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# Quasi-median graphs, their generalizations, and tree-like equalities

Boštjan Brešar[a,*], Sandi Klavžar[b], Riste Škrekovski[c]

[a] *FERI, University of Maribor, Smetanova 17, 2000 Maribor, Slovenia*
[b] *Department of Mathematics, PeF, University of Maribor, Koroška cesta 160, 2000 Maribor, Slovenia*
[c] *Department of Mathematics, University of Ljubljana, Jadranska 19, 1111 Ljubljana, Slovenia*

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**Abstract**

Three characterizations of quasi-median graphs are proved, for instance, they are partial Hamming graphs without convex house and convex $Q_3^-$ such that certain relations on their edge sets coincide. Expansion procedures, weakly 2-convexity, and several relations on the edge set of a graph are essential for these results. Quasi-semimedian graphs are characterized which yields an additional characterization of quasi-median graphs. Two equalities for quasi-median graphs are proved. One of them asserts that if $\alpha_i$, $i \geq 0$, denotes the number of induced Hamming subgraphs of a quasi-median graph, then $\sum_{i\geq 0} (-1)^i \alpha_i = 1$. Finally, an Euler-type formula is derived for graphs that can be obtained by a sequence of connected expansions from $K_1$.


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*Keywords:* Quasi-median graph; Partial Hamming graph; Quasi-semimedian graph; Expansion procedure; Tree-like equality; Euler-type formula

## 1. Introduction

Median and quasi-median graphs are well studied classes of graphs, cf. [1, 7, 12, 17, 19–21, 25, 32]. Quasi-median graphs have been introduced by Mulder [25] as a natural nonbipartite extension of median graphs. Chung et al. [12] and independently Wilkeit [32] proved that they are the weak retracts of Hamming graphs. On the other hand, Hamming graphs are the regular quasi-median graphs [25]. Chastand [6] extended the

* Corresponding author.
*E-mail addresses:* bostjan.bresar@uni-mb.si (B. Brešar), sandi.klavzar@uni-mb.si (S. Klavžar), skreko@fmf.uni-lj.si (R. Škrekovski).

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above retraction result to infinite graphs. In [1] a survey of characterizations of quasi-median graphs is given including some new ones.

Quasi-median graphs have an interesting application in location theory. Namely, they are precisely the graphs for which a certain dynamic location problem provides a finite solution, see [11, 12] or [19] for more details. From the algorithmic point of view it is an easy observation that quasi-median graphs can be recognized in polynomial time. Feder's general approach of [14] yields an $O(mn)$ algorithm, where $m$ is the number of edges and $n$ the number of vertices of a given graph. The fastest known recognition algorithm is due to Hagauer [17] and is of complexity $O(M(m, n) + m \log n)$, where $M(m, n)$ denotes the complexity of recognizing median graphs. (Currently $M(m, n) = O(n^{1.41}(\log n)^{2.82})$, see [19].)

Partial cubes, that is, isometric subgraphs of hypercubes, were first investigated by Graham and Pollak [15], see also [10, 13, 33]. Nonbipartite extensions of this class are isometric subgraphs of Hamming graphs, called partial Hamming graphs, see [8, 16, 31]. Since (weak) retracts are isometric subgraphs, quasi-median graphs are partial Hamming graphs. In addition, quasi-median graphs are also quasi-semimedian graphs, the class of graphs that forms a nonbipartite extension of semimedian graphs introduced in [18].

In this paper we consider the quasi-median graphs and their generalizations: weakly modular graphs, partial Hamming graphs, quasi-semimedian graphs, and graphs that can be obtained from $K_1$ by connected expansions.

In the next section we introduce necessary concepts and recall some known results. We follow with a section in which quasi-median graphs are introduced and relevant characterizations are given. Quasi-semimedian graphs are also presented and a result of [18] is extended from semimedian to quasi-semimedian graphs. We continue with a section containing three characterizations of quasi-median graphs. We show that quasi-median graphs are precisely partial Hamming graphs which include no convex house or $Q_3^-$, and for which certain relations on their edge sets coincide. We also prove that quasi-median graphs can be described as quasi-semimedian graphs which contain no convex house or $Q_3^-$. In Section 5 we study quasi-semimedian graphs, in particular we give their characterization. This in turn enables us to obtain another characterization of quasi-median graphs. In the last section we first prove that for a quasi-median graph $G$ the following holds:

$$\sum_{i \geq 0} (-1)^i \alpha_i = 1 \qquad \text{and} \qquad -t = \sum_{i \geq 0} (-1)^i i \alpha_i .$$

Here $t$ is a dimension of $G$ and $\alpha_i$ the number of induced Hamming subgraphs of $G$ of degree $i$. These results generalize such equalities for median graphs [27]. We conclude the paper by proving that for a graph $G$ that can be obtained by a sequence of connected expansions from $K_1$, $2n - m - k \leq 2$ holds, where we have equality if and only if $G$ is $C_t \,\square\, K_2$-free ($t \geq 3$) and $K_4$-free. This result extends all such previously known Euler-type formulae.

## 2. Preliminaries

The *interval* $I(u, v)$ between vertices $u$, $v$ of a connected graph $G$ is the set of vertices of all shortest paths between $u$ and $v$ in $G$. A graph $G$ is a *median graph* if $|I(u, v) \cap I(v, w) \cap I(w, u)| = 1$ for all triples of vertices $u, v, w$ of $G$.

A graph $G$ satisfies the *triangle property* if for any vertices $u, x, y \in V(G)$ where $d(u, x) = d(u, y) = k \geq 2$ such that $xy \in E(G)$, there exists a common neighbour $v$ of $x$ and $y$ with $d(u, v) = k - 1$. A graph $G$ satisfies the *quadrangle property* if for any $u, x, y, z \in V(G)$ such that $d(u, x) = d(u, y) = d(u, z) - 1$ and $d(x, y) = 2$ with $z$ a common neighbour of $x$ and $y$, there exists a common neighbour $v$ of $x$ and $y$ such that $d(u, v) = d(u, x) - 1$. A graph which fulfils the quadrangle property and the triangle property is called *weakly modular*.

