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A note on eccentricity based topological indices of honeycomb, oxide and 2-power interconnection networks

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Abstract

In a series of papers, Imran et al. reported the total eccentricity index, the average eccentricity index, the eccentricity-based Zagreb indices, the atom-bond connectivity index, and the geometric arithmetic index of honeycomb networks, hypertrees, X-trees, and oxide networks. In this paper, it is demonstrated that these computations contain flaws. Corrected, and in many cases also simplified, formulas are also obtained.

Subject Classification: *05C05, 05C10, 05C12, 05C90, 92E10, 68R10.*

Keywords: *Eccentricity-based topological index, Honeycomb network, Hypertree, X-tree, Oxide network.*

1. Introduction

Chemical graph theory has become very beneficial because of its applications in mathematical chemistry. In theoretical chemistry, molecular structure descriptors (also called topological indices) are used for modeling physico-chemical, pharmacological, toxicological, biological, and other properties of chemical compounds [13]. Among topological descriptors, connectivity indices play an important and prominent role. Applying M-polynomials which were introduced in [6] (see also [3, 7, 19, 20]), Cancan et al. [4] obtained topological indices which help to predict physico-chemical properties of the underlying dendrimers, and help to the study of the properties of the materials of ship building. Chemical-based experiments

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indicate that there is strong relationship between the characteristics of chemical compounds and drugs and their molecular structures. Topological indices calculated for these chemical structures help us to understand the physical features, chemical reactivity and biological activity [29]. Ahmad et al. [1] obtained the degree based topological indices of line graphs of benzene rings embedded in P-type-surface in 2D network which may help to study the properties of benzene rings.

Different eccentricity based indices are being used for the modeling of biological activities of chemical compounds, and proved to provide a high degree of predictability as compared to some other well-known indices in case of anticonvulsant, anti-inflammatory, and diuretic activities. The research of these topological indices is very active, we point to the following selected list of related recent investigations [2, 5, 14, 15, 27, 28, 30, 31, 32, 33]. In this direction, in a series of papers [16, 17, 18], different eccentricity based topological indices were investigated on honeycomb networks, hypertrees, X-trees, and oxide networks. In this paper we demonstrate that these computations contain errors, and present corrected formulas. In many cases the new formulas are also simpler than the original ones, that is, they are given in a closed form.

The paper is organized as follows. In the rest of the introduction, definitions needed in this paper are stated. In Section 2 we consider honeycomb networks and correct results from [17], in Section 3 we correct results from [16] on oxide networks, and in Section 4 we correct results from [18] on two classes of 2-power interconnection networks, such as hypertree networks and X-tree networks.

The distance $d(u,v)$ between two vertices u and v in a connected graph G is the length of a shortest path between them. For any vertex $v \in V(G)$, the eccentricity of v is defined as $ec(v) = \max\{d(v,u) : u \in V(G)\}$. The degree of a vertex v will be denoted by $d(v)$. The eccentricity based topological indices of interest in this paper are defined for a connected graph G as follows, where GA stands for geometric-arithmetic and ABC for atom bond connectivity.

- *Total-eccentricity index* [8] : $\zeta(G) = \sum_{u \in V(G)} ec(u)$
- *Average eccentricity index* [16] : $avec(G) = \frac{1}{n} \sum_{v \in V(G)} ec(v)$
- *First Zagreb eccentricity index* [11] : $M_1^*(G) = \sum_{uv \in E(G)} (ec(u) + ec(v))$
- *Second Zagreb eccentricity index* [11] : $M_1^{**}(G) = \sum_{v \in V(G)} ec(v)^2$

- *Third Zagreb eccentricity index* [11] : $M_2^*(G)=\sum_{uv\in E(G)}(ec(u)\cdot ec(v))$
- *Fourth GA eccentricity index* [10] : $GA_4(G)=\sum_{uv\in E(G)}\frac{2\cdot\sqrt{ec(u)\cdot ec(v)}}{ec(u)+ec(v)}$
- *Fifth multiplicative ABC index* [9] : $ABC_5(G)=\sum_{uv\in E(G)}\sqrt{\frac{ec(u)+ec(v)-2}{ec(u)\cdot ec(v)}}$
- *Eccentric connectivity index* [26] : $\xi(G)=\sum_{v\in V(G)}d(v)\cdot ec(v)$

2. Honeycomb networks

In this section we consider honeycomb networks and correct results from [17].

A unit honeycomb network is a hexagon, denoted by HC_1. The honeycomb network HC_2 is obtained from HC_1 by adding six hexagons around its boundary edges. Inductively, honeycomb network HC_n is obtained from HC_{n-1} by adding a layer of hexagons around the boundary edges of HC_{n-1}, see Fig. 1. The number of vertices and edges of HC_n is $6n^2$ and $9n^2-3n$, respectively [21].

