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On the packing chromatic number of Cartesian products, hexagonal lattice, and trees☆

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Abstract

The packing chromatic number $\chi_\rho(G)$ of a graph G is the smallest integer k such that the vertex set of G can be partitioned into packings with pairwise different widths. Several lower and upper bounds are obtained for the packing chromatic number of Cartesian products of graphs. It is proved that the packing chromatic number of the infinite hexagonal lattice lies between 6 and 8. Optimal lower and upper bounds are proved for subdivision graphs. Trees are also considered and monotone colorings are introduced.

Keywords: Packing chromatic number; Cartesian product of graphs; Hexagonal lattice; Subdivision graph; Tree; Computational complexity

1. Introduction

Let $G = (V(G), E(G))$ be a graph and let d be a positive integer. Then $X \subseteq V(G)$ is a *d-packing* if vertices of X are pairwise at distance more than d. The *d-packing number* of G, denoted by $\rho_d(G)$, is the maximum cardinality of a d-packing that occurs in G. The integer d is called the *width* of the packing X. For instance, packings of width 1 are precisely the independent sets of G and those of width 2 correspond to the centers of pairwise disjoint closed neighborhoods in G. Note that if X is a packing of width d, then X is also a packing of width $d-1$. Several applications of packings were proposed, for instance, in resource placement in communication networks and error-correcting codes, cf. [8].

Consider now the following optimization problem. For a graph G find the smallest integer k such that $V(G)$ can be partitioned into k packings $X_1, \ldots, X_k$ with pairwise different widths. Since such a partition has packings of k distinct widths and because the objective is to minimize k, we can assume that X_i is an i-packing for each i. (Of course, $\cup_{i=1}^k X_i = V(G)$.) We call the integer k the *packing chromatic number* of G and denote it by $\chi_\rho(G)$. Throughout the paper, packing colorings will be shortly called colorings. A (packing) coloring of G of size $\chi_\rho(G)$ will be called a $\chi_\rho(G)$*-coloring*.

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The concept of the packing chromatic number was introduced by Goddard et al. in [5] under the name *broadcast chromatic number* where an application to frequency assignments was indicated. In a given network the signals of two stations that are using the same broadcast frequency will interfere unless they are located sufficiently far apart. The distance the signals will propagate is directly related to the power of those signals. Within this model all stations located at vertices in the i-packing, X_i, are allowed to broadcast at the same frequency with a power that will not allow the signals to interfere at distance i.

We believe that the concept could have several additional applications, as, for instance, in resource placements and biological diversity. For instance, different species in a certain area require different amounts of territory. Moreover, the concept is both a packing and a coloring (i.e., a partitioning) concept. Hence, we decided to use “packing chromatic number” instead.

A special type of packing chromatic number, called eccentric chromatic number, has been studied for trees by Sloper [11]. He proved that the infinite 3-regular tree has packing chromatic number 7. In [5] it was shown that for the infinite square grid $\mathscr{G}$, $\chi_\rho(\mathscr{G}) \leqslant 22$. The authors in addition asked what is the packing chromatic number of the hexagonal lattice.

We proceed as follows. In the rest of this section necessary definitions are given. In the next section we study Cartesian products of graphs and obtain several lower and upper bounds for the packing chromatic number. Then, in Section 3, we bound the packing chromatic number of the infinite hexagonal lattice between 6 and 8. We continue with another graph operation, the subdivision graph $S(G)$ of a graph G, and prove that for any connected graph G with at least three vertices, $\omega(G) + 1 \leqslant \chi_\rho(S(G)) \leqslant \chi_\rho(G) + 1$. In the section preceding the final we consider the packing chromatic number of trees. Monotone colorings are introduced and it is surprisingly observed that there exist trees that do not admit such colorings. Moreover, examples of trees are given each of which contains two specified leaves that are colored with at least 2 in any optimal coloring. These examples indicate that this coloring problem might not be polynomial for trees.

The *Cartesian product* $G\Box H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$, vertices (g, h) and (g', h') being adjacent whenever $gg' \in E(G)$ and $h = h'$, or $g = g'$ and $hh' \in E(H)$. Recall that the Cartesian product operation is associative and commutative, see [7]. The subgraph of $G\Box H$ induced by $\{g\} \times V(H)$ is isomorphic to H. It is called an *H-fiber* and denoted by ${}^gH$. Similarly, one defines the *G-fiber* G^h for a vertex h of H.