A subgraph $H$ of a graph $G$ is called *isometric* if $d_H(u, v) = d_G(u, v)$ for all $u$, $v \in V(H)$, where $d_G(u, v)$ denotes the length of a shortest path in $G$ from $u$ to $v$. A connected subgraph $H$ of $G$ is called *convex* if for every two vertices from $H$ all shortest paths are contained in $H$. It is easy to see that the intersection of two convex subgraphs is also convex. A *convex closure* of a subgraph $H$ of $G$ is defined as the smallest convex subgraph of $G$ which contains $H$. A subgraph $H$ of a graph $G$ is called *gated* in $G$ if for every $x \in V(G)$ there exists a vertex $u$ in $H$ such that $u \in I(x, v)$ for all $v \in V(H)$. Note that if for some $x$ such a vertex $u$ in $V(H)$ exists, it must be unique.

An induced connected subgraph $H$ of a graph $G$ is 2-*convex* if for any two vertices $u$ and $v$ of $H$ with $d_G(u, v) = 2$, every common neighbour of $u$ and $v$ belongs to $H$. We call an induced subgraph $H$ of a graph $G$ *weakly* 2-*convex* if for any two vertices $u, v \in V(H)$ with $d_H(u, v) = 2$, every common neighbour of $u$ and $v$ belongs to $H$. The path on five vertices is a weakly 2-convex but not 2-convex subgraph of $C_6$. Chepoi [9] and Bandelt and Chepoi [2, Lemma 1] observed that a connected subgraph of a weakly modular graph is weakly 2-convex if and only if it is convex. In addition, a convex subgraph is gated if and only if it is triangle-closed, where a subgraph $H$ of a graph $G$ is *triangle-closed* if $H$ contains a triangle as soon it contains one of its edges. For further reference we thus state:

**Lemma 2.1.** *Let $G$ be a weakly modular graph. For an induced subgraph $H$ of $G$ the following assertions are equivalent:*

(i) *$H$ is gated.*
(ii) *$H$ is convex and triangle-closed.*
(iii) *$H$ is connected, triangle-closed, and weakly* 2-*convex.*

The equivalence between (i) and (ii) has also been noticed in [1, Lemma 2]. It is easy to see that an isometric subgraph is weakly 2-convex if and only if it is 2-convex. Therefore, we can also deduce a result of Vesel [30] which claims that subgraphs of pseudo-median graphs are gated precisely when they are 2-convex, triangle-closed, and isometric.

*The Cartesian product* $G = G_1 \square G_2 \square \cdots \square G_k$ of graphs $G_1, G_2, \ldots, G_k$ has vertices $V(G) = V(G_1) \times V(G_2) \times \cdots \times V(G_k)$ and vertices $u = (u_1, \ldots, u_k)$, $v = (v_1, \ldots, v_k)$ of $G$ are adjacent if there exists an index $j$ $(1 \leq j \leq k)$ such that $u_j v_j \in E(G_j)$ and $u_i = v_i$ for all $i \in \{1, 2, \ldots, k\} \setminus \{j\}$. If all the factors in a Cartesian product are complete graphs then $G$ is called a *Hamming graph* and in particular if all $k$ factors are $K_2$

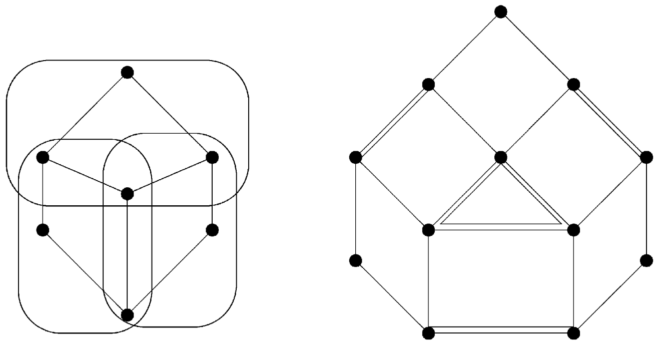

Fig. 1. Connected expansion.

then $G$ is a *hypercube* denoted $Q_k$. Isometric subgraphs of hypercubes are called *partial cubes* and isometric subgraphs of Hamming graphs are *partial Hamming graphs*.

Next we introduce several relations defined on the edge set of a graph $G$ that are essential for our investigations. For an edge $ab$ of a graph $G$ let $W_{ab} = \{x \in V(G) : d(x,a) < d(x,b)\}$. Then Djoković's relation $\sim$ is defined as follows [13]: edges $xy$, $ab \in E(G)$ are in relation $\sim$ if $x \in W_{ab}$ and $y \in W_{ba}$. The relation is reflexive and symmetric but it is in general not transitive, cf. $K_{2,3}$. It is well known that $\sim$ is a transitive relation for partial Hamming graphs (see [31]).

A relation $\approx$ was introduced in [3] (denoted there by $\triangle$) on the edge set of a connected graph as follows. Edges $e$, $f$ are in relation $\approx$, if $e \sim f$ or there exist edges $e'$, $f' \in E(G)$ of the same clique, such that $e \sim e'$ and $f \sim f'$. (Note the meaning of our notation: $\approx$ is used because, roughly speaking, we extend the relation $\sim$ by double applications of it over cliques.) Obviously, $\approx$ is reflexive, symmetric, and $\sim \subseteq \approx$. The relation $\approx$ is transitive for partial Hamming graphs [3]. It is illustrated in Fig. 1, where we infer that the marked edges, obtained in an expansion step, form an equivalence class of this relation.

Edges $e$ and $f$ are in relation $\delta$ if $e = f$ or $e$ and $f$ are opposite edges of an induced square in $G$. (By a square we mean a 4-cycle.) We say that edges $e$ and $f$ are in relation $\kappa$ if $e$ and $f$ belong to a common complete subgraph of $G$.

Finally, a graph obtained from $K_2 \square K_3$ by deletion of a vertex is called a *house*, $Q_3^-$ denotes the 3-cube minus a vertex, $K_4 - e$ is the complete graph on four vertices minus an edge, and $\langle X \rangle$ stands for the subgraph induced by the vertex set $X$.

## 3. Quasi-(semi)median graphs

Recall that for an edge $ab$ of a graph $G$, $W_{ab} = \{x \in V(G) : d(x,a) < d(x,b)\}$. In addition let

$$U_{ab} = \{x \in W_{ab} : x \text{ has a neighbour } y \text{ in } W_{ba}\}.$$

A graph is *quasi-median* if every clique (that is, a maximal complete subgraph) in a graph is gated and for any edge $ab$, $U_{ab}$ is convex. We will need the following characterization of quasi-median graphs due to Chung et al. [12].