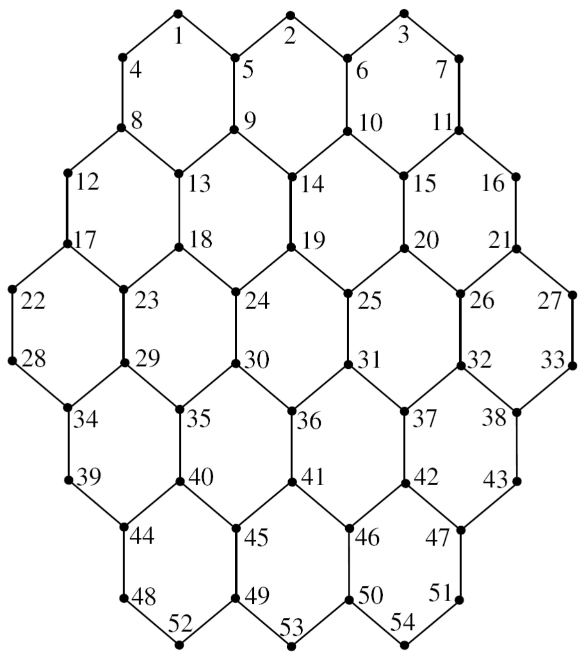

Figure 1

Honeycomb network HC_3

In the following formulas (1)-(7), the results reported in [17, Theorems 2.2.1-2.8.1] are listed.

$$\zeta(HC_n) = 6\sum_{m=1}^{n}\sum_{k=n}^{2n-1} m(2k+1) + 12\sum_{m=1}^{n-1}\sum_{k=n}^{2n-2} m(k+1) \tag{1}$$

$$avec(HC_n) = \frac{1}{n^2}\left\{\sum_{m=1}^{n}\sum_{k=n}^{2n-1} m(2k+1) + 2\sum_{m=1}^{n-1}\sum_{k=n}^{2n-2} m(k+1)\right\} \tag{2}$$

$$M_1^*(HC_n) = 12\sum_{k=n}^{2n-1}(2k+1) + 6\sum_{m=1}^{n-1}\sum_{k=n}^{2n-2} m(12k+13) \tag{3}$$

$$M_1^{**}(HC_n) = 6\sum_{m=1}^{n}\sum_{k=n}^{2n-1} m(2k+1)^2 + 24\sum_{m=1}^{n-1}\sum_{k=n}^{2n-2} m(k+1)^2 \tag{4}$$

$$M_2^*(HC_n) = 6\sum_{k=n}^{2n-1}(2k+1)^2 + 12\sum_{m=1}^{n-1}\sum_{k=n}^{2n-2} m(6k^2+13k+7) \tag{5}$$

$$GA_4(HC_n) = 6 + 12\sum_{m=1}^{n-1}\sum_{k=n}^{2n-2}\left\{m\frac{\sqrt{(2k+1)(2k+2)}}{4k+3} + 2m\frac{\sqrt{(2k+2)(2k+3)}}{4k+5}\right\} \tag{6}$$

$$ABC_5(HC_n) = 12\sum_{k=n}^{2n-1}\frac{\sqrt{k}}{2k+1} + 6\sum_{m=1}^{n-1}\sum_{k=n}^{2n-2}\left\{m\sqrt{\frac{4k+1}{(2k+1)(2k+2)}} + 2m\sqrt{\frac{4k+3}{(2k+2)(2k+3)}}\right\} \tag{7}$$

Consider now the honeycomb network HC_3. It contains 54 vertices and 72 edges, and by a direct calculation we find that:

- $ec(1) = ec(2) = ec(3) = ec(4) = ec(7) = ec(12) = ec(16) = ec(22) = ec(27) = ec(28) = ec(33) = ec(39) = ec(43) = ec(48) = ec(52) = ec(53) = ec(54) = ec(51) = 11$;
- $ec(5) = ec(6) = ec(8) = ec(11) = ec(17) = ec(21) = ec(34) = ec(38) = ec(44) = ec(47) = ec(49) = ec(50) = 10$;
- $ec(9) = ec(10) = ec(13) = ec(15) = ec(23) = ec(26) = ec(29) = ec(32) = ec(40) = ec(42) = ec(45) = ec(46) = 9$;
- $ec(14) = ec(18) = ec(20) = ec(35) = ec(37) = ec(41) = 8$ and
- $ec(19) = ec(24) = ec(25) = ec(30) = ec(31) = ec(36) = 7$.

From these values we obtain:

$$\zeta(HC_3) = (11\times 18)+(10\times 12)+(9\times 12)+(8\times 6)+(7\times 6) = 516 \quad (8)$$

$$avec(HC_3) = \frac{1}{54}((11\times 18)+(10\times 12)+(9\times 12)+(8\times 6)+(7\times 6)) = 9.55 \quad (9)$$

$$M_1^*(HC_3) = 6(7+7)+6(5+5)+6(6+5)+12(6+7) = 366 \quad (10)$$

$$M_1^{**}(HC_3) = 12(7^2)+6(6^2)+6(5^2) = 954 \quad (11)$$

$$M_2^*(HC_3) = 6(7+7)+6(5+5)+6(6+5)+12(6+7) = 366 \quad (12)$$

$$GA_4(HC_3) = 2\frac{\sqrt{(11\times 11)}}{(11+11)}+2\frac{\sqrt{(10\times 11)}}{(10+11)}+2\frac{\sqrt{(10\times 9)}}{(10+9)}+2\frac{\sqrt{(9\times 9)}}{(9+9)} +2\frac{\sqrt{(9\times 8)}}{(9+8)}+2\frac{\sqrt{(8\times 7)}}{(8+7)}+2\frac{\sqrt{(7\times 7)}}{(7+7)} = 71.918 \quad (13)$$

$$ABC_5(HC_3) = \sqrt{\frac{11+11-2}{11\times 11}}+\sqrt{\frac{10+11-2}{10\times 11}}+\sqrt{\frac{9+10-2}{9\times 10}}+\sqrt{\frac{9+9-2}{9\times 9}} +\sqrt{\frac{9+8-2}{9\times 8}}+\sqrt{\frac{8+7-2}{8\times 7}}+\sqrt{\frac{7+7-2}{7\times 7}} = 31.629 \quad (14)$$

On the other hand, (1) ([17, Theorem 2.2.1]) gives $\zeta(HC_3)=1296$, (2) ([17, Theorem 2.3.1]) gives $avec(HC_3)=24$, (3) ([17, Theorem 2.4.1]) gives $M_1^*(HC_3)=2304$, (4) ([17, Theorem 2.5.1]) gives $M_1^{**}(HC_3)=11988$, (5) ([17, Theorem 2.6.1]) gives $M_2^*(HC_3)=10686$, (6) ([17, Theorem 2.7.1]) gives $GA_4(HC_3)=113.8315$, and (7) ([17, Theorem 2.8.1]) gives $ABC_5(HC_3)=55.9638$. Comparing these values with (8)-(14) we see that the formulas stated in (1)-(7) are not correct. In the next theorem we fix the result as follows.

