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Research Article

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Coloring the vertices of a graph with mutual-visibility property

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Abstract: Given a graph G, a mutual-visibility coloring of G is a coloring of the vertices of G satisfying the following. Two vertices $x, y \in V(G)$ can be colored with the same color, if there is a shortest x, y-path whose internal vertices have different colors than x and y. The smallest number of colors among all mutual-visibility colorings of G is the mutual-visibility chromatic number of G, which is denoted by $\chi_\mu(G)$. Relationships between $\chi_\mu(G)$ and its two parent ones, the chromatic number and the mutual-visibility number, are presented. Graphs of diameter two are considered, and in particular, the asymptotic growth of the mutual-visibility number of the Cartesian product of complete graphs is determined. A greedy algorithm that finds a mutual-visibility coloring is designed, and several possible scenarios on its efficiency are discussed. Several bounds are given in terms of other graph parameters such as the diameter, the order, the maximum degree, the degree of regularity of regular graphs, and/or the mutual-visibility number. For the corona products, it is proved that the value of its mutual-visibility chromatic number depends on that of the first factor of the product. Graphs G for which $\chi_\mu(G) = 2$ are also considered.

Keywords: graph coloring, mutual-visibility set, mutual-visibility number, mutual-visibility chromatic number, graph product, diameter two graph

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1 Introduction

Given a connected graph $G = (V(G), E(G))$ and a set of vertices $S \subseteq V(G)$, it is said that two vertices $x, y \in S$ are *S-visible* if there is a shortest x, y-path P such that $V(P) \cap S = \{x, y\}$. The set S is called a *mutual-visibility set* of G if any two vertices of S are S-visible. The cardinality of a largest mutual-visibility set of G is the *mutual-visibility number* of G, denoted by $\mu(G)$. By a *μ-set* of G, we mean a mutual-visibility set of G with cardinality $\mu(G)$.

The concepts described earlier were introduced and studied for the first time by Di Stefano in 2022 [1] motivated in part by some model of robot navigation in networks avoiding collisions between themselves. Soon after, the variety of mutual-visibility problems in graphs consisting of (total, dual, outer) mutual-visibility sets was presented in [2]. In just a few years, the area has blossomed, not only because of the original computer

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science motivation on robot visibility, but perhaps above all for the reason, that the visibility problems are intrinsically connected with several classical combinatorial problems, as, for instance, the Zarankiewicz problem, see [3], and Turán type problems, see [4–6]. Additional interesting contributions to the problem are [7–16].

When we look at the mutual-visibility problem from a practical point of view, we might need more than just one mutual-visibility set; instead we wish to partition the vertex set of a graph into mutual-visibility sets. Therefore, we present here a new research direction on the mutual-visibility problem as follows. Given the connected graph G, we color its vertices by using the following rule. Two vertices $x, y \in V(G)$ are colored with the same color, if there exists a shortest x, y-path whose internal vertices have different colors than x and y. Clearly, if x and y are adjacent, then they can be colored equal. Such coloring shall be called a *mutual-visibility coloring* of G. The smallest number of colors needed in a mutual-visibility coloring of G is the *mutual-visibility chromatic number* of G, and will be denoted by $\chi_\mu(G)$. In order to justify the terminology, note that in a mutual-visibility coloring, each color class forms a mutual-visibility set. It is straightforward to observe that $\chi_\mu(G) = 1$ if and only if G is a complete graph. In this sense, the smallest nontrivial value for $\chi_\mu(G)$ is precisely 2.

The rest of the article is structured as follows. Section 2 contains some preliminary first results, including a lower bound of $\chi_\mu(G)$ as a function of the mutual-visibility number of the graph, as well as a relationship with the classical chromatic number for graphs of diameter two. In Section 3, we study the case of graphs with diameter two. Particularly, the asymptotic growth of the mutual-visibility chromatic number of the Cartesian product of complete graphs is established. A greedy algorithm will be described in Section 4. It will be proved there that although the algorithm is optimal in many cases, it can also return arbitrarily bad results as a consequence of an erroneous choice of the mutual-visibility sets at each stage of the algorithm. Section 5 describes several bounds as a function of various structural parameters such as the diameter, the order, the maximum degree, the degree of regularity of regular graphs, and/or the mutual-visibility number. A realizability result is also proved, showing that for every feasible value, there is a graph that attains it as a mutual-visibility chromatic number. This section also contains a Nordhauss-Gaddum-type upper bound for the mutual-visibility chromatic number. Section 6 explores the corona product of two graphs G and H, by showing that its mutual-visibility chromatic number is either the mutual-visibility chromatic number of the first factor of the product, or that one plus one. Section 7 is focused on finding all those graphs G for which $\chi_\mu(G) = 2$, since this is the smallest possible value that such parameter can achieve. Finally, we close this work with Section 8, where we outline some remarks and open problems that might be of interest to continue this investigation.

2 Preliminaries

Along our whole exposition, for a positive integer k, we shall write $[k] = \{1, \ldots, k\}$. Let G be a graph and let G_i, $i \in [k]$, be its connected components. Then, it is evident that

$$\chi_\mu(G) = \sum_{i=1}^{k} \chi_\mu(G_i).$$

For this reason, from now on, all the graphs considered are connected unless stated otherwise. Also, all our graphs have neither loops nor multiple edges.

A first basic connection between $\chi_\mu(G)$ and $\mu(G)$ is as follows. Let $\mathcal{P} = \{P_i : i \in [\chi_\mu(G)]\}$ be a partition of $V(G)$ into mutual-visibility sets. Clearly, $|P_i| \leq \mu(G)$ for all $i \in [\chi_\mu(G)]$. This already implies the following lower bound, which we state as a lemma for subsequent use.

Lemma 2.1. *If G is a graph, then*

$$\chi_\mu(G) \geq \left\lceil \frac{n(G)}{\mu(G)} \right\rceil.$$

On the other hand, the following arguments justify the term chromatic in our definition and relate our investigation with the classical one of vertex coloring. A *proper coloring* of the vertex set of a graph G is an assignment of labels (or colors) to the vertices of G in such a way that each two adjacent vertices have different labels (colors). The *chromatic number* of G, denoted $\chi(G)$, represents the smallest number of colors among all possible proper colorings of G. This parameter is one of the classical ones in graph theory, and there are lots of variations of it. For more information on this fact, see, for instance, the book of Jensen and Toft [17].

It can be observed that a proper coloring of an arbitrary graph G does not induce a mutual-visibility coloring in general. However, if we consider graphs of diameter two, then such relationship becomes true. That is, let G be a connected graph of diameter two. Since a set of vertices S having the same color in any proper coloring G forms an independent set (it induces an edgeless graph), we hence deduce that such S must be a mutual-visibility set of G. Thus, the partition of $V(G)$ induced by the proper coloring, represents also a partition into mutual-visibility sets for G. This leads to the following result.