**Theorem 3.1** ([12]). *A graph $G$ is quasi-median if and only if $G$ is weakly modular and does not contain $K_4 - e$ or $K_{2,3}$ as an induced subgraph.*

*Semimedian graphs* were introduced in [18] as partial cubes for which every set $U_{ab}$ is connected. A natural nonbipartite extension of semimedian graphs are *quasi-semimedian graphs* introduced as partial Hamming graphs for which every set $U_{ab}$ is connected [3]. Note that in [3] these graphs were called semi-quasi-median since they lie between partial Hamming graphs and quasi-median graphs, just as semimedian graphs lie between partial cubes and median graphs. Clearly, bipartite quasi-semimedian graphs are precisely semimedian graphs which is reflected in their new name—quasi-semimedian graphs.

It was shown in [18] that a bipartite graph is a semimedian graph if and only if $\delta^* = \sim$. This result can be extended to quasi-semimedian graphs as follows.

**Proposition 3.2.** *A graph is quasi-semimedian if and only if it is a partial Hamming graph with $\delta^* = \sim$.*

**Proof.** Let $G$ be quasi-semimedian. Since in partial Hamming graphs $\sim$ is transitive, and we always have $\delta \subseteq \sim$, it follows that $\delta^* \subseteq \sim$. On the other hand, if $ab \sim uv$ for $ab, uv \in E(G)$ then $u \in U_{ab}$, and since $U_{ab}$ is connected there exists a path from $u$ to $a$ which lies entirely in $U_{ab}$. We now easily deduce that $ab\delta^*uv$.

Conversely, let $\delta^* = \sim$ and suppose that $U_{ab}$ is not connected for $ab \in E(G)$. Then there exists an edge $uv$ in relation $\sim$ with $ab$ such that any path in $W_{ab}$ between $u$ and $a$ has at least one vertex in $W_{ab} \setminus U_{ab}$. We claim that then $uv$ is not in relation $\delta^*$ with $ab$. Indeed, if $uv$ were in relation $\delta^*$ with $ab$, then the vertices of one side of edges which are in relation $\delta^*$ with $ab$ would induce a path in $U_{ab}$ between $u$ and $a$. □

Odd cycles are examples of graphs for which $\delta^* = \sim$ holds. Indeed, both relations are trivial. As odd cycles of length at least 5 are not partial Hamming graphs (on the other hand, they can be embedded as induced subgraphs into Hamming graphs), we must assume in the above proposition that $G$ is a partial Hamming graph.

Let us present a class of quasi-semimedian graphs that are not quasi-median. Take the Cartesian product of $k$ paths, and select a set of $k$-cubes such that for any two $k$-cubes their edges are from different $\sim$ equivalence classes. Then to each $k$-cube of this set add all possible edges between its vertices, that is, each $Q_k$ is transformed into $K_{2^k}$. Note that the resulting graph is not quasi-median (unless the product of paths is in some sense trivial), but it is a partial Hamming graph which can be derived from the definitions of both classes (alternatively, one can use an expansion procedure described below to see that they are partial Hamming graphs). By Proposition 3.2 this partial Hamming graph is quasi-semimedian.

The notion of expansion was first introduced by Mulder in [24]; all other notions of expansion were derived from this. For our purposes, we recall the following general expansion, introduced by Chepoi [8] in the following way.

**Definition 3.3.** Let $G$ be a connected graph and let $W_1, W_2, \ldots, W_k$ be subsets of $V(G)$ such that:

(1) $W_i \cap W_j \neq \emptyset$ for all $i, j \in \{1, 2, \ldots, k\}$;
(2) $\cup_{i=1}^{k} W_i = V(G)$;
(3) there are no edges between sets $W_i \backslash W_j$ and $W_j \backslash W_i$ for all $i, j \in \{1, 2, \ldots, k\}$;
(4) subgraphs $\langle W_i \rangle$, $\langle W_i \cup W_j \rangle$ are isometric in $G$ for all $i, j = 1, 2, \ldots, k$.

Then to each vertex $x \in V(G)$ we associate a set $\{i_1, i_2, \ldots, i_t\}$ of all indices $i_j$, where $x \in W_{i_j}$. A graph $G^*$ is called an *expansion of $G$ relative to the sets $W_1, W_2, \ldots, W_k$* if it is obtained from $G$ in the following way:

(5) replace each vertex $x$ of $G$ with a clique with vertices $x_{i_1}, x_{i_2}, \ldots, x_{i_t}$;
(6) if an index $i_s$ belongs to both sets $\{i_1, i_2, \ldots, i_t\}$, $\{i'_1, i'_2, \ldots, i'_l\}$ corresponding to adjacent vertices $x$ and $y$ in $G$ then let $x_{i_s} y_{i_s} \in E(G^*)$.

Moreover, by imposing extra conditions to the above definition, we obtain some special expansions. If $W_i \cap W_j$ induce connected subgraphs, then this is called a *connected expansion*. If, in addition, $W_i \cap W_j = U$ for all $i, j = 1, 2, \ldots, k$ where $\langle U \rangle$ is a gated subgraph in $G$, and all subgraphs $\langle W_i \rangle$ are also gated, then this is called a *gated expansion*. If the number $k$ of subsets involved in the expansion equals 2, then the expansion is called *binary*. An example of a (connected) expansion is given on Fig. 1.

The following theorem collects expansion theorems that are of interest to us. The first result is due to Chepoi [8], the second to Mulder [25], cf. also Bandelt et al. [1], while the last one is given in [18] for the bipartite case and extended in [3] to the general case.

**Theorem 3.4.** *Let $G$ be a graph.*

(i) *$G$ is a partial Hamming graph if and only if it can be obtained from $K_1$ by a sequence of expansions.*
(ii) *$G$ is a quasi-median graph if and only if $G$ can be obtained from $K_1$ by a sequence of gated expansions.*
(iii) *If $G$ is a quasi-semimedian (resp. semimedian) graph then it can be obtained from $K_1$ by a sequence of (resp. binary) connected expansions.*

## 4. Characterizing quasi-median graphs

For a relation $R$, let $R^*$ stand for its transitive closure. We can prove straightforwardly that in quasi-semimedian graphs the relation $\approx$ equals $(\delta \cup \kappa)^*$. Hence, this is also true for quasi-median graphs. The reverse implication need not be true in general. Nevertheless, these relations are important for the main result of this section:

**Theorem 4.1.** *The following assertions are equivalent for a connected graph $G$:*

(i) *$G$ is a quasi-median graph.*
(ii) *$G$ is a partial Hamming graph with $\approx = (\delta \cup \kappa)^*$, and $G$ has neither a $Q_3^-$ nor a house as a convex subgraph.*
(iii) *$G$ is a quasi-semimedian graph, and $G$ has neither a $Q_3^-$ nor a house as a convex subgraph.*

For the proof of this theorem we need a lemma. It states that sets $W_i$ from Definition 3.3 enjoy the so-called *Helly property*. (It is well known that this property holds for gated subsets [29], hence the present lemma is seemingly a stronger variation of this result.)