Theorem 2.1 : *Let HC_n be the n-dimensional honeycomb network. Then we have the following.*

(*i*) $\zeta(HC_n) = n(20n^2-3n+1)$;

(*ii*) $avec(HC_n) = \dfrac{20n^2-3n+1}{6n}$;

(*iii*) $M_1^*(HC_n) = 3n(20n^2-11n+3)$;

(*iv*) $M_1^{**}(HC_n) = n(68n^3-20n^2+7n-1)$;

(*v*) $M_2^*(HC_n) = 6n^2(17n^2-13n+5)$;

Table 1

Vertex partition of honeycomb network for *n*-levels based on eccentricity of each vertex with existence of their frequencies.

$\varepsilon(u)$	Frequency	Range of m and n
$2n+2m-1$	$6m$	$1 \le m \le n, n \ge 1$
$2n+2m$	$6m$	$1 \le m \le n-1, n > 1$

Table 2

Corrected edge partition of honeycomb network for *n*-levels based on eccentricity of end vertices with existence of their frequencies.

$(\varepsilon(u), \varepsilon(v))$	Frequency	Range of m and n
$(2n+2m-1, 2n+2m-1)$	6	$1 \le m \le n, n \ge 1$
$(2n+2m-1, 2n+2m)$	$6m$	$1 \le m \le n-1, n > 1$
$(2n+2m, 2n+2m+1)$	$12m$	$1 \le m \le n-1, n > 1$

(*vi*) $$GA_4(HC_n) = 6n + \sum_{m=1}^{n-1} 12m \frac{\sqrt{(2n+2m-1)(2n+2m)}}{4n+4m-1} + \sum_{m=1}^{n-1} 24m \frac{\sqrt{(2n+2m)(2n+2m+1)}}{4n+4m+1};$$

(*vii*) $$ABC_5(HC_n) = \sum_{m=1}^{n} 6 \frac{\sqrt{4n+4m-4}}{2n+2m-1} + \sum_{m=1}^{n-1} 12m \sqrt{\frac{4n+4m-1}{(2n+2m)(2n+2m-1)}} + \sum_{m=1}^{n-1} 24m \sqrt{\frac{4n+4m+1}{(2n+2m)(2n+2m+1)}}.$$

Proof: Based on the vertex and edge partitions with respect to eccentricity as given in Tables 1 and 2, respectively, we can compute as follows.

(*i*) $$\begin{aligned} \zeta(HC_n) &= 6\sum_{m=1}^{n} m(2(n+m-1)) + \sum_{m=1}^{n-1} 6m(2(n+m-1)+2) \\ &= n(20n^2 - 3n + 1); \end{aligned}$$

$$(ii) \quad avec(HC_n) = \frac{1}{6n^2}\left\{\sum_{m=1}^{n} 6m(2n+2m-1) + \sum_{m=1}^{n-1} 6m(2(n+m))\right\}$$
$$= \frac{20n^2-3n+1}{6n};$$

Using the same approach, we prove equations (*iii*), (iv), and (*v*).

$$(vi) \quad GA_4(HC_n) = \sum_{m=1}^{n} 6\times 2\frac{\sqrt{(2(n+m-1)+1)(2(n+m-1)+1)}}{2(n+m-1)+1+2(n+m-1)+1}$$
$$+\sum_{m=1}^{n-1} 12m\frac{\sqrt{(2(n+m-1)+1)(2(n+m-1)+2)}}{2(n+m-1)+1+2(n+m-1)+2}$$
$$+\sum_{m=1}^{n-1} 12m\times 2\frac{\sqrt{(2(n+m-1)+2)(2(n+m-1)+3)}}{2(n+m-1)+2+2(n+m-1)+3}$$
$$= 6n+\sum_{m=1}^{n-1} 12m\frac{\sqrt{(2n+2m-1)(2n+2m)}}{4n+4m-1}$$
$$+\sum_{m=1}^{n-1} 24m\frac{\sqrt{(2n+2m)(2n+2m+1)}}{4n+4m+1};$$

$$(vii) \quad ABC_5(HC_n) = \sum_{m=1}^{n} 6\times\sqrt{\frac{2(n+m-1)+1+2(n+m-1)+1-2}{(2(n+m-1)+1)(2(n+m-1)+1)}}$$
$$+\sum_{m=1}^{n-1} 6m\times 2\sqrt{\frac{2(n+m-1)+1+2(n+m-1)+2}{(2(n+m-1)+1)(2(n+m-1)+2)}}$$
$$+\sum_{m=1}^{n-1} 12m\times 2\sqrt{\frac{2(n+m-1)+2+2(n+m-1)+3}{(2(n+m-1)+2)(2(n+m-1)+3)}}$$
$$= \sum_{m=1}^{n} 6\frac{\sqrt{4n+4m-4}}{2n+2m-1} + \sum_{m=1}^{n-1} 12m\sqrt{\frac{4n+4m-1}{(2n+2m)(2n+2m-1)}}$$
$$+\sum_{m=1}^{n-1} 24m\sqrt{\frac{4n+4m+1}{(2n+2m)(2n+2m+1)}}.$$

□

3. Oxide networks

In this section we consider oxide networks and correct results from [16].

