contains all the vertices of $Q \setminus (R_1 \cup R_k)$. Hence, the presence of a vertex of S in this set would prevent visibility between a vertex of $S \cap V(R_1)$ and a vertex of $S \cap V(R_k)$. Thus, $|S \cap V(R_i)| \leq 2$ for each $i \in \{3, \dots, k-1\}$.

Finally, let Q' be the path induced by $\{u_1^2, \dots, u_{s+2}^2\}$. Since Q' is convex in $G(a,b)$, we have $|S \cap V(Q')| \leq 2$. Consequently, $|S| = \mu(G(a,b)) \leq 2(k-1) + 2 = 2k = a$.

On the other hand, consider the set

$$S(a,b) = \bigcup_{i \in [k] \setminus \{2\}} \{u_1^i, u_2^i\} \cup \{u_1^2, u_{s+2}^2\}.$$

It is straightforward to verify that $S(a,b)$ is a visibility set of $G(a,b)$ of cardinality $2k = a$. Hence, $\mu(G(a,b)) = a$.

We next consider the monophonic visibility of $G(a,b)$. Lemma 2.4 guarantees that $G(a,b)$ admits a $\mu_{\rm m}$-set containing no cut-vertices. Let S' be such a $\mu_{\rm m}$-set. Suppose, for a contradiction, that $|S'| \geq b+1$. For $i \in [k] \setminus \{2\}$ and $j \in \{1,2\}$, let $R_i^j = \{u_j^i, v_j^i\}$, and for $j \in \{1, \dots, s+2\}$, let $R_2^j = \{u_j^2, v_j^2\}$. We refer to these sets as the layers of $G(a,b)$. Since there are exactly b layers and $|S'| \geq b+1$, there exists a layer R_i^j such that $R_i^j \subseteq S'$.

Since S' contains no cut-vertex, we have $S' \cap Q \subseteq \{v_1^1, v_2^k\} \cup \{v_2^2, \dots, v_{s+1}^2\}$. Moreover, if $R_1^1 \subseteq S'$, then $u_2^1 \notin S'$, while if $R_k^2 \subseteq S'$, then $u_1^k \notin S'$. These observations imply that S' cannot contain three distinct layers.

Let R_i^j and $R_{i'}^{j'}$ be two distinct layers contained in S'. By symmetry, it suffices to consider the cases $R_1^1, R_k^2 \subseteq S'$, or $R_2^j, R_2^{j'} \subseteq S'$ with $2 \leq j < j' \leq s+1$, or $R_1^1, R_2^{j'} \subseteq S'$ with $2 \leq j' \leq s+1$. In each case, we obtain $|S'| \leq b$, a contradiction.

Hence, S' contains exactly one layer, say R_i^j. Again by symmetry, it suffices to consider the cases $R_1^1 \subseteq S'$, $R_k^2 \subseteq S'$, and $R_2^j \subseteq S'$ with $2 \leq j \leq s+1$. If $R_1^1 \subseteq S'$ or $R_k^2 \subseteq S'$, then $|S'| \leq 2 + (b-2) = b$, while if $R_2^j \subseteq S'$ with $2 \leq j \leq s+1$, then $|S'| \leq 2 + \max\{2 + j - 1,\ s + 2 - j + 2(k-2)\} \leq b$, again contradicting the assumption $|S'| \geq b+1$.

In all above cases, we obtain $|S'| = \mu_{\rm m}(G(a,b)) \leq b$. On the other hand, consider the set

$$S'(a,b) = \bigcup_{i \in [k] \setminus \{2\}} \{u_1^i, u_2^i\} \cup \{u_1^2, \dots, u_{s+2}^2\}.$$

The graph $G(a,b)$ thus admits the monophonic visibility set $S'(a,b)$, whose cardinality is $2k + s = b$. Consequently, $\mu_{\rm m}(G(a,b)) = b$.

Case 3: $a = 5$.
Let $b \geq 5$ and let G' be the graph obtained by adding a pendant edge to one vertex of degree 3 of the grid $P_{b-1} \,\square\, P_2$. Having in mind Lemma 7.1 and Theorem 6.1(i) it is straightforward to infer that $\mu(G') = 5$ and $\mu_{\rm m}(G') = b$.