**Lemma 4.2.** *Let $G$ be a connected graph and let $W_i$, $i = 1, \dots, k$ be subsets of $V(G)$ which satisfy Definition 3.3. Then $\cap_{i=1}^{k} W_i \neq \emptyset$.*

**Proof** (Induction on $k$). The claim is true for $k = 2$. Suppose that the claim holds for $k \geq 2$, and let $W_i$, $i = 1, \dots, k+1$, be the subsets of $V(G')$ that satisfy the conditions in Definition 3.3. Observe that the sets $W_i$ for $i = 1, \dots, k$ satisfy the conditions in Definition 3.3 also in a graph induced by $\cup_{i=1}^{k} W_i$ hence by induction $\cap_{i=1}^{k} W_i$ is nonempty. Set $U = \cap_{i=1}^{k} W_i$. Suppose that $U \cap W_{k+1} = \emptyset$, and let $x \in W_{k+1} \cap [\cup_{i=1}^{k} W_i]$ be a vertex as close to $U$ as possible. Then there exist indices $j, \ell \in \{1, \dots, k\}$ such that $x \in [W_{k+1} \cap W_j] \backslash W_\ell$ and let $y$ be a vertex of $U$ closest to $x$. Since, by definition the subgraph induced by $W_{k+1} \cup W_\ell$ is isometric, it follows by Definition 3.3(3) that there exists a vertex $z \in W_{k+1} \cap W_\ell$ such that $z \in I(x, y)$. Hence, we have $d(x, y) = d(x, z) + d(z, y)$, thus $z$ is closer to $U$ than $x$, moreover $z \in W_{k+1} \cap [\cup_{i=1}^{k} W_i]$. This is a contradiction to the choice of $x$. $\Box$

**Proof** (Of Theorem 4.1). For (i) $\Rightarrow$ (ii) we only need to observe that a graph, having convex $Q_3^-$ or a convex house, cannot be quasi-median.

(ii) $\Rightarrow$ (iii): By Proposition 3.2 it is enough to prove that $\delta^* = \sim$, and we know already that $\delta^* \subseteq \sim$.

Let $ab \sim uv$. Using (ii) and the fact $\sim \subseteq \approx$, it follows that $ab(\delta \cup \kappa)^* uv$. Let $ab = x_0 y_0, x_1 y_1, x_2 y_2, \dots, x_k y_k = uv$ be a sequence of edges such that $x_i y_i (\delta \cup \kappa)\, x_{i+1} y_{i+1}$ for $i = 0, \dots, k-1$. Assume that $ab$ and $uv$ are selected such that consecutive edges of the above sequence are in relation $\kappa$ as few times as possible and, among such sequences, $k$ is as small as possible. Clearly, if $\kappa$ is not involved at all, we are done. Otherwise, by the minimality assumptions, $ab$ and $x_1 y_1$ are in the same clique, $x_2 y_2$ is not in it, and $x_1 y_1$ and $x_2 y_2$ are opposite edges of an induced square.

Assume first that $a$, $b$, $x_1$, and $y_1$ are pairwise different vertices. Then the vertices $a$, $x_1$, $y_1$, $x_2$, $y_2$ as well as $b$, $x_1$, $y_1$, $x_2$, $y_2$ induce houses, and as there is no convex house in $G$, any of these two houses gives a convex $K_3 \Box K_2$. Let $x_1'$, respectively $y_1'$, be the vertices of the convex closure of the two houses. As $G$ contains no $K_4 - e$ it follows that $x_1' \neq y_1'$. By the same argument $x_1' y_1'$ is an edge which is the opposite edge of a square containing $ab$ and lies in the same clique as $x_2 y_2$. As $ab \sim uv$ and $ab \sim x_1' y_1'$, transitivity implies $x_1' y_1' \sim uv$. By minimality, $x_1' y_1' \delta^* uv$, and since $ab \delta x_1' y_1'$ we conclude that $ab \delta^* uv$.

Let now $a = x_1$ (and, of course, $b \neq y_1$). Then the vertices $a$, $b$, $y_1$, $x_2$, $y_2$ induce a house whose convex closure is $K_3 \Box K_2$. Let $x_2'$ be the remaining vertex of the $K_3 \Box K_2$. Then $x_2 x_2' \kappa x_2 y_2$ and so $x_2 x_2' (\delta \cup \kappa)^* uv$. By the minimality we infer that $x_2 x_2' \delta^* uv$ and as $ab \delta x_2 x_2'$ we conclude again that $ab \delta^* uv$.

(iii) $\Rightarrow$ (i): We will prove that $G$ is a quasi-median graph by showing that $G$ can be obtained by a sequence of gated expansions. From Theorem 3.4 (iii) we know that $G$ can be obtained from $K_1$ by a sequence of connected expansions.

We first claim that for each expansion the sets $W_i$ corresponding to it have the same pairwise intersections, i.e. $W_i \cap W_j = W_1 \cap W_2$ for all pairs of indices $1 \leq i < j \leq k$.

Note that this also means that the common intersection of all sets $W_i$ is the same set, which we shall call $U$.

The claim is trivial for $k = 2$, so let $W_1, W_2, \ldots, W_k$, $k \geq 3$, be the sets corresponding to the expansion. By Lemma 4.2 these sets have a common nonempty intersection. Suppose that $W_i \cap W_j \neq W_i \cap W_l$ for some indexes $i, j, l \in \{1, \ldots, k\}$. Then, since the expansion is connected, there exists a vertex $x \in W_i \cap W_j \backslash W_l$ which is adjacent to a vertex $y \in W_i \cap W_j \cap W_l$. Let $x_i, y_i, x_j, y_j, y_l$ be vertices of the graph $G$ that is obtained from $G'$ by this expansion, so that the indices of vertices and sets naturally corresponds. Obviously these vertices form a convex house $G$. Now, if this is not the last expansion in the sequence further expansions cannot change that we have a convex house in a graph. Indeed, this is obvious if the house lies entirely in one of the $W_i$'s of an expansion. If not, then by Definition 3.3(3) the intersection of two sets $W_i, W_j$ in which the house is lying must include two vertices of the triangle of the house which are a cutset of the house. Clearly we also obtain the convex house in the graph obtained by this expansion. This contradiction proves the claim.