Proposition 2.2. *If G is a graph of diameter two, then $\chi_\mu(G) \leq \chi(G)$. Moreover, this bound is tight.*

Proof. The bound clearly follows from the previous comments. To see that it is tight, consider the complete bipartite graphs $K_{r,t}$ with $(r,t) \neq (1,1)$, for which $\chi_\mu(K_{r,t}) = 2 = \chi(K_{r,t})$. □

We close this section with some other extra terminology and notation that shall be further used.

Let G be a graph. The *maximum degree* of G is denoted by $\Delta(G)$, and its order by $n(G)$. Given a vertex $v \in V(G)$, the *open neighborhood* of v is denoted by $N_G(v)$. If $X \subseteq V(G)$ and $F \subseteq E(G)$, then the subgraphs induced by X and by F are, respectively, denoted by $G[X]$ and $G[F]$. The *complement* of G is denoted by $\overline{G}$.

The *distance* between vertices $u, v \in V(G)$ is denoted by $d_G(u,v)$. The *diameter* $\mathrm{diam}(G)$ of G is the maximum distance between its vertices. The graph G is *geodetic* if each pair of its vertices is connected by a unique shortest path. A subgraph H of G is a *geodetic subgraph* if it has the same property, i.e., each pair of vertices of H is connected in G by a unique shortest path. A subgraph H of G is a *convex subgraph* if for any vertices x and y of H, every shortest x,y-path in G lies completely in H.

The *Cartesian product* $G \,\square\, H$ and the *strong product* $G \boxtimes H$ of graphs G and H both have the vertex set $V(G) \times V(H)$. The vertices (g,h) and (g',h') are adjacent in $G \,\square\, H$ if either $gg' \in E(G)$ and $h = h'$, or $g = g'$ and $hh' \in E(H)$. By a *layer* of $G \,\square\, H$, we mean a subgraph induced by all the vertices in which one coordinate is fixed. Note that a layer is either isomorphic to G or to H. The vertices (g,h) and (g',h') are adjacent in $G \boxtimes H$ if either one of the two condition for the Cartesian product holds, or $gg' \in E(G)$ and $hh' \in E(H)$.

3 Diameter two graphs

In order to continue the flow initiated in Section 2 with graphs of diameter two, we next consider this class of graphs with a bit more detail. In the main result, we determine the asymptotic growth of $\chi_\mu(K_n \,\square\, K_n)$, before that we prove the following.

Theorem 3.1. *If G is a graph with $\mathrm{diam}(G) = 2$ and each pair of vertices at distance two lies in a 4-cycle, then $\chi_\mu(G) = O(\sqrt{\Delta(G)})$.*

Proof. Consider an arbitrary partition $\mathcal{E} = \{E_1, \ldots, E_\ell\}$ of $E(G)$ such that no part of it contains C_4 as a subgraph. Based on $\mathcal{E}$, we form a partition $\mathcal{V} = \{V_1, \ldots, V_\ell\}$ of $V(G)$ as follows. Let $v \in V(G)$, and let v lie in $G[E_i]$, $i \in I$. Then, we put v into V_k, where $k = \min_{i \in I} i$. Note that some of the sets V_i might be empty, and let j be the largest index such that $V_j \neq \emptyset$. Assume $\mathcal{V}' = \{V_1, \ldots, V_j\}$. Then, $\mathcal{V}'$ is a partition of $V(G)$. Note also that since $G[E_i]$, $i \in [j]$, contains no C_4, the same holds for $G[V_i]$.

We claim that $\mathcal{V}'$ forms a mutual-visibility coloring. For this sake, consider a nonempty part $V_i \in \mathcal{V}$ and let $x, y \in V_i$. If $xy \in E(G)$, then they are V_i-visible. Assume next that $d_G(x,y) = 2$. Then, by our assumption, x and y lie in a C_4. But now at least one vertex of this C_4 does not belong to V_i, which in turn implies that

x and y are again V_i-visible. Hence, V_i is a mutual-visibility set for each $i \in [j]$, and thus, $\mathcal{V}'$ forms a mutual-visibility coloring.

To complete our argument, we recall from [18, Theorem 1] that every graph G admits a decomposition of $E(G)$ into $O(\sqrt{\Delta(G)})$ parts, such that none of them contains C_4 as a subgraph. □

The next result implies that the upper bound of Theorem 3.1 is asymptotically tight.

Theorem 3.2. *If $n \geq 2$ is large enough, then $\chi_\mu(K_n \,\square\, K_n) = \Theta(\sqrt{n})$.*

Proof. First, note that $K_n \,\square\, K_n$ fulfills the assumptions of Theorem 3.1; and hence, $\chi_\mu(K_n \,\square\, K_n) = O(\sqrt{n})$.

To prove that $\chi_\mu(K_n \,\square\, K_n) = \Omega(\sqrt{n})$, we recall from [3, Corollary 3.7] that if $m, n \geq 2$, then $\mu(K_m \,\square\, K_n) = z(m, n;\ 2, 2)$, where $z(m, n;\ 2, 2)$ is the maximum number of 1s that an $m \times n$ binary matrix can have, provided that it contains no 2×2 submatrix of 1s. (To determine the value $z(m, n;\ 2, 2)$ is an instance of the Zarankiewicz's problem, see [19].) When n is sufficiently large, the value $z(n, n;\ 2, 2)$ can be bounded as follows (see [20,21]):

$$z(n, n;\ 2, 2) \leq \frac{1}{2} n(1 + \sqrt{4n - 3}).$$

Since $\mu(K_m \,\square\, K_n) = z(m, n;\ 2, 2)$, this implies that, provided that n is large enough, a largest mutual-visibility set of $K_n \,\square\, K_n$ is of order $n^{3/2}$. Therefore, we need $\Theta(\sqrt{n})$ mutual-visibility sets to partition $V(K_n \,\square\, K_n)$, i.e., $\chi_\mu(K_n \,\square\, K_n) = \Omega(\sqrt{n})$. □

Theorem 3.1 also shows that the bound given in Proposition 2.2 is in general not achieved, since it is well known from [22] that $\chi(K_n \,\square\, K_n) = n$.

4 Greedy mutual-visibility coloring algorithm

A natural greedy algorithm for coloring a graph with mutual-visibility sets is to select at each step a largest mutual-visibility set of G among the vertices that are not yet colored, and color the selected set with a new color. This is formalized in Algorithm 1.