The n-dimensional oxide network is denoted by OX_n and is obtained from the n-dimensional silicate network (cf. [22]) which is in turn a

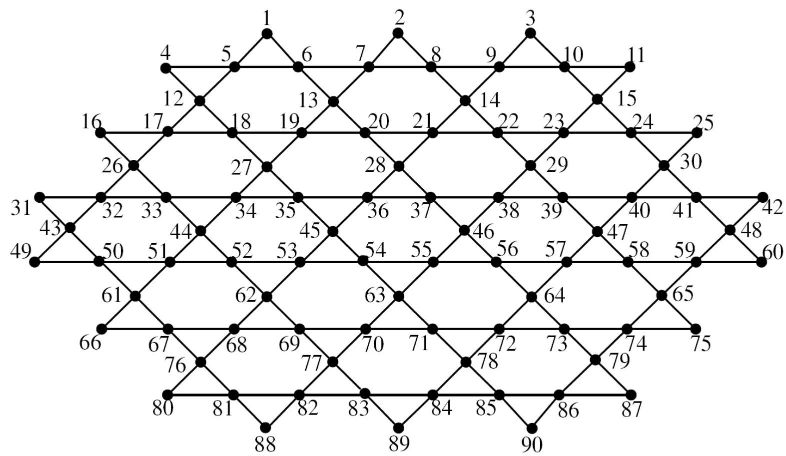

Figure 2

Oxide network OX_3

(silicate) sheet of tetrahedrons. The corner vertices of a tetrahedron are oxygen nodes and the center vertices are silicon nodes. When all the silicon vertices are deleted from a silicate network, an oxide network is constructed. The structure of OX_n should be understand from Fig. 2, where the oxide network OX_3 is shown. The number of vertices in OX_n is $9n^2+3n$ and the number of edges is $18n^2$.

In the following formulas (15)-(21), the results reported in [16, Theorems 1-7] are listed.

$$\zeta(OX_n) = 6\sum_{m=1}^{n}\sum_{k=n}^{2n-1}\{6mk+4m-2k-1\} \quad (15)$$

$$avec(OX_n) = \frac{2}{3n^2+n}\sum_{m=1}^{n}\sum_{k=n}^{2n-1}\{6mk+4m-2k-1\} \quad (16)$$

$$M_1^{*}(OX_n) = 12\sum_{m=1}^{n}\sum_{k=n}^{2n-1}\{8mk+5m-2k-1\}+12\sum_{m=1}^{n-1}\sum_{k=n}^{2n-2}m(4k+5) \quad (17)$$

$$M_1^{**}(OX_n) = 6\sum_{m=1}^{n}\sum_{k=n}^{2n-1}\{2m(6k^2+8k+3)-(2k+1)^2\} \quad (18)$$

$$M_2^{*}(OX_n) = 12\sum_{m=1}^{n}\sum_{k=n}^{2n-1}\{2m(8k^2+8k+3)-(4k^2+2k+1)\} +24\sum_{m=1}^{n-1}\sum_{k=n}^{2n-2}m(2k+3)(k+1) \quad (19)$$

$$GA_4(OX_n) = 12\sum_{m=1}^{n}\sum_{k=n}^{2n-1}\left\{\frac{2m-1}{2}+2m\frac{\sqrt{(2k+1)(2k+2)}}{4k+3}\right\} +24\sum_{m=1}^{n-1}\sum_{k=n}^{2n-2}m\sqrt{\frac{(2k+2)(2k+3)}{4k+5}} \quad (20)$$

$$ABC_5(OX_n) = 12\sum_{m=1}^{n}\sum_{k=n}^{2n-1}\left\{\frac{(2m-1)\sqrt{k}}{2k+1}+m\sqrt{\frac{4k+1}{(2k+1)(2k+2)}}\right\} +12\sum_{m=1}^{n-1}\sum_{k=n}^{2n-2}m\sqrt{\frac{4k+3}{(2k+2)(2k+3)}} \quad (21)$$

Consider now the oxide network OX_3 which contains 90 vertices and 162 edges. In OX_3, we have the following six different values of eccentricities.

- $ec(1) = ec(2) = ec(3) = ec(4) = ec(16) = ec(25) = ec(31) = ec(42) = ec(49) = ec(60) = ec(66) = ec(75) = ec(80) = ec(87) = ec(88) = ec(89) = ec(90) = 12$;
- $ec(5) = ec(6) = ec(7) = ec(8) = ec(9) = ec(10) = ec(11) = ec(12) = ec(15) = ec(17) = ec(24) = ec(26) = ec(30) = ec(32) = ec(41) = ec(43) = ec(48) = ec(50) = ec(59) = ec(61) = ec(65) = ec(67) = ec(74) = ec(76) = ec(79) = ec(81) = ec(82) = ec(83) = ec(84) = ec(85) = ec(86) = 11$;
- $ec(13) = ec(14) = ec(18) = ec(20) = ec(21) = ec(22) = ec(23) = ec(33) = ec(40) = ec(51) = ec(58) = ec(68) = ec(73) = ec(77) = ec(78) = 10$;
- $ec(19) = ec(27) = ec(29) = ec(34) = ec(39) = ec(44) = ec(47) = ec(52) = ec(57) = ec(62) = ec(64) = ec(69) = ec(70) = ec(71) = ec(72) = 9$;
- $ec(28) = ec(35) = ec(38) = ec(53) = ec(56) = ec(63) = 8$ and
- $ec(36) = ec(37) = ec(45) = ec(46) = ec(54) = ec(55) = 7$.