Now, let us assume that in one of the expansions of the sequence, the subgraph induced by $U = \bigcap_{i=1}^{k} W_i$ is not gated. Assume first that this happens in the last expansion step. Thus $G'$ is quasi-median and $G$ is obtained from $G'$ by an expansion relative to the sets $W_i$, $i = 1, \ldots, k$, having a common intersection $U$. Since $G'$ is quasi-median it is a weakly modular graph, $\langle U \rangle$ is its triangle closed subgraph (this again follows from nonexistence of convex houses), therefore by Lemma 2.1, $\langle U \rangle$ is not weakly 2-convex. Thus there exist vertices $u, v \in U$ such that $d_{\langle U \rangle}(u, v) = 2$, and there is $x \in V(G)\backslash U$ which is a common neighbour of $u$ and $v$. Let $w \in U$ be a common neighbour of $u$ and $v$. Let $u', v', w', u'', v'', w''$ be vertices in $G$ corresponding to vertices $u$, $v$, and $w$. (There can be more than two such triples, but we need just two.) Then $u', v', w', u'', v'', w''$ and $x$ form a convex $Q_3^-$ in $G$ which is a contradiction. Similarly as above one can check that if this was not the last expansion step, further expansions cannot change that we have a convex $Q_3^-$ in a graph. □

From the above theorem we immediately obtain the following characterization of median graphs:

**Corollary 4.3** ([4]). *A graph $G$ is a median graph if and only if $G$ is a semimedian graph that contains no convex $Q_3^-$.*

**Proof.** Use that median graphs are precisely bipartite quasi-median graphs, that bipartite partial Hamming graphs are precisely partial cubes, and that in bipartite graphs $\delta = \delta \cup \kappa$ and $\sim = \approx$. Then apply the first two assertions of Theorem 4.1. □

To get another characterization of quasi-median graphs, we recall the following result.

**Theorem 4.4** ([3]). *A connected graph $G$ is a partial Hamming graph if and only if*

(i) *the relation $\approx$ is transitive,*
(ii) *for edges $ab, xy \in E(G)$: if $ab \sim xy$ then $W_{ab} = W_{xy}$, and*
(iii) *$G$ has no isometric cycles $C_{2n+1}$ for $n \geq 2$.*

Note that condition (iii) of the above theorem can be replaced by (cf. [3]):

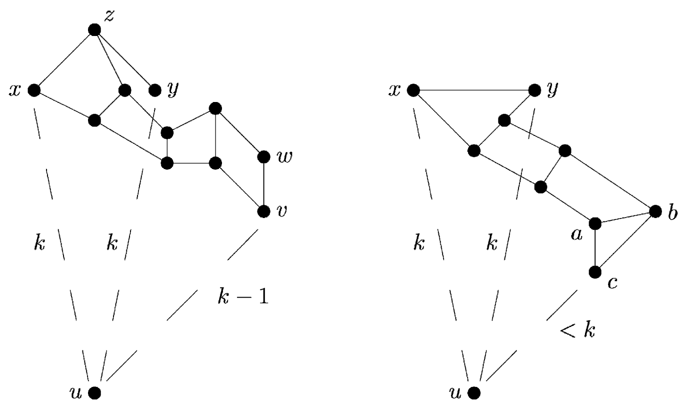


Fig. 2. Semi-quadrangle and semi-triangle property.

(iii′) If $P$ is a path connecting the endpoints of an edge $xy$, then $P$ contains an edge $ab$ with $xy \approx ab$.

Combining Theorem 4.4 with 4.1 we get:

**Corollary 4.5.** *A connected graph $G$ is a quasi-median graph if and only if*

(i) $\approx = (\delta \cup \kappa)^*$,
(ii) *for edges $ab, xy \in E(G)$: if $ab \sim xy$ then $W_{ab} = W_{xy}$,*
(iii) *$G$ has no isometric cycles $C_{2n+1}$ for $n \geq 2$, and*
(iv) *$G$ has no convex $Q_3^-$ and no convex house.*

## 5. Characterizing quasi-semimedian graphs

In this section we examine quasi-semimedian graphs more closely. We extend a result from [18] by characterizing quasi-semimedian graphs among partial Hamming graphs. Then we prove a characterization of quasi-semimedian graphs which, together with Theorem 4.1, gives another characterization of quasi-median graphs. We begin with a new concept—a semi-quadrangle property. It generalizes the concept of the quadrangle property and will be used in a characterization of quasi-semimedian graphs.

A graph $G$ satisfies the *semi-quadrangle property* if for any $u, x, y, z \in V(G)$ such that $d(u, x) = d(u, y) = d(u, z) - 1$ and $d(x, y) = 2$ with $z$ a common neighbour of $x$ and $y$, there exists an edge $wv$ such that $vw\delta^* xz$ and $d(u, v) = d(u, x) - 1$, cf. Fig. 2. (Note that in the definition of the quadrangle property a part of the condition that uses $\delta^*$ is changed to $vw\delta xz$ and $w = y$.)

For our next result we recall:

**Lemma 5.1** ([8]). *Let $G$ be a partial Hamming graph, and $K$ a clique in $G$. Then for any vertex $u \in V(G)$ the distances from $u$ to vertices of $K$ are either equal or there exists a unique $x \in K$ that is closer to $u$ than other vertices of $K$.*

**Proposition 5.2.** *A graph is quasi-semimedian if and only if it is a partial Hamming graph that satisfies the semi-quadrangle property.*

**Proof.** Let $G$ be quasi-semimedian and let vertices $u, x, y, z$ be as above. Let $P_1$ be a shortest path from $x$ to $u$, and $P_2$ a shortest path from $u$ to $y$. Without loss of generality, we may assume that $u$ is the only common vertex of $P_1$ and $P_2$. By Theorem 4.4 (iii′) there exists an edge $ab$ which lies on a path $x \to P_1 \to u \to P_2 \to y \to z$ and is in relation $\approx$ with $xz$. Suppose that there exists a clique with edges $e, f$ such that $xz \sim e$ and $f \sim ab$. Since $ab$ is on a shortest path from $u$ to $z$, one of the vertices $a$ or $b$ is closer to $u$ than the other. Hence by Theorem 4.4 (ii) one of the endvertices of $f$ is closer to $u$ than the other, and by Lemma 5.1 we deduce that this endvertex of $f$ is closer to $u$ than both endvertices of $e$. This is a contradiction to $u \in W_{xz}$, since by Theorem 4.4 (ii) $u$ should be closer to one endvertex of $e$. Thus, the remaining option is that $xz \sim ab$. Since $G$ is quasi-semimedian, we derive by Proposition 3.2 that $xz\delta^*ab$, and the semi-quadrangle property now easily follows.