Algorithm 1: Greedy mutual-visibility coloring

```
Input: Connected graph G.
Output: Mutual-visibility coloring of G.
V = V(G), c = 1
while V ≠ ∅:
    determine a largest mutual-visibility set X of G, where X ⊆ V
    color the vertices of X with c
V = V\X, c = c + 1
```

It was proved in the seminal paper [1, Theorem 3.1] that the decision mutual-visibility problem is NP-complete. Hence, Algorithm 1 is not efficient in general. Of course, the algorithm is not always optimal either, but is optimal in many cases. It will follow from our subsequent results that it works optimally on block graphs, since provided that at each step of the algorithm, the set of simplicial vertices is selected (a vertex is *simplicial* if the set of its neighbors induces a complete graph). By [1, Theorem 4.2], these sets are indeed largest mutual-visibility sets. Moreover, in this case, Algorithm 1 is polynomial since the set of simplicial vertices of an arbitrary graph can be determined in polynomial time. For another example, consider the strong grids

$H_k = P_{2k} \boxtimes P_{2k}$, $k \geq 2$. It follows from [12, Theorem 4.4] that $\mu(H_k) = 8k - 4$. Moreover, the set of all vertices of H_k, which are not of maximum degree (i.e., the set of boundary vertices of H_k) forms a μ-set of H_k. (It can be actually proved that this set is the unique μ-set of H_k.) Then, Algorithm 1 will color these vertices with color 1, and proceeding by induction, the algorithm will color H_k with k colors. On the other hand, the main diagonal of H_k is the unique shortest path between its end vertices. Because this path contains $2k$ vertices, we infer that $\chi_\mu(H_k) \geq k$, which in turn implies that Algorithm 1 returns an optimal coloring.

The next result provides another example which demonstrates that Algorithm 1 is optimal, and at the same time, it gives a family of graphs for which the bound of Lemma 2.1 is sharp.

Proposition 4.1. *If* $t = 6k$, $k \geq 3$, *then* $\chi_\mu(C_t \,\square\, C_t) = 2k$.

Proof. By [13, Proposition 3.3], we have $\mu(C_t \,\square\, C_t) = 3t$. Hence, by Lemma 2.1, we obtain $\chi_\mu(C_t \,\square\, C_t) \geq \lceil t^2/3t \rceil = t/3 = 2k$.

In the proof of [13, Proposition 3.3], a set M of cardinality $3t$ is constructed and proved to be a mutual-visibility set of $C_t \,\square\, C_t$. We will not repeat here the explicit (slightly complicated) definition of this set, but instead identify its key properties. In each layer with respect to the first factor of $C_t \,\square\, C_t$, the set M has exactly three vertices that are uniformly spaced at distance $2k$, i.e., the three vertices from M that lie in a given layer are pairwise at distance $2k$. By the transitivity of $C_t \,\square\, C_t$, M can be, respectively, shifted $2k - 1$ times, each time by 1 in the first coordinate, to construct sets $M_2, M_3, \ldots, M_{2k}$. Using the transitivity of $C_t \,\square\, C_t$ again, each of the sets $M_2, M_3, \ldots, M_{2k}$ is a mutual-visibility set. Since $V(C_t \,\square\, C_t) = (\bigcup_{i=2}^{2k} M_i) \cup M$, we have thus found a mutual-visibility coloring of $C_t \,\square\, C_t$ using $2k$ colors. We conclude that $\chi_\mu(C_t \,\square\, C_t) \leq 2k$ and we are done. □

We next show that Algorithm 1 is not optimal in general. For $k \geq 1$, let G_k be the graph obtained from C_4 by amalgamating k private 4-cycles to each of the four edges of the original C_4 (see Figure 1). Setting $V(C_4) = [4]$, we denote the remaining vertices as can be seen from the figure. Note that $n(G_k) = 8k + 4$.

Proposition 4.2. *If* $k \geq 1$, *then* $\mu(G_k) = 8k$ *and* $\chi_\mu(G_k) = 2$.

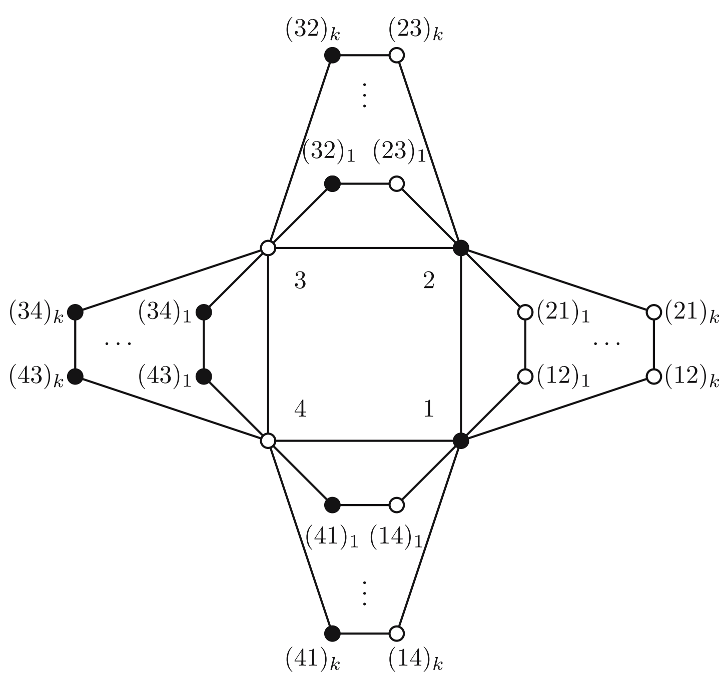

Figure 1: Graph G_k.

Proof. It is straightforward to check that the set $V(G_k)\setminus[4]$ is a mutual-visibility set; and hence, $\mu(G_k) \geq 8k$. On the other hand, consider the following four paths: (i) $(14)_1, 1, (12)_1$, (ii) $(21)_1, 2, (23)_1$, (iii) $(32)_1, 3, (34)_1$, and (iv) $(43)_1, 4, (41)_1$. Each of these paths is the unique shortest path between its end vertices; hence, an arbitrary mutual-visibility set can contain at most two vertices from each of these four disjoint paths. It follows that $\mu(G_k) \leq n(G_k) - 4 = 8k$. We can conclude that $\mu(G_k) = 8k$.

Assume now that

$$X = \{1, 2\} \cup \{(32)_i, (41)_i, (34)_i, (43)_i : i \in [k]\},$$

and let $X' = V(G_k)\setminus X$. See Figure 1, where the black vertices are the vertices of X. By examining all the possibilities (and taking into account the symmetries), we can verify that X is a mutual-visibility set of G_k. Using the symmetry again, we obtain that X' is likewise a mutual-visibility set. Since $V(G_k) = X \cup X'$, we conclude that $\chi_\mu(G_k) = 2$. □

The graphs G_k of Proposition 4.2 thus demonstrate that Algorithm 1 is not optimal in general. Indeed, in view of the proof of the proposition, the algorithm first colors all the vertices from the set $V(G_k)\setminus[4]$ with color 1. Then, the vertices 1, 2, 3, and 4 are yet to be colored. Since they induce a geodetic subgraph of G_k isomorphic to C_4, the algorithm colors three of these vertices with color 2 and the remaining vertex with color 3. As $\chi_\mu(G_k) = 2$, the conclusion follows.