From these values we obtain:

$$\zeta(OX_3) = (7\times 6)+(9\times 18)+(11\times 30)+(8\times 6)+(10\times 12) +(12\times 18) = 918 \quad (22)$$

$$avec(OX_3) = \frac{1}{90}((7\times 6)+(9\times 18)+(11\times 30)+(8\times 6)+(10\times 12) +(12\times 18)) = 10.2 \quad (23)$$

$$GA_4(OX_3) = 36\frac{\sqrt{(12\times 11)}}{(12+11)}+30\frac{\sqrt{(11\times 11)}}{(11+11)}+24\frac{\sqrt{(11\times 10)}}{(11+10)} +24\frac{\sqrt{(10\times 9)}}{(10+9)}+18\frac{\sqrt{(9\times 9)}}{(9+9)}+12\frac{\sqrt{(9\times 8)}}{(9+8)} +12\frac{\sqrt{(8\times 7)}}{(8+7)}+6\frac{\sqrt{(7\times 7)}}{(7+7)} = 83.9258 \quad (24)$$

$$ABC_5(OX_3) = 36\sqrt{\frac{11+12-2}{11\times 12}}+30\sqrt{\frac{11+11-2}{11\times 11}}+24\sqrt{\frac{10+11-2}{10\times 11}} +24\sqrt{\frac{9+10-2}{9\times 10}}+18\sqrt{\frac{9+9-2}{9\times 9}}+12\sqrt{\frac{9+8-2}{9\times 8}} +12\sqrt{\frac{8+7-2}{8\times 7}}+6\sqrt{\frac{7+7-2}{7\times 7}} = 69.192 \quad (25)$$

On the other hand, (15) ([16, Theorem 1]) gives $\zeta(OX_3)=2538$, (16) ([16, Theorem 2]) gives $avec(OX_3)=169.2$, (20) ([16, Theorem 6]) gives $GA_4(OX_3)=690.6036$, and (21) ([16, Theorem 7]) gives $ABC_5(OX_3)=190.1088$. Comparing these values with (22), (23), (24), and (25) we see that the formulas stated in (15), (16), (20), and (21) are not correct. In a similar way we see that the formulas (17), (18), and (19) are not correct. In the next theorem we fix the formulas.

Theorem 3.1 : *Let OX_n be the n-dimensional oxide network. Then, we have the following.*

(*i*) $\zeta(OX_n)=6n^2(5n+2)$;

(*ii*) $avec(OX_n)=\dfrac{2n(5n+2)}{3n+1}$;

(*iii*) $M_1^*(OX_n)=120n^3$;

(*iv*) $M_1^{**}(OX_n)=6n^3(17n+8)$;

(*v*) $M_2^*(OX_n)=6n^2(34n^2-1)$;

Table 3

Vertex partitions of oxide network for n-levels based on eccentricity of each vertex with existence of their frequencies.

$\varepsilon(u)$	Frequency	Range of m and n
$2n+2m-1$	$6(2m-1)$	$1 \le m \le n, n \ge 1$
$2n+2m$	$6m$	$1 \le m \le n, n \ge 1$

Table 4

Edge partitions of oxide network for n-levels based on eccentricity of end vertices with existence of their frequencies.

$(\varepsilon(u), \varepsilon(v))$	Frequency	Range of m and n
$(2n+2m-1, 2n+2m-1)$	$6(2m-1)$	$1 \le m \le n, n \ge 1$
$(2n+2m-1, 2n+2m)$	$12m$	$1 \le m \le n, n \ge 1$
$(2n+2m, 2n+2m+1)$	$12m$	$1 \le m \le n-1, n > 1$

$$(vi) \quad GA_4(OX_n) = 6n^2 + \sum_{m=1}^{n} 24m \frac{\sqrt{(2n+2m-1)(2n+2m)}}{4n+4m-1} + \sum_{m=1}^{n-1} 24m \frac{\sqrt{(2n+2m)(2n+2m+1)}}{4n+4m+1};$$

$$(vii) \quad ABC_5(OX_n) = \sum_{m=1}^{n} 12(2m-1)\frac{\sqrt{n+m-1}}{2n+2m-1} + \sum_{m=1}^{n} 12m\sqrt{\frac{4n+4m-3}{(2n+2m-1)(2n+2m)}} + \sum_{m=1}^{n-1} 12m\sqrt{\frac{4n+4m-1}{(2n+2m)(2n+2m+1)}}.$$

P***roof*** **:** Using the vertex and edge partitioned from Tables 3 and 4, we have the following computations.

$$(i) \qquad \zeta(OX_n) = \sum_{m=1}^{n} 6(2m-1)(2(n+m-1)+1) + \sum_{m=1}^{n} 6m(2(n+m-1)+2)$$

$$= \sum_{m=1}^{n} 6(2m-1)(2n+2m-1) + \sum_{m=1}^{n} 6m(2n+2m)$$
$$= 6n^2(5n+2);$$

$$(ii) \qquad avec(OX_n) = \frac{1}{9n^2+3n}\left\{\sum_{m=1}^{n} 6(2m-1)(2n+2m-1) + \sum_{m=1}^{n} 6m(2n+2m)\right\}$$
$$= \frac{1}{9n^2+3n}\{6n^2(5n+2)\}$$
$$= \frac{2n(5n+2)}{3n+1}$$

Using the same approach, we prove equations (iii), (iv), and (v).