The *diameter*, $\mathrm{diam}(G)$, of a connected graph G is the maximum distance between any two vertices of G. By $N_k[x]$ we denote the set $\{u \in V(G) \mid d(u, x) \leqslant k\}$. Finally, $\omega(G)$ denotes the order of a largest complete subgraph of G and $\alpha(G)$ the order of a largest independent set of G.

2. Cartesian products

In this section we consider the packing chromatic number of Cartesian product graphs. This investigation has been initiated in [5] where grids, that is, Cartesian products of two paths, and hypercubes (in a way the simplest Cartesian product graphs) are treated.

We first prove the following lower bound. Recall that $\mathrm{diam}(G\Box H) = \mathrm{diam}(G) + \mathrm{diam}(H)$, a fact that will be used throughout the next proof. To shorten the notation we will use $|G|$ for the order of a graph G.

Theorem 1. *Let G and H be connected graphs on at least two vertices. Then*

$$\chi_\rho(G\Box H) \geqslant (\chi_\rho(G) + 1)|H| - \mathrm{diam}(G\Box H)(|H| - 1) - 1.$$

Proof. Let $x \in \mathbb{Z}$ be selected such that $\chi_\rho(G) = \mathrm{diam}(G) + x$. We claim that

$$\chi_\rho(G\Box H) \geqslant \mathrm{diam}(G\Box H) + |H|(x - \mathrm{diam}(H) + 1) - 1. \quad (1)$$

Case 1: $x \leqslant 0$. Clearly, $(|H| - 1) \leqslant (|H| - 1)\,\mathrm{diam}(H)$ and hence

$$\begin{aligned}\chi_\rho(G\Box H) &\geqslant \chi_\rho(G)\\ &= \mathrm{diam}(G) + x\\ &\geqslant \mathrm{diam}(G) + |H|x \quad (\text{since } x \leqslant 0)\\ &\geqslant \mathrm{diam}(G) + |H|x + (|H| - 1) - (|H| - 1)\,\mathrm{diam}(H)\\ &= \mathrm{diam}(G\Box H) + |H|(x - \mathrm{diam}(H) + 1) - 1.\end{aligned}$$

Hence claim (1) is proved for $x \leqslant 0$.

Case 2: $x \geqslant 1$. If $x - \operatorname{diam}(H) + 1 < 0$, then, since $|H| > \operatorname{diam}(H)$,

$$\begin{aligned}\chi_\rho(G\Box H) &\geqslant \operatorname{diam}(G) + x\\ &\geqslant \operatorname{diam}(G) + 1\\ &\geqslant \operatorname{diam}(G) + (\operatorname{diam}(H) - |H|) + 1\\ &= \operatorname{diam}(G\Box H) - |H| + 1\\ &\geqslant \operatorname{diam}(G\Box H) + |H|(x - \operatorname{diam}(H) + 1) + 1\\ &\geqslant \operatorname{diam}(G\Box H) + |H|(x - \operatorname{diam}(H) + 1) - 1.\end{aligned}$$

Suppose next $x - \operatorname{diam}(H) + 1 = 0$. Then $\operatorname{diam}(H) = x + 1$ and therefore

$$\begin{aligned}\chi_\rho(G\Box H) &\geqslant \operatorname{diam}(G) + x\\ &= (\operatorname{diam}(G) + x + 1) + 0 - 1\\ &= \operatorname{diam}(G\Box H) + |H|(x - \operatorname{diam}(H) + 1) - 1.\end{aligned}$$

Let finally $x - \operatorname{diam}(H) + 1 \geqslant 1$. Assume that there exists a coloring of $G\Box H$ using at most $\operatorname{diam}(G\Box H) + |H|(x - \operatorname{diam}(H) + 1) - 2$ colors. Then each of the colors from the interval

$$I = [\operatorname{diam}(G\Box H), \operatorname{diam}(G\Box H) + |H|(x - \operatorname{diam}(H) + 1) - 2]$$

is used at most once. Note that I contains $|H|(x - \operatorname{diam}(H) + 1) - 1$ colors. Therefore, there exists a G-fiber, say G^h, that uses at most $x - \operatorname{diam}(H)$ colors from I. But then G^h is colored with at most $(\operatorname{diam}(G) + \operatorname{diam}(H) - 1) + (x - \operatorname{diam}(H)) = \operatorname{diam}(G) + x - 1 = \chi_\rho(G) - 1$ colors. This is not possible since G^h is isomorphic to G and claim (1) is proved also for $x \geqslant 1$.