Our next goal is to show that Algorithm 1 can return an arbitrarily bad result. For this sake, consider the class of graphs F_k, $k \geq 2$, obtained from the disjoint union of C_{4k} and P_{2k-1} by adding an edge between a vertex of C_{4k} and one vertex of degree one in P_{2k-1}. Let the vertices of C_{4k} be x_i, $i \in [4k]$, and let the vertices of P_{2k-1} be y_i, $i \in \{2, 3, \dots, 2k\}$, where the edges in both graphs are natural. Then, F_k is obtained from these C_{4k} and P_{2k-1} by adding the edge x_1y_2. F_k belongs to the family of the so-called frog graphs, see [3], which also gives rise to the notation for these graphs.

Proposition 4.3. *If $k \geq 2$, then $\chi_\mu(F_k) = 2k$.*

Proof. Consider the following partition of $V(F_k)$:

$$\{x_1, x_{2k+1}\}, \{y_2, x_2, x_{4k}\}, \{y_3, x_3, x_{4k-1}\}, \dots, \{y_{2k}, x_{2k}, x_{4k-(2k-2)}\} = \{y_{2k}, x_{2k}, x_{2k+2}\}.$$

Since each part of this partition is a mutual-visibility set of F_k, we infer that $\chi_\mu(F_k) \leq 2k$. On the other hand, we recall from [3, Theorem 4.4] that $\mu(F_k) = 3$. By Lemma 2.1, we then obtain

$$\chi_\mu(G) \geq \left\lceil \frac{n(F_k)}{\mu(F_k)} \right\rceil = \left\lceil \frac{6k-1}{3} \right\rceil = 2k,$$

and we are done. □

Consider now the following sequence of mutual-visibility sets of F_k:

$$\{x_1, x_2, x_{2k+1}\}, \{x_3, x_4, x_{2k+3}\}, \dots, \{x_{2k-1}, x_{2k}, x_{2k+(2k-1)}\} = \{x_{2k-1}, x_{2k}, x_{4k-1}\}.$$

Algorithm 1 legally colors the vertices from these respective sets with colors $1, 2, \dots, k$. At that stage of the algorithm, the vertices

$$x_{2k+2}, x_{2k+4}, \dots, x_{4k}, y_2, y_3, \dots, y_{2k}$$

are not yet colored. They all lie on a shortest x_{2k+2}, y_{2k}-path. Moreover, this path is the unique shortest path between x_{2k+2} and y_{2k}, therefore, a mutual-visibility set can contain at most two vertices out of them. It follows that the algorithm uses at least $\lceil (k + 2k - 1)/2 \rceil = \lceil (3k - 1)/2 \rceil$ colors for them; hence, overall at least $k + \lceil (3k - 1)/2 \rceil$ colors are used. This demonstrates that even if an optimal coloring uses color classes that are μ-sets, Algorithm 1 may (by selecting "wrong" μ-sets) return a coloring using arbitrary more colors than the mutual-visibility chromatic number.

5 General bounds

Since the mutual-visibility properties of graphs stands on a geodetic distance frame, it is natural to think that for a graph G, the parameter $\chi_\mu(G)$ relates to the diameter of G. However, such relationships become clear only for geodetic graphs. We open this section precisely with the following result on geodetic graphs.

Proposition 5.1. *Let G be a geodetic graph. Then,*

$$\chi_\mu(G) \geq \left\lceil \frac{\mathrm{diam}(G)+1}{2} \right\rceil.$$

Moreover, if G is a block graph, then the equality is achieved.

Proof. Let $P = \{v_i : i \in [\mathrm{diam}(G)+1]\}$ be a diametral path of G. Since G is geodetic, no three vertices in P may have the same color. Therefore, we need at least $\left\lceil \frac{\mathrm{diam}(G)+1}{2} \right\rceil$ colors in a mutual-visibility coloring of the vertices of P. Hence, we deduce that $\chi_\mu(G) \geq \left\lceil \frac{\mathrm{diam}(G)+1}{2} \right\rceil$.

Assume now that G is a block graph, which are well known to be geodetic. The equality holds when G is complete; hence, assume in the rest that $\mathrm{diam}(G) \geq 2$. Let S be the set of simplicial vertices of G. (Since S is a μ-set of G, this selection can be considered as the one from Step 5 of Algorithm 1). Then, $G - S$ is again a block graph. Let $x, y \in V(G)\backslash S$ such that $d_{G-S}(x,y) = \mathrm{diam}(G-S)$. Hence, x and y are cut vertices of G and there exist simplicial vertices $x', y' \in V(G)$, respectively adjacent to x, y. Therefore $\mathrm{diam}(G-S) = d_{G-S}(x,y) = d_G(x,y) = d_G(x',y') - 2 = \mathrm{diam}(G) - 2$. Since the set S forms a mutual-visibility set, we color its vertices with the same color. Continuing in the same manner on the block graph $G - S$, we arrive to a mutual-visibility coloring using $\left\lceil \frac{\mathrm{diam}(G)+1}{2} \right\rceil$ colors. □

Let P be the Petersen graph. Then, it can be checked that $\chi_\mu(P) = 2$; hence, the Petersen graph is a sporadic example for which the equality in Proposition 5.1 holds. The latter result also yields the mutual-visibility chromatic number of trees, a result worthy of special mention, and where $\mathrm{rad}(T)$ represents the *radius* of the tree T.

Corollary 5.2. *If T is a tree, then*

$$\chi_\mu(T) = \begin{cases} \mathrm{rad}(T)+1; & \mathrm{diam}(T) \ \textit{even}, \\ \mathrm{rad}(T); & \mathrm{diam}(T) \ \textit{odd}. \end{cases}$$

Apart from the trivial bound from Lemma 2.1, our parameter can be related to the classical mutual-visibility number in a different way. The next upper bound appears to be a basic one, but it shows to be very useful. We remark that the bound is somehow derived from the greedy algorithm described in Section 4.

Proposition 5.3. *If G is a connected graph, then*

$$\chi_\mu(G) \leq \left\lceil \frac{n(G)-\mu(G)+2}{2} \right\rceil.$$

Proof. Let $k = \left\lceil \frac{n(G)-\mu(G)}{2} \right\rceil$ and let S be a μ-set of G. Consider $\{S_j : j \in [k]\}$ as a partition of $V(G)\backslash S$ such that every S_j has cardinality two for every $j \in [k]-1$, and $|S_k| \leq 2$.

Now, it can be readily seen that the partition of $V(G)$ given by $\mathcal{P} = \{S\} \cup \{S_j : j \in [k]\}$ induces a mutual-visibility coloring of G, since G is connected and each S_j has cardinality at most 2. Thus, $\chi_\mu(G) \leq 1 + \left\lceil \frac{n(G)-\mu(G)}{2} \right\rceil = \left\lceil \frac{n(G)-\mu(G)+2}{2} \right\rceil$. □

Since the bound of Proposition 5.3 depends on the value of the mutual-visibility number, and computing such parameter is an NP-hard problem (cf. [1]), the following corollary gives us a tool to bound above the value of $\chi_\mu(G)$ in terms of the maximum degree of a graph. This is based on the fact that for any vertex of a graph G, its neighborhood forms a mutual-visibility set of G, which implies that $\mu(G) \geq \Delta(G)$.