If $G$ is not quasi-semimedian then by Proposition 3.2 there exist edges $xy, uv$ such that $xy \sim uv$, but $xy$ is not in relation $\delta^*$ with $uv$. In addition, we may choose $xy$ and $uv$ in such a way that the distances between their endvertices are as small as possible. Now, the semi-quadrangle property does not hold for vertices $x, u$, a neighbour of $v$ which lies on a shortest path to $y$, and $v$. □

Proposition 5.2 is analogous to the following characterization of median graphs from [21]: $G$ is a median graph if and only if $G$ is a partial cube satisfying the quadrangle property. Also, it implies the following characterization of semimedian graphs.

**Corollary 5.3.** *A graph is semimedian if and only if it is a partial cube that satisfies the semi-quadrangle property.*

We now introduce yet another concept—semi-triangle property. A graph $G$ satisfies a *semi-triangle property* if for any vertices $u, x, y \in V(G)$ where $d(u, x) = d(u, y) = k \geq 2$ such that $xy \in E(G)$, there exists a triangle with vertices $a, b, c$ such that $xy\delta^*ab$, and $d(u, a) - 1 = d(u, b) - 1 = d(u, c) < k$, cf. Fig. 2. (Note that in the definition of ordinary triangle property we have $a = x$ and $b = y$.) A graph is *semi-weakly-modular* if it satisfies both the semi-quadrangle and the semi-triangle property.

It is not hard to see that quasi-semimedian graphs are semi-weakly-modular. Indeed, let $G$ be a quasi-semimedian graph, and vertices $u, x, y$ as above. Let $u'$ be the last vertex on a shortest path from $u$ to $x$ for which $d(u', x) = d(u', y)$. Thereby, there exist neighbours $a, b$ of $u'$ such that $a \in W_{xy}$ and $b \in W_{yx}$. Wilkeit showed:

**Lemma 5.4** ([31]). *If $G$ is a partial Hamming graph then: if a vertex $w \in V(G)$ has the same distance to adjacent vertices $x$ and $y$ of $G$, then any two neighbours $a \in W_{xy}$ and $b \in W_{yx}$ of $w$ are adjacent.*

From this we infer that $a, b$ and $u'$ are in a triangle, and obviously $d(u, a) - 1 = d(u, b) - 1 = d(u, u') < d(u, x)$. Finally, by Proposition 3.2 it follows that $ab\delta^*xy$, so the semi-triangle property holds.

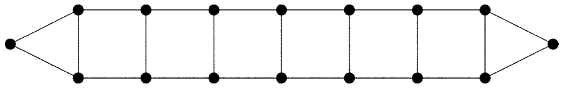

Fig. 3. Graph $H_7$.

In the search for an analogue of Theorem 3.1 we first observe that excluding graphs $K_4 - e$ and $K_{2,3}$ is not enough. For this sake consider graphs $H_n$ obtained from grid graphs $P_2 \Box P_n$ by attaching a triangle to each of both the edges with endvertices of degree 2, cf. Fig. 3. The graphs $H_n$ are semi-weakly-modular but not quasi-semimedian. Moreover, they are not even partial Hamming graphs.

Whenever $H_n$ is an induced subgraph of a graph $G$ we shall denote its vertices by $H_n(u, v)$ where $u$ and $v$ are the unique vertices of degree 2 in $H_n$. In the following theorem we prove that one must exclude graphs $H_n$ as induced subgraphs for which $I_G(u, v) \cap H_n(u, v) \neq \{u, v\}$ holds. Also, instead of just excluding subgraphs $K_{2,3}$ we need a stronger condition taken from Theorem 4.4 (ii). Note that this condition implies transitivity of the relation $\sim$, cf. [31].

**Theorem 5.5.** *A graph G is quasi-semimedian if and only if*

(i) *$G$ is semi-weakly-modular,*
(ii) *for every induced $H_n$, $n \geq 1$, we have $I_G(u, v) \cap H_n(u, v) = \{u, v\}$,*
(iii) *for edges $ab, xy \in E(G)$: if $ab \sim xy$ then $W_{ab} = W_{xy}$.*

**Proof.** By the above discussion we only need to prove that conditions are sufficient, and by Proposition 5.2 it is enough to show that conditions (i)–(iii) ensure that $G$ is a partial Hamming graph. Moreover, by Theorem 4.4 we only need to prove conditions (i) and (iii) of that theorem.

First we claim that the condition of Lemma 5.1 holds for $G$. (In the proof we shall recall that $\delta^* \subseteq \sim$, since $\sim$ is transitive.) Assume that there is a clique with vertices $x, y, z$, and a vertex $u \in V(G)$ such that $d(u, x) = d(u, y) = d(u, z) - 1$. Let $u$ be a vertex closest to $z$ with this property, hence the neighbour $x'$ on a shortest path from $u$ to $x$ is in $W_{xy}$. By the semi-triangle property there exists a triangle with vertices $a, b, c$ such that $ab\delta^*xy$, and $d(u, c) = d(u, a) - 1 = d(u, b) - 1$. Note that, by condition (iii) of the theorem, since $x' \in W_{xy}$, also $x' \in W_{ab}$. It is clear that we have an induced subgraph $H_n$ with vertex set $H_n(c, z)$, hence $I_G(c, z) \cap H_n(c, z) = \{c, z\}$. Thus $z \in W_{ca}$, and again by (iii) we infer that $z \in W_{ux'}$ (because $ux' \sim ca$), which is a contradiction.

Secondly, we prove that the condition of Lemma 5.4 holds for $G$. Let $w \in V(G)$ be a vertex having the same distance to adjacent vertices $x$ and $y$ of $G$, and let $u \in W_{xy}$ and $v \in W_{yx}$ be the neighbours of $w$. By the semi-triangle property there exists a triangle with vertices $a, b, c$ such that $ab\delta^*xy$, and $d(w, c) = d(w, a) - 1 = d(w, b) - 1$. By condition (iii) of the theorem we have $u \in W_{ab}$, $v \in W_{ba}$, and by the claim of the previous paragraph $u \in W_{ac}$. Hence, again using (iii), we deduce $d(u, v) = d(u, w) = 1$ as claimed.