To complete the proof just substitute $x = \chi_\rho(G) - \operatorname{diam}(G)$ into the claim and use again the fact $\operatorname{diam}(G\Box H) = \operatorname{diam}(G) + \operatorname{diam}(H)$. □

By the commutativity of the Cartesian product, the proof of Theorem 1 also gives

$$\chi_\rho(G\Box H) \geqslant (\chi_\rho(H) + 1)|G| - \operatorname{diam}(G\Box H)(|G| - 1) - 1.$$

Having in mind that $\chi_\rho(K_n) = n$ and $\operatorname{diam}(G\Box K_n) = \operatorname{diam}(G) + 1$, Theorem 1 yields:

Corollary 2. *Let $n \geqslant 2$. Then for any graph G,*

$$\chi_\rho(G\Box K_n) \geqslant n\chi_\rho(G) - (n - 1)\operatorname{diam}(G).$$

For $n = 2$ Corollary 2 reduces to [5, Proposition 7.1].

We next consider upper bounds for $\chi_\rho(G\Box H)$. As the study of grids already indicates, the problem does not reveal some particular advantage of the product structure. Hence, in general we could only apply the following obvious (see also [5]) bound:

$$\chi_\rho(G\Box H) \leqslant |G||H| - \alpha(G\Box H) + 1. \tag{2}$$

Since the computation of the independence number of Cartesian products is difficult, see [1,6,9,10], (2) becomes more useful after applying some lower bound for $\alpha(G\Box H)$. For instance, substituting the following old result of Vizing [15]:

$$\alpha(G\Box H) \geqslant \alpha(G)\alpha(H) + \min\{|G| - \alpha(G), |H| - \alpha(H)\}$$

into (2) we get

$$\chi_\rho(G\Box H) \leqslant |G||H| - \alpha(G)\alpha(H) - \min\{|G| - \alpha(G), |H| - \alpha(H)\} + 1. \tag{3}$$

Inequality (3) for $2 \leqslant m \leqslant n$ gives $\chi_\rho(K_m\Box K_n) \leqslant m(n-1)+1$. On the other hand, Theorem 1 implies $\chi_\rho(K_m\Box K_n) \geqslant n(m - 1) + 1$ as well as $\chi_\rho(K_m\Box K_n) \geqslant m(n - 1) + 1$. We conclude that $\chi_\rho(K_m\Box K_n) = m(n - 1) + 1$.

For another application of (2) recall from [6] that for any graph G and any bipartite graph H,

$$\alpha(G\Box H)\geqslant\alpha_2(G)|H|/2, \tag{4}$$

where $\alpha_2(G)$ denotes the 2-independence number of G (that is, the size of the largest union of two independent sets in $V(G)$). Therefore, using (4) in (2) we obtain the bound

$$\chi_\rho(G\Box H)\leqslant|G||H|-\alpha_2(G)|H|/2+1$$

that holds for any graph G and any bipartite graph H.

We conclude the section with the following question. If the answer to it is positive, then the product structure plays at least some role in the problem of determining the packing chromatic number of Cartesian product graphs.

Problem 3. *Let G and H be graphs. Is it then true that*

$$\chi_\rho(G\Box H)\leqslant\max\{\chi_\rho(G)|H|,\chi_\rho(H)|G|\}?$$

Note that if one of the factor graphs is complete, the answer is positive. Indeed, suppose G is complete, then $\chi_\rho(G)=|G|$ and hence $|H|\cdot\chi_\rho(G)=|H||G|$.

3. Hexagonal lattice

The hexagonal lattice plays a central role in many network applications, as, for instance, in frequency assignments [4,12,13]. The packing chromatic number of the infinite square grid is bounded by 22, see [5]. As the infinite hexagonal lattice is a spanning subgraph of the square grid, the packing chromatic number of the infinite hexagonal lattice is also finite. But more can be said.

Theorem 4. *Let $\mathscr{H}$ be the infinite hexagonal lattice. Then $6\leqslant\chi_\rho(\mathscr{H})\leqslant 8$.*

Proof. The upper bound follows from the coloring of the infinite hexagonal lattice with 8 colors, whose pattern can be seen in Fig. 1. In this figure it is assumed that uncolored vertices receive color 1, and that in rows below (and above) the pattern starts repeating (hence in the lowest row on the figure, in which colors are not assigned, one uses $\ldots 3\ 8\ 2\ 3\ 7\ 2\ldots$ again).