Corollary 5.4. *If G is a connected graph, then*

$$\chi_\mu(G) \leq \left\lceil \frac{n(G) - \Delta(G) + 2}{2} \right\rceil.$$

Let us point out that the condition for G to be connected is essential for applying this result. For example, consider the infinite family of graphs $G(p,q) = K_p \cup \overline{K_q}$. In such case, for every $q \geq 3$, it holds that

$$\chi_\mu(G(p,q)) = q + 1 > \left\lceil \frac{n(G(p,q)) - \Delta(G(p,q)) + 2}{2} \right\rceil$$
$$= \left\lceil \frac{p + q - (p-1) + 2}{2} \right\rceil = \left\lceil \frac{q+3}{2} \right\rceil.$$

For graphs containing a vertex adjacent to all other vertices, the following consequence is deduced.

Corollary 5.5. *If G is not complete and $\Delta(G) = n(G) - 1$, then $\chi_\mu(G) = 2$.*

Proof. Since $\Delta(G) = n(G) - 1$ and G is not complete, $\mu(G) = n(G) - 1$. Therefore, Lemma 2.1 and Corollary 5.4 became equalities and we may derive that $\chi_\mu(G) = 2$. □

Observe that the maximum possible value of the upper bound from Corollary 5.4 is $\left\lceil \frac{n(G)}{2} \right\rceil$, which appears when $\Delta(G) = 2$. We next characterize the graphs achieving precisely this maximum value.

Proposition 5.6. *If G is a connected graph, then $\frac{n(G)}{2} < \chi_\mu(G) = \left\lceil \frac{n(G)}{2} \right\rceil$ if and only if G is a path of odd order.*

Proof. Let $n = n(G)$ and $\Delta = \Delta(G)$. First, let us suppose that $G = P_{2k+1}$. By Lemma 2.1 and Proposition 5.3, we have that $\chi_\mu(G) = \left\lceil \frac{n}{2} \right\rceil$. On the other hand, assume now that $\frac{n}{2} < \chi_\mu(G) = \left\lceil \frac{n}{2} \right\rceil$, which implies that n is an odd integer. By Corollary 5.4, we deduce that

$$\left\lceil \frac{n}{2} \right\rceil = \left\lceil \frac{n - \Delta + 2}{2} \right\rceil.$$

Since n is odd, we have that $\frac{n+1}{2} = \frac{n+1}{2} + \left\lceil \frac{1-\Delta}{2} \right\rceil$, and therefore, $\Delta \leq 2$. Clearly, $\Delta = 1$ implies that $G = K_2$, because G is connected, which is not possible because n is odd. Hence, it is derived that $\Delta = 2$ and G must be either an odd cycle, or a path of odd order.

If G is a cycle, then it is known that $\mu(G) = 3$. Let S be a mutual visibility set of order 3 in the cycle. As $|V(G)\backslash S| = n - 3$ is even, we may consider a partition of $V(G)\backslash S$ into $(|V(G)\backslash S|)/2$ sets of cardinality 2, which are mutual-visibility sets. Then,

$$\chi_\mu(G) \leq 1 + \frac{|V(G)\backslash S|}{2} = 1 + \frac{n-3}{2} = \frac{n-1}{2} < \frac{n+1}{2} = \chi_\mu(G),$$

which is a contradiction. Therefore, we conclude that G must be a path with an odd number of vertices. □

The following realization result complements Proposition 5.6.

Proposition 5.7. *Let n and k be positive integers such that* $2 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$. *Then, there is a graph G with* $n = n(G)$ *and*

$$\left\lceil \frac{n(G)}{\mu(G)} \right\rceil \leq \chi_\mu(G) = k \leq \left\lfloor \frac{n(G)}{2} \right\rfloor.$$

Proof. If $k = \left\lfloor \frac{n}{2} \right\rfloor$, then it is sufficient to consider the path graph $G = P_{2k}$. From now on, we may assume that $2 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$. Let G be the graph obtained by joining a leaf of a path having $p = 2k - 1$ vertices with $q = n - 2k + 1$ isolated vertices (see Figure 2).

Since $4 \leq 2k \leq n - 2$, the graph G is well defined and we have that $3 \leq p, q \leq n - 3$. It is straightforward to check that the set of vertices $S = \{v_1, v_{p+1}, \ldots, v_{p+q}\}$ is a mutual-visibility set of maximum cardinality $q + 1$. Hence, $\mu(G) = q + 1$. Moreover, G is a tree having $\operatorname{diam}(G) = d(v_1, v_{p+1}) = p = 2k - 1$ and $\operatorname{rad}(G) = d(v_k, v_{p+1}) = k$. Therefore, by applying Corollary 5.2, we have that $\chi_\mu(G) = k$. □

Next, we show an upper bound for regular graphs that improves the one given by Proposition 5.3.

Proposition 5.8. *If G is an r-regular graph,* $r \geq 2$, *with girth* $g(G) \geq 6$, *then*

$$\chi_\mu(G) \leq \left\lceil \frac{n(G) - r^2 + 4}{2} \right\rceil.$$

Proof. Set $n = n(G)$. Let $v \in V(G)$, and set $S = N_G(v)$ and $T = N_G(N_G(v)) - v$. Since $g(G) \geq 6$, we infer that $S \cap T = \emptyset$, that $G[T]$ is an edgeless graph, and $|T| = r(r-1)$. And, clearly, $|S| = r$.

The set S is a mutual-visibility set since it is an open neighborhood of a vertex. We further claim that T is a mutual-visibility set as well. Consider arbitrary vertices $x, y \in T$, and note that $d_G(x, y) \leq 4$. We wish to see that x and y are T-visible. Since $G[T]$ is an edgeless graph, there is nothing to show if $d_G(x, y) \leq 2$. The fact that $G[T]$ is an edgeless graph also implies that if $d_G(x, y) = 3$, then no shortest x, y-path contains a vertex of T as an interior vertex. Finally, if $d_G(x, y) = 4$, then there exits a shortest x, y-path containing v; and hence, also in this case, x and y are T-visible. We can conclude that T is a mutual-visibility set.