We next prove condition (i) of Theorem 4.4. Suppose not: then there exist edges $e$, $f$, and $g$ such that $e \approx f$ and $f \approx g$ but $e$ and $g$ are not in relation $\approx$. Now, in both cases where relation $\approx$ holds, it is clear that it is not equal to $\sim$. Hence there exist cliques

$C$ and $C'$, and edges $e'$ of $C$ and $g'$ of $C'$, such that $f \sim e'$ and $f \sim g'$. Now, it is not hard to prove that $f\delta^*e'$ and $f\delta^*g'$. (Use the semi-quadrangle property in the cycle formed by $f$, $e'$ and shortest paths between their endvertices, and then use the induction on the distance between endvertices of two edges in relation $\sim$.) Thus we have at least one $H_n$ in $G$, moreover, using a shortest $\delta^*$ sequence we can choose a $H_n$ which is induced. (The number of induced $H_n$'s in $G$ depends on the sizes of $C$ and $C'$.) By the condition of Lemma 5.1 each vertex of $C$ is either closest to exactly one of the vertices of $C'$, or is at the same distance to all of them. If the latter holds for a vertex $z$ of $C$, then obviously $H_n(z, w) \cap I(z, w) \neq \{z, w\}$ for any $w$ of $C'$, and we are through in this case. On the other hand, if all vertices of $C$ and $C'$ have their unique closest vertices, then we deduce that $e'$ and $g'$ are in relation $\sim$, hence $e \approx g$, and so $\approx$ is transitive in $G$.

Finally, we prove the condition (iii) of Theorem 4.4. Suppose that the odd cycle $C : x \to z_1 \to z_2 \to \cdots \to z_{2k+1} = x$ is isometric. Then the condition of Lemma 5.4 can be used for $x, z_k, z_{k+1}$, and $z_1, z_{2k}$, which says that $z_1$ and $z_{2k}$ are adjacent. This proves that $G$ is a partial Hamming graph, and thus quasi-semimedian. □

Combining Theorem 5.5 with 4.1 we obtain yet another characterization of quasi-median graphs:

**Corollary 5.6.** *A connected graph $G$ is a quasi-median graph if and only if*

(i) *$G$ is semi-weakly-modular,*
(ii) *$G$ has no induced $H_n$, $n \geq 1$, for which $I_G(u, v) \cap G_n(u, v) \neq \{u, v\}$,*
(iii) *for edges $ab, xy \in E(G)$: if $ab \sim xy$ then $W_{ab} = W_{xy}$, and*
(iv) *$G$ has no convex $Q_3^-$ and no convex house.*

## 6. Tree-like equalities for quasi-median graphs

Median graphs simultaneously generalize trees and hypercubes. Moreover, they are considered to be the class which reflects all important properties shared by these two classes (see Mulder's metaconjecture [26]). Soltan and Chepoi [28] and Škrekovski [27] proved tree-like equalities for median graphs which shed a surprising light on the metaconjecture. Indeed, let $q_r$ be the number of subgraphs of a median graph isomorphic to $Q_r$, and let $k$ be the number of its equivalence classes with respect to the relation $\sim$. Then

$$\sum_{i\geq 0}(-1)^i q_i = 1 \qquad \text{and} \qquad k = -\sum_{i\geq 0}(-1)^i i q_i .$$

Note that the second one applied to trees tells that the number of equivalence classes with respect to the relation $\sim$ equals the number of edges—a less known characterizing property of trees. These relations also imply the Euler-type formulae from [22, 23], and they were widely generalized in [5]. In the following result we will extend the above equalities to the quasi-median graphs by using subgraphs which are isomorphic to Hamming graphs.

(We note here that invariant (with respect to automorphisms) Hamming subgraphs of quasi-median graphs were studied by Chastand and Polat [7].)

For a Hamming graph $H = K_{k_1} \Box K_{k_2} \Box \cdots \Box K_{k_n}$, with $k_i > 1$ for all $i$, we say that $n$ is the *dimension* of $H$. The *dimension* of a partial Hamming graph $G$ is the dimension of a Hamming graph of smallest dimension into which $G$ can be isometrically embedded. Alternatively, the dimension of $G$ is the number of expansion steps with which $G$ is obtained from $K_1$, which in turn coincides with the number of $\approx$ classes in $G$ (using Lemma 4.2). Note that the dimension of $Q_n$ is $n$.

**Theorem 6.1.** *Let $G$ be a quasi-median graph of dimension $t$ and let $\alpha_i$ $(i \geq 0)$ be the number of induced Hamming subgraphs of $G$ of degree $i$. Then*

$$\sum_{i \geq 0} (-1)^i \alpha_i = 1 \qquad \textit{and} \qquad -t = \sum_{i \geq 0} (-1)^i i \alpha_i .$$

**Proof.** The proof is by induction on the number of vertices. The claim is obviously true for $G \cong K_1$. So, we may assume that $G$ is constructed by a gated expansion from a quasi-median graph $G'$ with respect to $U, W_1, \ldots, W_n$. Let $\alpha_i^0$ (resp. $\alpha_i'$) be the number of induced subgraphs of $\langle U \rangle$ (resp. $G'$) isomorphic to some Hamming graph of degree $i$. Denote by $t_0$ and $t'$ the dimensions of $\langle U \rangle$ and $G'$, respectively. Since $G'$ and $\langle U \rangle$ are quasi-median graphs, by induction, we assume that the above two relations are valid for these two graphs. It is not hard to observe that

$$\alpha_k = \alpha_k' - \alpha_k^0 + \sum_{i \geq 0}^{k} \alpha_{k-i}^0 \binom{n}{i+1} .$$

Recall that $\binom{n}{i} = 0$ whenever $i > n$. In what follows, we will use the following two identities:

$$\sum_{i \geq 1} (-1)^i \binom{n}{i} = -1 \qquad \text{and} \qquad \sum_{i \geq 0} (-1)^i i \binom{n}{i} = 0.$$