Case 4: $a \geq 7$ is odd.
Then $a - 1 \geq 6$ is even and we may consider the graph $G(a-1, b-1)$ from Case 2. We construct the graph $X(a,b)$ by attaching a pendant edge (v_2^k, z) to $G(a-1, b-1)$. The graph $X(a,b)$ is clearly bipartite. By Lemma 7.1, we have $\mu_{\rm m}(G(a-1, b-1)) \leq$

$\mu_{\rm m}(X(a,b)) \le \mu_{\rm m}(G(a-1,b-1))+1$. Considering the sets $S = S(a-1,b-1)\cup\{z\}$ and $S' = S'(a-1,b-1)\cup\{z\}$, we obtain $\mu(X(a,b)) = a$ and $\mu_{\rm m}(X(a,b)) = b$. □

8 Concluding remarks

We first discuss two possible variations of the monophonic visibility problem.

- Related to [17], one may want to define all-path visibility, where a set $S \subseteq V(G)$ is an all-path visibility set if for every two vertices $u,v \in S$ there exists an *arbitrary* (u,v)-path with all internal vertices from $V(G)\setminus S$. However, if such a path is non-induced, then there always exists an induced (u,v)-path with the same property. Consequently, monophonic visibility coincides with the all-path visibility.

- The concept of total visibility set of a graph was introduced in [11], and in particular further studied in [4–6,20]. Analogously, we may define a total monophonic visibility set S in G as a subset of $V(G)$ such that every two vertices of G are S-m-visible. However, if G is not a complete graph, this definition implies that $\overline{S} = V(G)\setminus S$ induces a connected subgraph, and that every $v \in S$ has a neighbor in $\overline{S}$. Therefore, $\overline{S}$ is a connected dominating set in G. As the other direction is also easy to verify, we obtain that $S \subseteq V(G)$ is a total monophonic visibility set if and only if the complement $\overline{S}$ is a connected dominating set.

  We may observe that the reduction in the proof of Theorem 3.1 also works for the total version. Hence we can conclude that the decision problem for the connected domination number is NP-complete over the class of bipartite graphs of diameter 4. It was earlier proved in [23] that it is NP-complete over the class of bipartite graphs without induced cycles of length at least 6.

In [1] it was proved that determining the monophonic general position number is a $W[1]$-hard problem (parameterized by the size of the solution) over the class of graphs of diameter 2. We have proved in Section 3 that determining the monophonic visibility number is NP-hard for bipartite graphs of diameter 4. However, the question of $W[1]$-hardness remains open for the monophonic visibility number, and hence we pose it as an open problem for further study.

Problem 8.1 *Is it $W[1]$-hard to determine the monophonic visibility number when the problem is parameterized by the solution size?*

In Section 6, we concentrated on the monophonic visibility numbers of Cartesian grids. Determining an exact formula for $\mu_{\rm m}(P_m \,\square\, P_n)$ appears to be difficult. Theorem 6.2 provides lower and upper bounds on $\mu_{\rm m}(P_m \,\square\, P_n)$, and the upper bound can be slightly improved as described in Remark 6.3. We ask for a more significant improvement in the inequality (7).

Problem 8.2 *Improve the lower and upper bounds in Theorem 6.2.*

Theorem 6.1 provides exact values for $\mu_{\rm m}(P_2 \,\square\, P_n)$ and $\mu_{\rm m}(P_3 \,\square\, P_n)$ for every $n \geq 4$. A natural next step is to study the grids of the form $P_4 \,\square\, P_n$. Note that Theorem 6.2 (together with Remark 6.3) yields $\left\lfloor \frac{7}{3}n \right\rfloor \leq \mu_{\rm m}(P_4 \,\square\, P_n) \leq \frac{8}{3}n$ for every $n \geq 4$.

Problem 8.3 *Determine the value of* $\mu_{\rm m}(P_4 \,\square\, P_n)$ *for every* $n \geq 4$.

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Declaration of interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Our manuscript has no associated data.

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