The set of vertices $V(G) \setminus \{S, T\}$ has cardinality $n - r^2$. Let v_i, with $i \in [n - r^2]$, be the vertices of such set. Let $P_i = \{v_{2i-1}, v_{2i}\}$ for each $i \in [\frac{n-r^2}{2}]$ if $n - r^2$ is even, or $i \in [\frac{n-r^2-1}{2}]$ if $n - r^2$ is odd. Now, consider the partition of $V(G) \setminus \{S, T\}$ given by $\mathcal{P} = \{P_i : i \in [\frac{n-r^2}{2}]\}$, if $n - r^2$ is even or $\mathcal{P} = \{P_i : i \in [\frac{n-r^2-1}{2}]\} \cup \{\{v_{n-r^2}\}\}$,

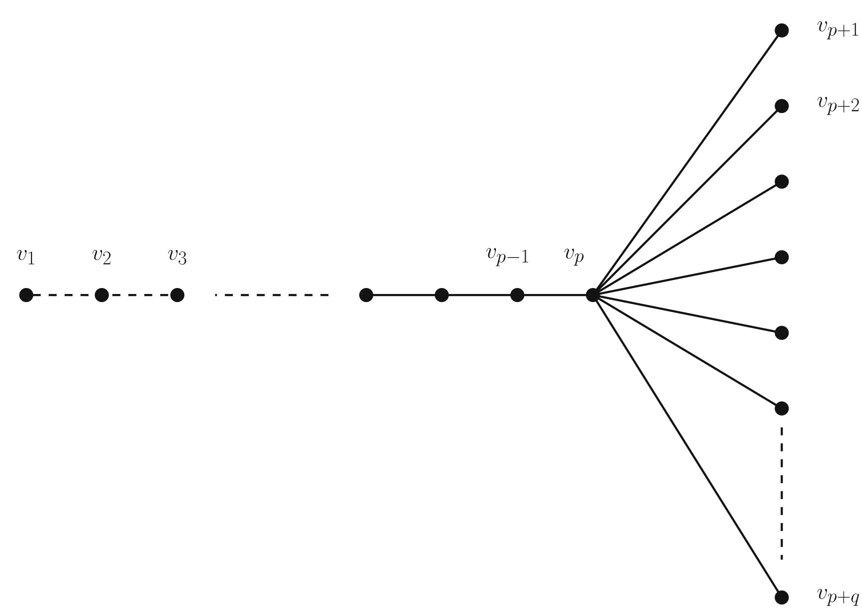

Figure 2: A graph G of order $n = p + q$ with $\chi_\mu(G) = k = \frac{p+1}{2}$.

if $n - r^2$ is odd. Since every single element of $\mathcal{P}$ has cardinality at most 2, they are mutual-visibility sets of G. So, we can color S and T with two different colors and we may assign one more distinct color to each set P_i. Hence,

$$\chi_\mu(G) \le 2 + \left\lceil \frac{n - r^2}{2} \right\rceil = \left\lceil \frac{n - r^2 + 4}{2} \right\rceil,$$

which we needed to prove. □

We can note that the bound given by Proposition 5.8 improves the one described in Proposition 5.3 for regular graphs, because $r^2 - 2 \ge r = \Delta(G)$ for every r-regular graph G.

We end this section with a Nordhaus-Gaddum-type upper bound related to the mutual-visibility chromatic number of a graph.

Proposition 5.9. *If G is a connected graph such that $\overline{G}$ is also connected, then*

$$\chi_\mu(G) + \chi_\mu(\overline{G}) \le \left\lceil \frac{n(G) - \mu(G) + 2}{2} \right\rceil + \left\lceil \frac{\delta(G) + 3}{2} \right\rceil.$$

Proof. Since G is connected, Proposition 5.3 yields $\chi_\mu(G) \le \left\lceil \frac{n(G) - \mu(G) + 2}{2} \right\rceil$. Also, as $\overline{G}$ is connected, Corollary 5.4 gives $\chi_\mu(\overline{G}) \le \left\lceil \frac{n(G) - \Delta(\overline{G}) + 2}{2} \right\rceil = \left\lceil \frac{\delta(G) + 3}{2} \right\rceil$. Therefore,

$$\chi_\mu(G) + \chi_\mu(\overline{G}) \le \left\lceil \frac{n(G) - \mu(G) + 2}{2} \right\rceil + \left\lceil \frac{\delta(G) + 3}{2} \right\rceil,$$

which is the desired bound. □

It can be readily seen that the bound given by Proposition 5.9 is tight for paths of odd order $G = P_{2k+1}$ with $k \ge 2$. Since such P_{2k+1} is a bipartite graph (and its complement is connected), its complement can be partitioned into two cliques, each of which forms a mutual-visibility set (in the complement). Thus, it holds that $\chi_\mu(\overline{G}) = 2$. By Proposition 5.6, we know that $\chi_\mu(P_{2k+1}) = k + 1$. Therefore, equality holds in Proposition 5.9.

Table 1: Summary of bounds for the mutual-visibility chromatic number $\chi_\mu(G)$

Graph class/condition	**Bound for $\chi_\mu(G)$**	**Reference**
Geodetic graph	$\chi_\mu(G) \ge \left\lceil \frac{\mathrm{diam}(G) + 1}{2} \right\rceil$	Proposition 5.1
Block graph	Equality in above bound	Proposition 5.1
Tree with even diameter	$\chi_\mu(T) = \mathrm{rad}(T) + 1$	Corollary 5.2
Tree with odd diameter	$\chi_\mu(T) = \mathrm{rad}(T)$	Corollary 5.2
Connected graph (general bound)	$\chi_\mu(G) \le \left\lceil \frac{n(G) - \mu(G) + 2}{2} \right\rceil$	Proposition 5.3
Connected graph (degree bound)	$\chi_\mu(G) \le \left\lceil \frac{n(G) - \Delta(G) + 2}{2} \right\rceil$	Corollary 5.4
$\Delta(G) = n(G) - 1$, G not complete	$\chi_\mu(G) = 2$	Corollary 5.5
$G = P_{2k+1}$ (odd path)	$\chi_\mu(G) = \left\lceil \frac{n(G)}{2} \right\rceil$	Proposition 5.6
G is r-regular, $r \ge 2$, $g(G) \ge 6$	$\chi_\mu(G) \le \left\lceil \frac{n(G) - r^2 + 4}{2} \right\rceil$	Proposition 5.8
G and $\overline{G}$ connected	$\chi_\mu(G) + \chi_\mu(\overline{G}) \le \left\lceil \frac{n(G) - \mu(G) + 2}{2} \right\rceil + \left\lceil \frac{\delta(G) + 3}{2} \right\rceil$	Proposition 5.9
Same as above (general bound)	$\chi_\mu(G) + \chi_\mu(\overline{G}) \le \left\lceil \frac{n(G) + 5}{2} \right\rceil$	Corollary 5.10

In addition to the comments mentioned earlier, Proposition 5.9 also leads to the following conclusion by taking into account that $\mu(G) \geq \Delta(G) \geq \delta(G)$ for any graph G.

Corollary 5.10. *If G is a connected graph such that $\overline{G}$ is also connected, then*

$$\chi_\mu(G) + \chi_\mu(\overline{G}) \leq \left\lceil \frac{n(G) + 5}{2} \right\rceil.$$

Note that again the case of paths of odd order $G = P_{2k+1}$ can be used to show the tightness of the upper bound of Corollary 5.10. We end this section, by including a summary table containing all the bounds we have obtained (Table 1).