We begin the proof of the lower bound by assuming that $\chi_\rho(\mathscr{H})=5$ (from the arguments used in the proof it will be also clear that $\chi_\rho(\mathscr{H})$ cannot be less than 5). Clearly, there must be a vertex to which the color 1 is assigned. Let u_1 be a vertex in $\mathscr{H}$ with $c(u_1)=1$. Note that all neighbors of u_1 have pairwise different colors, and let v be the neighbor of u_1 with the largest color. Hence $c(v)\in\{4,5\}$. Next, let u_6 be the neighbor of u_1 with the smallest color and let u_2 be the remaining neighbor of u_1. See Fig. 2 in which some other vertices of $\mathscr{H}$ close to u_1 are also denoted. Observe that $c(u_6)\in\{2,3\}$ and $c(u_2)\in\{3,4\}$. Let α be the unique color from $\{2,3,4,5\}\setminus\{c(v),c(u_2),c(u_6)\}$. Note that $\{c(u_5),c(u_4)\}=\{1,\alpha\}$. We distinguish two cases.

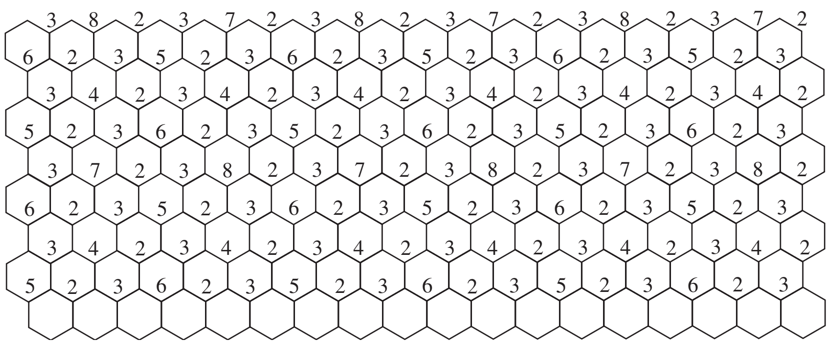

Fig. 1. The 8-coloring of the hexagonal lattice.

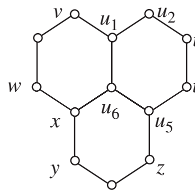

Fig. 2. A subgraph of the hexagonal lattice from the proof.

Case 1: $c(u_5)=\alpha$. This forces $c(u_4)=1$, which implies $c(u_3)=2=c(u_6)$, which in turn implies that $c(x)=1$. Since $d(u_5, y)=3$, and $\alpha \geqslant 3$, the only choice is that $c(u_2)=3$, so that there is a free color available for y. Hence $c(y)=3$. But now w cannot be colored unless a color greater than 5 is used.

Case 2: $c(u_5)=1$. This forces $c(u_4)=\alpha$. Now vertex z cannot be colored unless $c(u_2)=3$, in which case $c(z)=3$, and which also implies that $c(u_6)=2$, which in turn implies $\alpha \geqslant 4$. Since at least one of the vertices x, y must use a color distinct from 1, this vertex cannot be colored, unless a color greater than 5 is used.

We derive $\chi_\rho(\mathscr{H}) \geqslant 6$. $\square$

By a similar, but much more tedious analysis as in the above proof one can establish that $\chi_\rho(\mathscr{H}) \geqslant 7$.

A similar question can be posed for the infinite triangular lattice. In fact we conjecture that the packing chromatic number is not finite in this case.

Problem 5. *Let $\mathscr{T}$ be the infinite triangular lattice. Is it true that for any integer k*

$$\chi_\rho(\mathscr{T}) > k?$$

4. Subdivisions

The *subdivision graph* $S(G)$ of a graph G is obtained from G by subdividing every edge of G. Then $V(S(G)) = V(G) \cup E(G)$. It was observed in [5, Proposition 3.2] that for a connected bipartite graph G with at least two edges, $\chi_\rho(S(G)) = 3$. In this section we consider the subdivision graph of an arbitrary graph. To deal with the general case, complete graphs are treated first.