$$(vi) \qquad GA_4(OX_n) = \sum_{m=1}^{n} 6(2m-1)\frac{2\sqrt{(2(n+m-1)+1)(2(n+m-1)+1)}}{4n+4m-2}$$
$$+\sum_{m=1}^{n} 12m\frac{2\sqrt{(2(n+m-1)+1)(2(n+m-1)+2)}}{4n+4m-1}$$
$$+\sum_{m=1}^{n-1} 12m\frac{2\sqrt{(2(n+m-1)+2)(2(n+m-1)+3)}}{4n+4m+1}$$
$$= 6n^2 + \sum_{m=1}^{n} 24m\frac{\sqrt{(2n+2m-1)(2n+2m)}}{4n+4m-1}$$
$$+\sum_{m=1}^{n-1} 24m\frac{\sqrt{(2n+2m)(2n+2m+1)}}{4n+4m+1};$$

$$(vii) \quad ABC_5(OX_n) = \sum_{m=1}^{n} 6(2m-1)\sqrt{\frac{2n+2m-1+2n+2m-1-2}{(2n+2m-1)^2}}$$
$$+\sum_{m=1}^{n} 12m\sqrt{\frac{2n+2m-1+2n+2m-2}{(2n+2m-1)(2n+2m)}}$$
$$+\sum_{m=1}^{n-1} 12m\sqrt{\frac{2n+2m+2n+2m+1-2}{(2n+2m)(2n+2m+1)}}$$
$$= \sum_{m=1}^{n} 12(2m-1)\frac{\sqrt{n+m-1}}{2n+2m-1}$$
$$+\sum_{m=1}^{n} 12m\sqrt{\frac{4n+4m-3}{(2n+2m-1)(2n+2m)}}$$
$$+\sum_{m=1}^{n-1} 12m\sqrt{\frac{4n+4m-1}{(2n+2m)(2n+2m+1)}}. \qquad \square$$

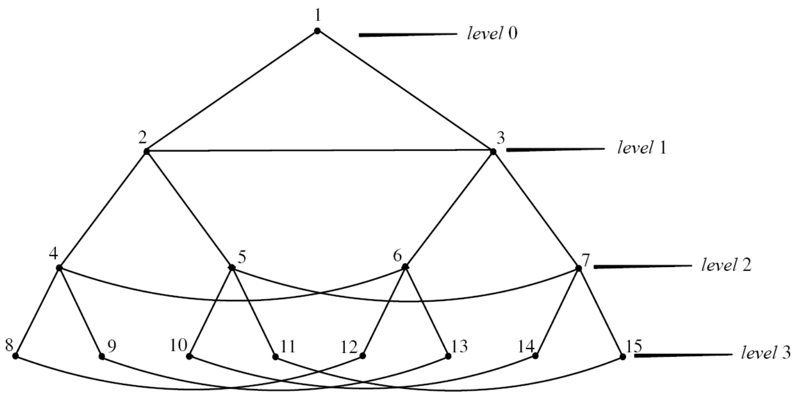

Figure 3

Hypertree network $HT(3)$

4. 2-power interconnection networks

In this section we consider 2-power interconnection networks and correct results from [18]. In the first subsection we deal with hypertree networks, and in the second subsection with X-tree networks.

4.1 *Hypertree networks*

A hypertree $HT(k)$ [12] is constructed as follows. Start with a complete binary tree T_k (a binary tree in which level i, $0 \le i \le k$, contains 2^i vertices). Label the vertices of T_k as follows. The root has label 1, and the children of a vertex x are labeled by $2x$ and $2x+1$. The hypertree $HT(k)$ is then obtained by adding edges in each level i, $1 \le i \le k$, between vertices whose label difference is 2^{i-1}. See Fig. 3 for $HT(3)$.

In the following formulas (26)-(30), the results reported in [18, Theorems 1-5] are listed.

$$\xi(HT(k)) = 2k + 4\sum_{i=1}^{k-1}\sum_{p=k}^{2k-2} 2^i p + 2^{k+1}(2k-1) \qquad (26)$$

$$\zeta(HT(k)) = k + \sum_{i=1}^{k-1}\sum_{p=k}^{2k-2} 2^i p + 2^k(2k-1) \qquad (27)$$

$$M_1^*(HT(k)) = 6k + \sum_{i=1}^{k-1}\sum_{p=k}^{2k-2}(2p+1)2^{i+1} + \sum_{i=1}^{k-1}\sum_{p=k}^{2k-2}(2p+2)2^i \qquad (28)$$

$$M_1^{**}(HT(k)) = k^2 + \sum_{i=1}^{k-1} \sum_{p=k}^{2k-2} 2^i p^2 + 2^k (2k-1)^2 \quad (29)$$

$$M_2^{*}(HT(k)) = 3k^2 + \sum_{i=1}^{k-1} \sum_{p=k}^{2k-2} p(p+1)2^{i+1} + \sum_{i=k} \sum_{p=k} (p+1)^2 2^i \quad (30)$$

We note that the above results (28) and (30) have been already published in [9], and that the same results have also been investigated in [23].

We next show that (26), (27), and (29) do not hold. For this sake consider the hypertree network $HT(3)$ which has 15 vertices, 21 edges, and three different vertex eccentricities:

- $ec(1) = ec(2) = ec(3) = 3$;
- $ec(4) = ec(5) = ec(6) = ec(7) = 4$ and
- $ec(8) = ec(9) = ec(10) = ec(11) = ec(12) = ec(13) = ec(14) = ec(15) = 5$.