6 Corona product graphs

Let G be a graph whose vertex set is $V(G) = \{v_1, \dots, v_n\}$, and let H be a graph. The *corona product* $G \odot H$ of G and H is the graph defined as follows. We take one copy of G and $n(G)$ disjoint copies of H, denoted $H^1, \dots, H^n$. Next, for every integer $i \in [n]$, we add all the possible edges between any vertex $v_i \in V(G)$ and all the vertices of H^i.

Proposition 6.1. *If G is a connected graph of order at least two and H is a graph, then*

$$\chi_\mu(G) \leq \chi_\mu(G \odot H) \leq \chi_\mu(G) + 1.$$

Proof. To prove the lower bound, let us consider a mutual-visibility coloring of $G \odot H$ given by the collection of sets $\mathcal{P} = \{P_i\}$, with $i \in [\chi_\mu(G \odot H)]$. Since G is a convex subgraph of $G \odot H$, it holds that the collection $\mathcal{P}' = \{P_i \cap V(G)\}$ is a mutual-visibility coloring of G. Therefore, $\chi_\mu(G) \leq \chi_\mu(G \odot H)$.

Next, we prove the upper bound. If G is a complete graph K_n, then it can be readily observed that $\chi_\mu(K_n \odot H) = 2 = \chi_\mu(K_n) + 1$, since the vertex set $V(K_n)$ and the complement of it in $K_n \odot H$ are both mutual-visibility sets of $K_n \odot H$. Since $G \odot H$ contains a geodetic subgraph isomorphic to P_5, it follows that $\chi(G \odot H) > 2$. Furthermore, a 3-mutual-visibility coloring can be explicitly constructed as follows: assign

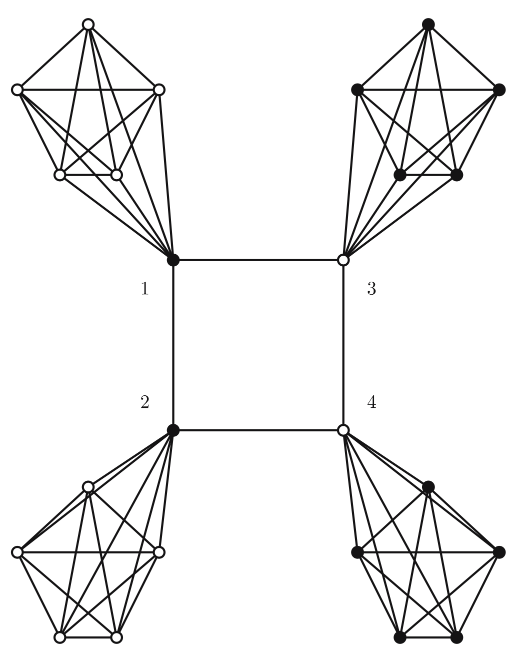

Figure 3: A mutual-visibility coloring of $C_4 \odot K_5$ with $\chi_\mu(C_4) = 2$ colors.

the same color to all vertices in the three copies of H, and then color the vertices of the base path P_3 using two additional colors so that the mutual-visibility condition is satisfied. This shows that $\chi(G \odot H) = 3$.

Moreover, if G is the graph P_3, then $G \odot H$ contains a geodetic subgraph isomorphic to P_5; hence, it can be deduced that $\chi_\mu(P_3 \odot H) = 3 = \chi_\mu(P_3) + 1$. Thus, from now on, we may assume that G is a non-complete graph of order at least 4.

Let $\mathcal{P} = \{P_1, \ldots, P_r\}$ be a mutual-visibility coloring of G. We claim that $\mathcal{P} \cup W$, where $W = \bigcup_{i\in[n]} V(H^i)$, is a mutual-visibility coloring of $G \odot H$.

Clearly, any set $P_i \in \mathcal{P}$ is also a mutual-visibility set of $G \odot H$. On the other hand, if $x, y \in V(H^i)$ for some $i \in [n]$, then they are either adjacent or at distance two. Since they have a common neighbor (the vertex v_i), which is not in W, it holds that x and y are W-visible when they are not adjacent. Assume now that $x \in V(H^i)$ and $y \in V(H^j)$ for some distinct $i, j \in [n]$. Hence, they are clearly W-visible by using any shortest v_i, v_j-path. As a consequence, it follows that $\mathcal{P} \cup W$ is a mutual-visibility coloring of $G \odot H$ as claimed, and so, $\chi_\mu(G \odot H) \le \chi_\mu(G) + 1$. □

It is worth noting that the bounds given by Proposition 6.1 are tight. To see this, let us consider the cycle C_4 with vertex set $V(C_4) = [4]$. Let us denote by K_m^i the copy of K_m corresponding to the vertex i in $C_4 \odot K_m$. We can readily check that $\chi_\mu(C_4 \odot K_m) = \chi_\mu(C_4) = 2$, by considering the mutual-visibility coloring of $C_4 \odot K_m$ given by the sets of vertices $\{\{1, 2\} \cup V(K_m^3) \cup V(K_m^4), \{3, 4\} \cup V(K_m^1) \cup V(K_m^2)\}$ (Figure 3). Similarly, we can verify that $\chi_\mu(K_n \odot K_m) = \chi_\mu(K_n) + 1 = 2$ for any integers m, n with $n \ge 2$ and $m \ge 1$.

7 Graphs with mutual-visibility chromatic number two

Consider the family $\mathcal{A}$ of graphs G_A defined as follows. We begin with two graphs A_r and A_s and two complete graphs K_r and K_s. Next, we add all possible edges between the vertices of A_r and the vertices of K_r, as well as all the possible edges between the vertices of A_s and the vertices of K_s. Then, to obtain a graph $G_A \in \mathcal{A}$, we add some edges between the vertices of A_r and A_s in such a way that any two not adjacent vertices of A_r will have a common neighbor in $V(A_s)$, and that any two not adjacent vertices of A_s will have a common neighbor in $V(A_r)$ (Figure 4).

We also consider the family $\mathcal{B}$ of graphs G_B constructed as follows. We begin with two graphs B and B' satisfying that $|\mathrm{diam}(B) - \mathrm{diam}(B')| \le 2$. Next, for any two not adjacent vertices $x, y \in V(B)$ with $d_B(x, y) = d$, we select vertices $x', y' \in V(B')$ such that $d_{B'}(x', y') = d - 2$, and add the edges xx' and yy'. (Note that if $d = 2$, then $x' = y'$.) Similarly, for any two not adjacent vertices $x', y' \in V(B')$ with $d_{B'}(x', y') = d$, we select two vertices $x, y \in V(B)$ such that $d_B(x, y) = d - 2$, and add the edges xx' and yy' (Figure 5).

We have the following result.