Lemma 6. *For any $n \geqslant 3$, $\chi_\rho(S(K_n)) = n+1$.*

Proof. For $n=3$ we have $S(K_3)=C_6$ and clearly $\chi_\rho(C_6)=4$. Now let $n \geqslant 4$. Let $V(K_n)=\{v_1, \ldots, v_n\}$, and for the edge $v_i v_j$, $1 \leqslant i, j \leqslant n$, let v_{ij} denote the vertex of $S(K_n)$ obtained by subdividing $v_i v_j$. Let $c : V(S(K_n)) \to \{1, 2, \ldots\}$ be a coloring. We claim that there exists $v \in S(K_n)$ such that $c(v) \geqslant n+1$. If $c(v_i) > 1$ for all i, then $c(v_i) \neq c(v_j)$ for $i \neq j$ and hence, for some k, $c(v_k) > n$. So, assume without loss of generality that $c(v_1)=1$. Then it follows that $c(v_{1j}) \neq c(v_{1k})$ for $j \neq k$ which immediately implies the claim unless $\{c(v_{12}), c(v_{13}), \ldots, c(v_{1n})\} = \{2, 3, \ldots, n\}$. Assume without loss of generality that $c(v_{12})=2$. Since $d(v_{1j}, v_2) \leqslant 3$, we infer $c(v_2) \notin \{2, 3, \ldots, n\}$, and so $c(v_2)=1$. Consider now the set $X=\{v_{23}, v_{24}, \ldots, v_{2n}\}$, and note that $|X| \geqslant 2$. Since vertices from X are at distance at most 4 from the vertices v_{1j} we infer that they can only receive color 3, unless a color greater than n is used. But $c(v_{23})=3=c(v_{24})$ is again not possible, by which our claim is finally proved. $\square$

Note that for $n=2$ the above lemma does not hold since $\chi_\rho(S(K_2))=2$.

Theorem 7. *For any connected graph G with at least three vertices*,

$$\omega(G)+1 \leqslant \chi_\rho(S(G)) \leqslant \chi_\rho(G)+1.$$

Moreover, *the bounds are best possible*.

Proof. Let $c : V(G) \to \{1, \dots, k\}$ be a $\chi_\rho(G)$-coloring. Define $\widetilde{c} : V(S(G)) \to \{1, \dots, k+1\}$ by setting $\widetilde{c}(e)=1$ for any edge $e \in E(G)$ and $\widetilde{c}(v)=c(v)+1$ for any vertex $v \in V(G)$. Suppose $\widetilde{c}(x)=\widetilde{c}(y)=\alpha$ for some $x, y \in V(S(G))$. Clearly, if $\alpha=1$ then $d_{S(G)}(x, y)\geqslant 2 > \alpha$. Suppose $\alpha > 1$. Then $x, y \in V(G)$ and since $\widetilde{c}(x)=\widetilde{c}(y)$ we have $c(x)=c(y)$. Therefore, $d_G(x, y)\geqslant\alpha$ and hence $d_{S(G)}(x, y)=2d_G(x, y)\geqslant 2\alpha\geqslant\alpha+1$. Hence $\widetilde{c}$ is a coloring which proves the upper bound. By Lemma 6, $\chi_\rho(S(K_n)) = \chi_\rho(K_n) + 1$, $n\geqslant 3$, which shows that the upper bound is best possible.

For the proof of the lower bound consider first the case when $\omega(G) = 2$. Note that the only connected graphs with packing chromatic number 2 are the stars $K_{1,m}$. Since $S(G)$ can be a star only when $G=K_2$ we derive that $\chi_\rho(S(G)) > 2$ for every connected graph G with more than two vertices.

Finally, let $\omega(G) = n\geqslant 3$. Since for a subgraph H of G it is clear that $\chi_\rho(G)\geqslant\chi_\rho(H)$, we derive by Lemma 6 that $\chi_\rho(S(G))\geqslant\chi_\rho$
$(S(K_n)) = n + 1$. The lower bound is best possible since, by [5], $\chi_\rho(S(G)) = 3$ for any connected bipartite graph on at least three vertices. □

5. Trees and monotone colorings

Goddard et al. in [5] ask what is the computational complexity of the coloring on trees, more precisely, whether the problem is polynomial? In this section we indicate that the answer to this question could be negative. To be more precise, we show that two natural approaches do not work. We first introduce monotone colorings and show that there exist trees in which no optimal coloring is monotone. Another approach would be the following. For a tree T we first find all vertices having at least two leaves as neighbors and color these leaves with color 1. However, we give an infinite family of trees in which two such leaves are colored with at least 2 in any optimal coloring.