From here we get:

$$\xi(HT(3)) = 2(3) + (4 \times 3)2 + (4 \times 4)4 + (2 \times 5)8 = 174 \quad (31)$$

$$\zeta(HT(3)) = 3 + (3)2 + (4)4 + (5)8 = 65 \quad (32)$$

$$M_1^{**}(HT(3)) = 3^2 + (3)^2 2 + (4)^2 4 + (5)^2 8 = 291 \quad (33)$$

On the other hand, (26) ([18, Theorem 1]) gives $\xi(HT(3)) = 254$, (27) ([18, Theorem 2]) gives $\zeta(HT(3)) = 85$, and (29) ([18, Theorem 4]) gives $M_1^{**}(HT(3)) = 354$. Hence formulas (26), (27), and (29) are not correct. We correct them as follows.

Theorem 4.1 : *Let $HT(k)$ be the k-dimensional hypertree network. Then we have the following.*

(i) $\xi(HT(k)) = 12k \times 2^k - 6k - 14 \times 2^k + 16$;

(ii) $\zeta(HT(k)) = 4k \times 2^k - k - 4 \times 2^k + 4$;

(iii) $M_1^{**}(HT(k)) = 8k + 2^k \times (4k^2 - 12k + 11) + 2^k \times (2k-1)^2 - k^2 - 12.$

Table 5

Vertex partitions of hypertree for k-levels based on eccentricity of each vertex with existence of their frequencies.

$d(v)$	$\varepsilon(u)$	Frequency	Range of i
2	k	2^i	$i=0$
4	$k+i-1$	2^i	$1\le i\le k-1$
2	$2k-1$	2^i	$i=k$

Table 6

Edge partitions of hypertree for k-levels based on eccentricity of end vertices with existence of their frequencies.

$(\varepsilon(u),\varepsilon(v))$	Frequency	Range of i
(k,k)	3×2^i	$i=0$
$(k+i-1,k+i)$	2^{i+1}	$1\le i\le k-1$
$(k+i,k+i)$	2^i	$1\le i\le k-1$

Proof : Using the vertex and edge partitioned from Tables 5 and 6, we have the following computations.

$$(i)\quad \xi(HT(k)) = 2k+4\sum_{i=1}^{k-1}2^i(k+i-1)+2\sum_{i=k}(2k-1)2^i$$
$$= 12k\times 2^k-6k-14\times 2^k+16;$$

$$(ii)\quad \zeta(HT(k)) = k+\sum_{i=1}^{k-1}2^i(k+i-1)+\sum_{i=k}(2k-1)2^i$$
$$= 4k\times 2^k-k-4\times 2^k+4;$$

$$(iii)\quad M_1^{**}(HT(k)) = k^2+\sum_{i=1}^{k-1}2^i(k+i-1)^2+\sum_{i=k}2^i(2k-1)^2$$
$$= 8k+2^k\times(4k^2-12k+11)+2^k\times(2k-1)^2-k^2-12.$$

□

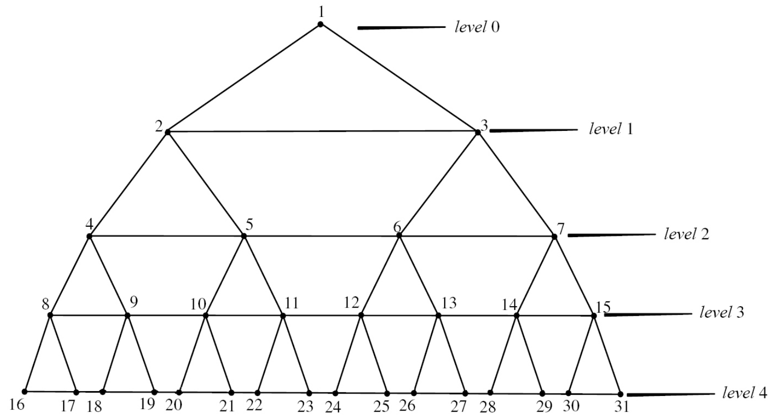

Figure 4
***X*-tree network *XT*(4)**

4.2 *X-tree networks*

The *X*-tree $XT(k)$ [18] is obtained from the complete binary tree T_k by joining the vertices in each level from left to right. The construction should be clear from Fig. 4, where $XT(4)$ is drawn.

In the following formulas (34)-(38), the results reported in [16, Theorems 6-10] are listed.

$$\begin{aligned}\xi(XT(k)) &= 2k+8\sum_{p=k}^{2k-2}p+4(2k-1)+5\sum_{p=k}^{2k-3}\sum_{i=1}^{k-2}2^i p+2^{k-1}(2k-2)3\\ &\cdot 5\sum_{p=k+2,k>3}^{2k-2}\sum_{i=1,k>3}^{k-3}(2(2^i-1)p)+3(2k-1)(2(2^{k-2}-1)) \qquad (34)\end{aligned}$$

$$\begin{aligned}\zeta(XT(k)) &= k+2\sum_{p=k}^{2k-2}p+2(2k-1)+\sum_{p=k}^{2k-3}\sum_{i=1}^{k-2}2^i p\\ &+\sum_{p=k+2,k>3}^{2k-2}\sum_{i=1,k>3}^{k-3}(2(2^i-1)p)+(2k-1)(2(2^{k-2}-1))+(2k-2)2^{k-1}\end{aligned} \qquad (35)$$

$$\begin{aligned}M_1^*(XT(k)) &= 12k+\sum_{i=1}^{k-2}\sum_{p=k+1}^{2k-2}(2p)(3(2^i-1))+(4k-3)(2^{k-1}+2)\\ &+\sum_{i=1}^{k-2}\sum_{p=k}^{2k-3}(2p+1)(2(3(2^{i-1})+1))+(4k-2)(2(2^{k-2}-1)) \qquad (36)\end{aligned}$$