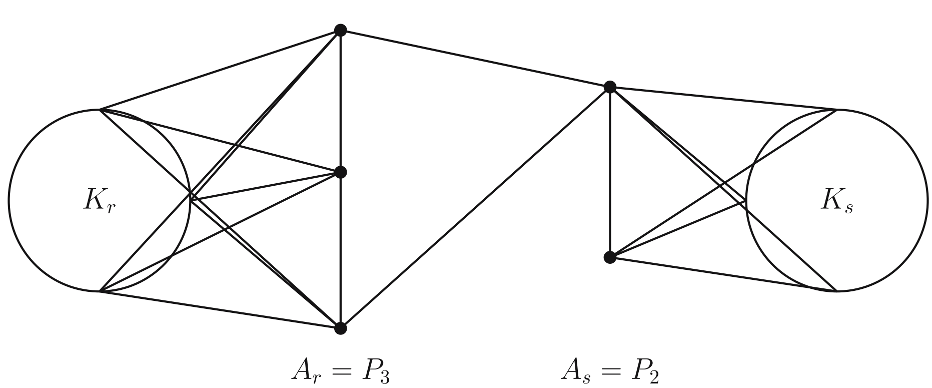

Figure 4: A graph belonging to the family $\mathcal{A}$.

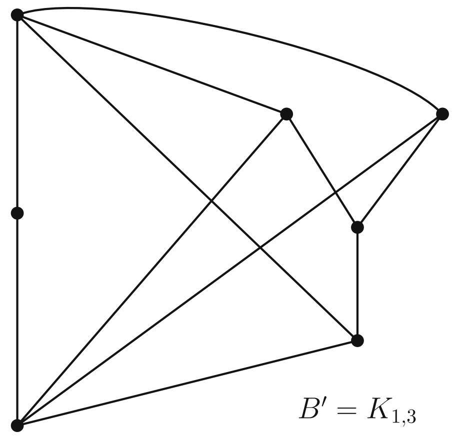

Figure 5: A graph belonging to the family $\mathcal{B}$.

Proposition 7.1. *If* $G \in \mathcal{A} \cup \mathcal{B}$, *then* $\chi_\mu(G) = 2$.

Proof. Assume $G = G_A \in \mathcal{A}$. According to the construction of G_A, we can readily see that any two vertices of the set $S = V(A_r) \cup V(K_r)$ are S-visible, since they are either adjacent or at distance two (in such case, $x, y \in V(A_r)$). In the latter case, by the construction, any two not adjacent vertices of $V(A_r)$ have a common neighbor $z \in V(A_s)$ and $z \notin S$. Thus, S is a mutual-visibility set of G. Similarly, the set $S' = V(A_s) \cup V(K_s)$ is also a mutual-visibility set of G. These facts allow us to conclude that $\chi_\mu(G_A) = 2$.

Assume next that $G = G_B \in \mathcal{B}$. In such situation, we claim that each set $V(B)$ and $V(B')$ are mutual-visibility sets of G_B. Note that for any two not adjacent vertices $x, y \in V(B)$ with $d_B(x, y) = d \geq 2$, there is a shortest x, y-path whose internal vertices are in $V(B')$ according to the construction of G_B. Thus, x and y are $V(B)$-visible and so, $V(B)$ is a mutual-visibility set of G_B as claimed. The same conclusion can be readily deduced for $V(B')$. Thus, $\chi_\mu(G_B) = 2$. □

Another infinite family of graphs G satisfying $\chi_\mu(G) = 2$ is given by the strong product graphs $H \boxtimes K_2$ for any non-complete graph H. From the proof of [12, Theorem 5.1], one can deduce that if $V(K_2) = \{u, v\}$, then the two sets $V(H) \times \{u\}$ and $V(H) \times \{v\}$ are mutual-visibility set of $H \boxtimes K_2$, and so, $\chi_\mu(H \boxtimes K_2) \leq 2$. The equality follows because H is not a complete graph. Yet another family of graphs G satisfying that $\chi_\mu(G) = 2$ is obtained from the last comments of Section 6, i.e., for any corona graph $K_n \odot K_m$ ($n \geq 2$ and $m \geq 1$) it holds that $\chi_\mu(K_n \odot K_m) = 2$.

The comments of the paragraph above and the results from Proposition 7.1 allow us to observe that there are graphs G satisfying that $\chi_\mu(G) = 2$ and whose structures are intrinsically different. Note that the graphs $G_A \in \mathcal{A}$ satisfy that $\operatorname{diam}(G_A) \leq 3$, while the graphs $G_B \in \mathcal{B}$ might have diameter as large as we would require.

8 Concluding remarks

In this section, we propose some open problems and suggestions that might be taken into account for further investigation of the concept introduced in this article. First, the following is a fundamental question to be investigated.

Problem 8.1. Investigate the computational complexity of computing the mutual-visibility chromatic number.

In Section 3, the value of the $\chi_\mu(G)$ is computed (asymptotically) for the case of two-dimensional Hamming graphs. It is already known that finding the mutual-visibility number of Hamming graphs is a very challenging problem (see, for instance, [3]). In this sense, the following problem might also be challenging, but worth of exploring.

Problem 8.2. Compute or at least bound the mutual-visibility chromatic number of Hamming graphs, and of the Cartesian product of at least two complete graphs in general.

In view of Proposition 2.2, and due to the fact that the chromatic number of graphs is one of the most classical parameters in graph theory, we pose the following problem.

Problem 8.3. Characterize the graphs G of diameter two for which $\chi_\mu(G) = \chi(G)$ holds. In particular, are there any additional such graphs besides the complete bipartite graphs? In addition, are there some other relationships between $\chi_\mu(G)$ and $\chi(G)$ for other graph classes?

Axenovich and Liu [7] proved that $\mu(Q_n) > 0.186 \cdot 2^n$ for any hypercube graph Q_n. This raises the following:

Question 8.4. Does there exist an absolute constant M such that $\chi_\mu(Q_n) < M$ holds for any integer n?

We remark here that, while this manuscript has been under review, this question has already been negatively answered in [23]. Section 7 deals the graphs G satisfying that $\chi_\mu(G) = 2$ and Corollary 5.10 shows a Nordhauss-Gaddum-type result for $\chi_\mu(G)$. In this sense, it seems to be worth of considering the following.

Problem 8.5. Characterize the graphs G (at least partially) satisfying that $\chi_\mu(G) = 2$, as well as, those connected graphs G whose complements are also connected, and for which $\chi_\mu(G) + \chi_\mu(\overline{G}) = \left\lceil \frac{n(G)+5}{2} \right\rceil$.

A variety of mutual-visibility problems in graph theory (variants called total, outer, and dual) was described in [2]. In this sense, it seems to be natural to consider the problem of coloring the vertices of a graph with total, outer, or dual mutual-visibility sets. Analogously, closely related to the mutual-visibility sets of graphs, there exist the general position sets [24–28]. Hence, coloring the vertices of a graph with general position sets is also of interest.

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