Let G be a graph. For a coloring $c : V(G) \to \{1, \dots, k\}$, and a color m, $1\leqslant m\leqslant k$, we denote by c_m the cardinality of the class of vertices, colored by m, that is, $c_m = |\{v \in V(G) : c(v) = m\}|$. Since $\rho_s(G)\leqslant\rho_r(G)$ whenever $s > r$, the following definition is natural. A $\chi_\rho(G)$-coloring is *monotone* if

$$c_1\geqslant c_2\geqslant \cdots \geqslant c_k.$$

At the first sight it seems plausible that any graph admits a monotone coloring. In fact, the following can be proved.

Proposition 8. *For any graph G and any m, where* $m\leqslant\lfloor\chi_\rho(G)/2\rfloor$, *there exists a* $\chi_\rho(G)$-*coloring* $c: V(G) \to \{1, \dots, k\}$ *such that* $c_m\geqslant c_n$ *for all* $n\geqslant 2m$.

Proof. Let G be a graph, m an integer, and c a $\chi_\rho(G)$-coloring. Suppose that for some m, n where $2m\leqslant n$ we have $c_m < c_n$. Let $X=\{v \in V(G) : c(v)=n\}$, and note that for any $x, y \in X$ we have $N_m[x]\cap N_m[y]=\emptyset$. By the pigeonhole principle there exist at least $c_n - c_m$ vertices $x \in X$ such that $N_m[x]$ does not have any vertex colored m. Now, we change the coloring c by replacing the color n with m for all such vertices of X. Clearly, the obtained coloring $\widetilde{c}$ is a $\chi_\rho(G)$-coloring and $\widetilde{c}_m\geqslant\widetilde{c}_n$. By successively applying such a recoloring whenever necessary, for all $n = 2m, \dots, k$, we obtain a desired $\chi_\rho(G)$-coloring. □

Note that Proposition 8 in particular implies that for any graph G there exists an optimal coloring in which color 1 is most frequent, that is, $c_1\geqslant c_i$ for $2\leqslant i\leqslant k$.

On the other hand, a bit surprisingly, there exist trees that admit no monotone optimal coloring. In Fig. 3 a class of trees T_k, $k\geqslant 2$, are presented, that consist of a 3-path on vertices u, v, x, y, each of u and v having two (black) leaves, and there are k paths of length 2 that emerge from y.

Proposition 9. *For any* $k\geqslant 2$, $\chi_\rho(T_k)=3$. *Moreover, there exists a unique optimal coloring of* T_k *with* $c_1=k+5$, $c_2 = 2$, $c_3 = k + 1$. *In particular*, $c_3 > c_2$.

Proof. In Fig. 3 a coloring of T_k is depicted that shows $\chi_\rho(T_k)\leqslant 3$. As obviously 2 colors are not enough, we infer $\chi_\rho(T_k) = 3$.

We now claim that the presented coloring is unique. Let c be a 3-coloring of T_k. Note that none of u, v can be colored 1. Indeed, if u or v is colored 1, its black neighbors must be colored 2 and 3, respectively, which is already not possible since then v or u, respectively, cannot be colored with colors from $\{1, 2, 3\}$. Now, suppose that u is colored 3. This

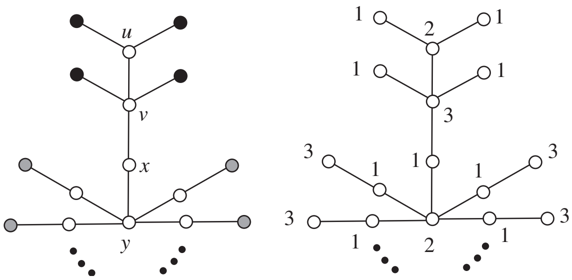

Fig. 3. Tree T_k with no monotone χ_ρ-coloring and the unique χ_ρ-coloring of it.

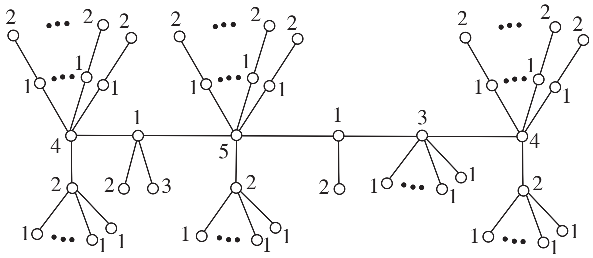

Fig. 4. A 5-coloring of T.