$$M_1^{**}(XT(k)) = k^2 + 2\sum_{p=k}^{2k-2} p^2 + 2(2k-1)^2 + \sum_{p=k}^{2k-3}\sum_{i=1}^{k-2} 2^i p^2 + \sum_{p=k+2,k>3}^{2k-2}\sum_{i=1}^{k-3}(2(2^i-1)p^2) + (2k-1)^2(2(2^{k-2}-1)) + (2k-2)^2 2^{k-1} \quad (37)$$

$$M_2^{*}(XT(k)) = 6k^2 + \sum_{p=k+1}^{2k-2}\sum_{i=1}^{k-2} 3(2^i-1)p^2 + \sum_{p=k}^{2k-3}\sum_{i=1}^{k-2} p(p+1)(2(3(2^{i-1})+1)) + (2k-1)(2k-2)(2^{k-1}+2) + (2k-1)^2(2(2^{k-2}-1)). \quad (38)$$

The results (36) and (38) have been already published in [9] and have been investigated in [24]. We now show that formulas (34), (35), and (37) are not correct. For this sake consider the X-tree network $XT(4)$ from Fig. 4 which has 31 vertices, 56 edges, and the following eccentricities of its vertices:

- $ec(1) = ec(2) = ec(3) = ec(5) = ec(6) = 4$;
- $ec(4) = ec(7) = ec(10) = ec(11) = ec(12) = ec(13) = 5$;
- $ec(8) = ec(9) = ec(14) = ec(15) = ec(20) = ec(21) = ec(22) = ec(23) = ec(24) = ec(25) = ec(26) = ec(27) = 6$ and
- $ec(16) = ec(17) = ec(18) = ec(19) = ec(28) = ec(29) = ec(30) = ec(31) = 7$.

From these values we derive:

$$\xi(XT(4)) = 2(4) + 4(4)(2) + 5(4)(2) + 4(5)(2) + 4(6)(2) + 5(6)(2) + 5(5)(4) + 2(7)(2) + 3(7)(6) + 8(6)(3) = 626 \quad (39)$$

$$\zeta(XT(4)) = 5(4) + 6(5) + 12(6) + 8(7) = 178 \quad (40)$$

$$M_1^{**}(XT(4)) = 5(4)^2 + 6(5)^2 + 12(6)^2 + 8(7)^2 = 1054 \quad (41)$$

On the other hand, (34) ([18, Theorem 6]) gives $\xi(XT(4)) = 2280$, (35) ([18, Theorem 7]) gives $\zeta(XT(4)) = 204$, and (37) ([18, Theorem 9]) gives $M_1^{**}(XT(4)) = 1168$. We conclude that formulas (34), (35), and (37) and not correct. Corrected results read as follows.

Table 7

Vertices partition of an *X*-tree (*k*-level) based on degree and eccentricity of each vertex with the existence of its frequencies.

$d(v)$	$\varepsilon(u)$	frequency	Range of i
2	k	2^i	$i=0$
4	$k+i-1$	2	$1\le i\le k-1$
2	$2k-1$	2^i	$i=1$
5	$k+i-1$	2^i	$1\le i\le k-2$
5	$k+i+1$	$2(2^i-1)$	$1\le i\le k-3, k>3$
3	$2k-1$	$2(2^i-1)$	$i=k-2$
3	$2k-2$	2^i	$i=k-1$

Theorem 4.2 : *Let* $XT(k)$ *be the k-dimensional X-tree network. Then, we have the following.*

(*i*) $\xi(XT(k)) = 16k\times 2^k - 7k - 22\times 2^k - 3k^2 + 30;$

(*ii*) $\zeta(XT(k)) = 4k\times 2^k - k - 5\times 2^k + 6;$

(*iii*) $M_1^{**}(XT(k)) = 12k - 20k\times 2^k + 17\times 2^k - k^2 + 8\times 2^k \times k^2 - 18.$

Proof : Using the vertex partition from Table 7, we have the following computations.

$$(i)\ \ \xi(XT(k)) = 2\sum_{i=0}2^i k + 4\sum_{i=1}^{k-1}2p + 2\sum_{i=1}2^i(2k-1) + 5\sum_{i=1}^{k-3}2(2^i-1)(k+i+1)$$
$$+3\sum_{i=k-2}(2k-1)2(2^i-1) + 3\sum_{i=k-1}(2k-2)2^i + 5\sum_{i=1}^{k-2}2^i(k+i-1)$$
$$= 16k\times 2^k - 7k - 22\times 2^k - 3k^2 + 30;$$

$$(ii)\ \zeta(XT(k)) = k + \sum_{i=1}^{k-1}2p + \sum_{i=1}(2k-1)2^i + \sum_{i=1}^{k-2}2^i(k+i-1)$$
$$+\sum_{i=1}^{k-3}2(2^i-1)(k+i+1) + \sum_{i=k-2}2(2k-1)(2^i-1) + \sum_{i=k-1}(2k-2)2^i$$
$$= 4k\times 2^k - k - 5\times 2^k + 6;$$

$$(iii)\quad M_1^{**}(XT(k)) = k^2+\sum_{i=1}^{k-1}2p^2+\sum_{i=1}2^i(2k-1)^2+\sum_{i=1}^{k-2}2^i(k+i-1)^2$$
$$+\sum_{i=1}^{k-3}2(2^i-1)(k+i+1)^2+\sum_{i=k-2}2(2^i-1)(2k-1)^2$$
$$+\sum_{i=k-1}2^i(2k-2)^2$$
$$= 12k-20k\times 2^k+17\times 2^k-k^2+8\times 2^k\times k^2-18.$$

□

Conflict of interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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