We can now derive

$$\begin{aligned} \sum_{k \geq 0} (-1)^k \alpha_k &= \sum_{k \geq 0} (-1)^k \alpha_k' - \sum_{k \geq 0} (-1)^k \alpha_k^0 + \sum_{k \geq 0} (-1)^k \sum_{i \geq 0}^{k} \alpha_{k-i}^0 \binom{n}{i+1} \\ &= 1 - 1 + \sum_{k \geq 0} \left( (-1)^{k-1} \alpha_k^0 \sum_{j \geq 1} (-1)^j \binom{n}{j} \right) \\ &= \sum_{k \geq 0} (-1)^k \alpha_k^0 = 1. \end{aligned}$$

Observe that $t = t' + 1$. For the second equality,

$$\begin{aligned}
\sum_{k\geq 0}(-1)^k k\alpha_k &= \sum_{k\geq 0}(-1)^k k\alpha'_k - \sum_{k\geq 0}(-1)^k k\alpha_k^0 \\
&\quad + \sum_{k\geq 0}\left((-1)^k k \sum_{i\geq 0}^{k} \alpha_{k-i}^0 \binom{n}{i+1}\right) \\
&= -t' + t_0 + \sum_{k\geq 0}\left((-1)^{k-1}\alpha_k^0\left(\sum_{j\geq 0}(-1)^{j+1}(k+j)\binom{n}{j+1}\right)\right) \\
&= -t' + t_0 + \sum_{k\geq 0}\left(\alpha_k^0 \sum_{j\geq 0}(-1)^{k+j}(k+j)\binom{n}{j+1}\right) \\
&= -t' + t_0 + \sum_{k\geq 0}\left((-1)^{k-1}(k-1)\alpha_k^0 \sum_{j\geq 0}(-1)^{j+1}\binom{n}{j+1}\right) \\
&\quad + \sum_{k\geq 0}\left((-1)^k\alpha_k^0 \sum_{j\geq 0}(-1)^{j+1}(j+1)\binom{n}{j+1}\right) \\
&= -t' + t_0 + \sum_{k\geq 0}(-1)^k(k-1)\alpha_k^0 + \sum_{k\geq 0}(-1)^k\alpha_k^0\, 0 \\
&= -t' + t_0 + \sum_{k\geq 0}(-1)^k(k-1)\alpha_k^0 \\
&= -t' + t_0 + \sum_{k\geq 0}(-1)^k k\,\alpha_k^0 - \sum_{k\geq 0}(-1)^k\alpha_k^0 \\
&= -t' + t_0 - t_0 - 1 = -t. \quad \square
\end{aligned}$$

The equalities of Theorem 6.1 cannot be extended to quasi-semimedian graphs, not even in the bipartite case. However, these relations imply an Euler-type formula which can be extended to a larger class of graphs. We are going to prove it for graphs that can be obtained by a connected expansion procedure. Note that these graphs include the class of quasi-semimedian graphs, and that this result extends all such previously known formulae [4, 22].

**Theorem 6.2.** *Let $G$ be a graph with $n$ vertices, $m$ edges and of dimension $k$, that is obtained by a sequence of connected expansions from $K_1$. Then $2n - m - k \leq 2$. Moreover equality holds if and only if $G$ is $C_t \,\square\, K_2$-free ($t \geq 3$) and $K_4$-free.*

**Proof.** The proof is by induction on $k$. Let $G$ be obtained from a graph $G'$ by a connected expansion with respect to $W_1, W_2, \ldots, W_r$. Let $W^* = \bigcup_{1\leq i<j\leq r}(W_i \cap W_j)$. For $i = 1, \ldots, r$ denote by $a_i$ and $b_i$ the number of vertices and edges, respectively, that lie in at least $i$ covering subsets of $G'$. Let $k'$ be the dimension of $G'$, then $k = k' + 1$.

Clearly, $a_1$ and $b_1$ are the number of vertices and edges, respectively, of $G'$. Moreover, $\sum_{i=2}^{r} b_i$ is the number of edges added by the expansion to the $\approx$ classes of $G'$, while $\sum_{i=1}^{k}(i-1)a_i$ is the number of edges of the new $\approx$ class. Hence,

$$n = \sum_{i=1}^{r} a_i \qquad \text{and} \qquad m = \sum_{i=1}^{r}(b_i + (i-1)a_i),$$

from which we obtain

$$\begin{aligned} 2n - m - k &= \sum_{i=1}^{r}(3-i)a_i - \sum_{i=1}^{r} b_i - (k'+1) \\ &= (2a_1 - b_1 - k') + \sum_{i=2}^{r}(3-i)a_i - \sum_{i=2}^{r} b_i - 1. \end{aligned}$$

By the induction hypothesis, $2a_1 - b_1 - k' \le 2$ holds for $G'$, therefore

$$\begin{aligned} 2n - m - k &\le \sum_{i=2}^{r}(3-i)a_i - \sum_{i=2}^{r} b_i + 1 \\ &= (a_2 - b_2 + 1) + \sum_{i=3}^{r}(3-i)a_i - \sum_{i=3}^{r} b_i \\ &\le a_2 - b_2 + 1. \end{aligned}$$

Now, $a_2$ and $b_2$ are the numbers of vertices and edges of $\langle W^* \rangle$, respectively. Since $\langle W^* \rangle$ is connected, we have $a_2 - b_2 \le 1$, which proves the theorem's inequality.

For the second part of the theorem observe that the equality will hold precisely when $G$ is obtained by an expansion procedure in such a way that in all the expansions the numbers $a_i$, $i \ge 4$, and $b_j$, $j \ge 3$, are zero and $a_2 - b_2 = 1$. This holds precisely when in each expansion step at most three covering sets are involved, no edge lies in all three covering sets (which means that their common intersection is a vertex), and $\langle W^* \rangle$ is a tree. Obviously no $K_4$, $C_3 \,\square\, K_2$, $C_4 \,\square\, K_2$, $C_5 \,\square\, K_2$ can then appear in $G$, and it is straightforward to check by induction on the dimension, that this holds also for $C_t \,\square\, K_2$ ($t \ge 6$). The converse is obvious. □

**Corollary 6.3.** *Let $G$ be a planar graph with $n$ vertices and of dimension $k$, that is obtained by a sequence of connected expansions from $K_1$. Let $f$ be the number of faces in its planar embedding. Then $f \ge n-k$, where equality holds if and only if $G$ is $C_t \,\square\, K_2$-free ($t \ge 3$) and $K_4$-free.*

**Proof.** Combine Theorem 6.2 with Euler's formula $n - m + f = 2$. □

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