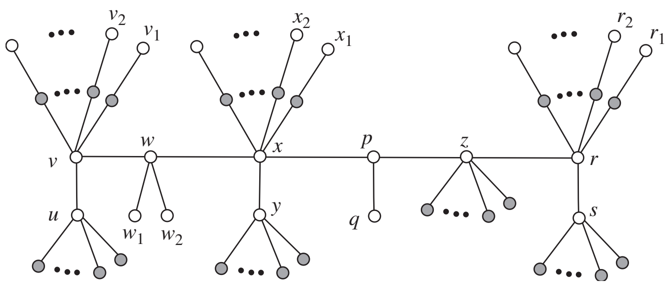

Fig. 5. Tree T from Proposition 10.

readily implies $c(v)=2$, and $c(x)=1$. But now y cannot be colored by any color from $\{1,2,3\}$. Hence, this forces $c(u)=2$, $c(v)=3$, $c(x)=1$, and $c(y)=2$, and also the black vertices can only be colored 1. Note that all neighbors of y must as well be colored 1, and finally this forces the remaining gray leaves to be colored 3. $\square$

In Fig. 4 a (family of) tree(s) T is depicted together with the 5-coloring of vertices. (The dotted lines mean there can be several 2-paths or leaves attached to a particular vertex, though for our purposes four of each, in fact, suffice.) It is not hard to see that this is a $\chi_\rho(T)$-coloring. An interesting property of this coloring is that there is a vertex colored 1 which is adjacent to two leaves, and hence the leaves receive greater colors. A question, whether for any tree there exists a $\chi_\rho(T)$-coloring such that every vertex, adjacent to at least two leaves, is colored by more than 1, has a negative answer.

The tree T is again shown in Fig. 5 and some of its vertices are denoted, in particular the previously mentioned vertex is denoted by w.

Proposition 10. *In any $\chi_\rho(T)$-coloring c of the tree T from Fig. 5, $c(w)=1$.*

Proof. We have already observed that $\chi_\rho(T) \leqslant 5$. Note that if w is colored by color 1 then all its neighbors must be colored differently, and so at least 5 colors must be used in this case. Suppose there is a 5-coloring c of T in which $c(w) \geqslant 2$ (this will eventually lead us to a contradiction). We may assume that gray vertices (see Fig. 5) receive color 1. (In fact, one of the gray vertices that are attached to the same vertex could be colored differently, say by $\alpha > 1$, yet by recoloring it to 1 we only create the possibility for some other vertices to be colored by α, and at the same time not lose anything.) For a similar reason we may assume that the leaves (denoted by indexed letters) adjacent to gray vertices receive either color 2 or 3. In addition, we may assume that all such vertices, having a unique common vertex in their distance 2-neighborhoods, are colored with the same color (for instance, all v_i's either by 2 or 3, etc.) Let P be the 4-path induced by vertices u, v, w, x, y. We distinguish four cases.

Case 1: $c(w) = 2$. Note that on P no other vertex can be colored 2, no vertex can be colored 1 (since they are adjacent to gray vertices that are colored 1), and at most one vertex can be colored 4 or 5, respectively. This implies $c(u) = c(y) = 3$. Since $\{c(v), c(x)\} = \{4, 5\}$ we derive $c(z) = 2$ which in turn implies $c(p) = 1$. But now q cannot be colored unless a new color is used, a contradiction.

Case 2: $c(w) = 3$. This implies $c(v_i) = 2$ for all i since the distance between w and v_i is 3. Hence $c(v) \neq 2$. Since on P we must now color two vertices by 2, this implies $c(u) = 2$. If also $c(x) = 2$, then as $\{c(v), c(y)\} = \{4, 5\}$, one cannot color z. The remaining case $c(y) = 2$ implies $\{c(v), c(x)\} = \{4, 5\}$ which in turn implies $c(p) = 1$, and so $c(z) = 2$. We derive again that q cannot be colored.

Case 3: $c(w) = 4$. Observe that on P one can use either color 2 or 3 twice.

If $c(u) = 2 = c(x)$ then $\{c(v), c(y)\} = \{3, 5\}$, which implies that $c(p) = 1$. Hence $c(q) = 3$, $c(y) = 5$, $c(v) = 3$, and so z cannot be colored.

If $c(u) = 2 = c(y)$ then $\{c(v), c(x)\} = \{3, 5\}$ which again implies that $c(p) = 1$. We easily infer $\{c(z), c(q)\} = \{2, 3\}$ and $c(x) = 5$, $c(v) = 3$. Now r can be colored only 2 in which case $c(q) = 2$, $c(z) = 3$. But now r_i's cannot receive colors from $\{1, 2, 3\}$ which yields a contradiction (namely, at most one r_i could be colored by 4 or 5).

If $c(v) = 2 = c(y)$ then $\{c(u), c(x)\} = \{3, 5\}$. We must color $c(u) = 5$ (to avoid a contradiction arising from v_i's forced to be colored by colors from $\{4, 5\}$), and so $c(x) = 3$. Hence $c(p) = 1$ and $c(q) = 2$ which yields a contradiction because z cannot be colored.

Finally, if $c(u) = 3 = c(y)$, we have $\{c(v), c(x)\} = \{2, 5\}$. But $c(v) \neq 2$ (otherwise v_i's cannot be colored), and also $c(x) \neq 2$ (otherwise x_i's cannot be colored).

Case 4: $c(w) = 5$. We deal with all subcases in the same way as in Case 3 (by reversing the role of colors 4 and 5), except when $c(v) = 2 = c(y)$. In this case $\{c(u), c(x)\} = \{3, 4\}$, and it follows that $c(u) = 4$ so that the v_i's can be colored. Hence, $c(x) = 3$, $c(p) = 1$, and $\{c(q), c(z)\} = \{2, 4\}$. The only way to color r is that $c(z) = 4$ and $c(r) = 2$. It follows that $c(r_i) = 3$. But then s cannot be colored, the final contradiction. □

6. Concluding remarks

In these concluding remarks we report on several developments that have transpired since our paper was submitted.

We have mentioned that by a tedious analysis one can establish that $\chi_\rho(\mathscr{H}) \geqslant 7$. Using a computer, Vesel [14] checked that this is indeed the case. Moreover, Fiala and Lidicky [2] found a coloring of $\mathscr{H}$ that uses only 7 colors, which implies that $\chi_\rho(\mathscr{H}) = 7$.

Problem 5 has also been solved affirmatively in the meantime. Finbow and Rall [3] have proved that the infinite triangular lattice has infinite packing chromatic number. Moreover, they proved that the same holds for the three-dimensional square lattice.

References

[1] B. Brešar, B. Zmazek, On the independence graph of a graph, Discrete Math. 272 (2003) 263–268.

[2] J. Fiala, B. Lidicky, Private communication, 2007.

[3] A. Finbow, D.F. Rall, On the packing chromatic number of some infinite graphs, Manuscript, 2007.

[4] S. Fitzpatrick, J. Janssen, R. Nowakowski, Distributive online channel assignment for hexagonal cellular networks with constraints, Discrete Appl. Math. 143 (2004) 84–91.

[5] W. Goddard, S.M. Hedetniemi, S.T. Hedetniemi, J.M. Harris, D.F. Rall, Broadcast chromatic numbers of graphs, Ars Combin., in press.

[6] J. Hagauer, S. Klavžar, On independence numbers of the Cartesian product of graphs, Ars Combin. 43 (1996) 149–157.

[7] W. Imrich, S. Klavžar, Product Graphs: Structure and Recognition, Wiley-Interscience, New York, 2000.

[8] P.K. Jha, Perfect r-domination in the Kronecker product of three cycles, IEEE Trans. Circuits and Systems I Fund. Theory Appl. 49 (2002) 89–92.
[9] S. Klavžar, Some new bounds and exact results on the independence number of Cartesian product graphs, Ars Combin. 74 (2005) 173–186.
[10] S.P. Martin, J.S. Powell, D.F. Rall, On the independence number of the Cartesian product of caterpillars, Ars Combin. 60 (2001) 73–84.
[11] C. Sloper, An eccentric coloring of trees, Australas. J. Combin. 29 (2004) 309–321.
[12] P. Šparl, J. Žerovnik, 2-Local 5/4-competitive algorithm for multicoloring triangle-free hexagonal graphs, Inform. Process. Lett. 90 (2004) 239–246.
[13] K.S. Sudeep, S. Vishwanathan, A technique for multicoloring triangle-free hexagonal graphs, Discrete Math. 300 (2005) 256–259.
[14] A. Vesel, Private communication, 2007.
[15] V.G. Vizing, Cartesian product of graphs, Vychisl. Sistemy 9 (1963) 30–43.