Strong Edge Geodetic Problem on Complete Multipartite Graphs and some Extremal Graphs for the Problem

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Abstract

A set of vertices X of a graph G is a strong edge geodetic set if to any pair of vertices from X we can assign one (or zero) shortest path between them such that every edge of G is contained in at least one on these paths. The cardinality of a smallest strong edge geodetic set of G is the strong edge geodetic number $\mathrm{sg_e}(G)$ of G. In this paper, the strong edge geodetic number of complete multipartite graphs is determined. Graphs G with $\mathrm{sg_e}(G) = n(G)$ are characterized and $\mathrm{sg_e}$ is determined for Cartesian products $P_n \,\square\, K_m$. The latter result in particular corrects an error from the literature.

Keywords: strong edge geodetic problem; complete multipartite graph; edge-coloring; Cartesian product of graphs

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1 Introduction

Covering vertices or edges of a graph by the smallest number of paths is a fundamental optimization problem and appears in the literature in several variations depending on the properties one requires from the paths. In the isometric path cover problem (alias geodetic cover problem) the aim is to cover all the vertices by a minimum number of shortest paths [4,5,10,11,15,21]. In the path cover problem we want to cover all the vertices by a minimum number of vertex disjoint paths [6,7,20]. Dual concepts have also been studied as for instance the k-path covers which are sets S of vertices of a graph G such that every path of order k in G contains at least one vertex from S, see [2,3,9]. In the edge version of the isometric path cover problem we want to cover all the edges by a minimum number of shortest paths [1,22,23]. In this paper we are interested in the strong edge geodetic problem introduced in [17] as follows.

Let $G = (V(G), E(G))$ be a graph. A set of vertices $X \subseteq V(G)$ is a *strong edge geodetic set* if to any pair of vertices u and v from X we can assign a shortest u, v-path P_{uv} such that every edge $xy \in E(G)$ is contained in at least one on the paths P_{uv}. The cardinality of a smallest strong edge geodetic set of G is the *strong edge geodetic number* $\mathrm{sg_e}(G)$ of G. Such a set is briefly called a $\mathrm{sg_e}(G)$*-set*.

In the seminal paper [17] it was proved, among other results, that the strong edge geodetic problem is $\mathcal{NP}$-complete. In [8] it was further proved that there is no approximation of the strong edge geodetic number with an approximation factor better that 781/780. Several additional results on the strong edge geodetic number were reported in [25, 26]. In the latter paper, the strong edge geodetic number was determined for Cartesian products $P_n \,\square\, P_k$, where $k \in \{2, 3, 4\}$.

The vertex version of the strong edge geodetic problem is known as the *strong geodetic problem* and was studied for the first time in [16, 18]. The strong geodetic problem is also $\mathcal{NP}$-complete and remains such even when restricted to bipartite graphs and multipartite graphs [13]. Moreover, determining whether a given set X is a strong geodetic set is $\mathcal{NP}$-hard [8] as well.

The strong geodetic number of complete bipartite (resp. mutipartite) graphs received a lot of attention. First, in [13] the problem was solved for balanced complete bipartite graphs $K_{n.n}$. Subsequently, using different approaches, a formula for arbitrary complete bipartite graphs was derived in [14] and in [12]. In [14], a lower bound for the strong geodetic number of a complete multipartite graph was given and it was conjectured that the strong geodetic number remains $\mathcal{NP}$-complete on complete mutipartite graphs. In [8] this conjecture was disproved by developing a polynomial algorithm for the strong geodetic number of complete mutipartite graphs. In this direction we emphasize that in [19] an $O(mn^2)$ algorithm for computing the strong geodetic number in outerplanar graphs was developed. Several additional interesting results on the strong geodetic problem were presented in [24]. Among other results, relations between the strong geodetic number and the connectivity and the diameter were established, and graphs with the strong geodetic number equal to 2, $n - 1$, and n were characterized.

Motivated by the efforts to determine the strong geodetic number of complete bipartite graphs, we determine in Section 2 the strong edge geodetic number of complete bipartite graphs and in Section 3 the strong edge geodetic number of complete multipartite graphs. Before that, we give some definitions at the end of this section. In Section 4 we characterize graphs G with $\mathrm{sg_e}(G) = n(G)$ and discuss the graphs with $\mathrm{sg_e}(G) = n(G)-1$. In particular we observe that Cartesian products $P_2 \,\square\, K_n$ belong to this family of graphs. This corrects [25, Theorem 13] where it is wrongly stated that $\mathrm{sg_e}(P_2 \,\square\, K_n) = 2n-2$. We then proceed by determining $\mathrm{sg_e}(P_m \,\square\, K_n)$ for all $m, n \geq 2$.

The order of a graph G is denoted by $n(G)$. A vertex u of a graph G is *universal* if $\deg_G(u) = n(G) - 1$. The *Cartesian product* $G \,\square\, H$ of graphs G and H is the graph with the vertex set $V(G) \times V(H)$, vertices (g,h) and (g',h') being adjacent if either $g = g'$ and $hh' \in E(H)$, or $h = h'$ and $gg' \in E(G)$. As usual, $\chi'(G)$ is the chromatic index of G. For a positive integer n, the set $\{1,\dots,n\}$ will be dented by $[n]$.

If U is a strong edge geodetic set, then we will denote by $\widehat{U}$ the set of associated paths with endpoints from U which cover all the edges of G. Clearly, $\widehat{U}$ is not unique, but unless stated otherwise, we will assume that $\widehat{U}$ has been selected and is fixed.

2 Complete bipartite graphs

In this section we prove the following result where in part edge colorings of complete graphs will be very useful.

Theorem 2.1 *If $n \geq m \geq 2$, then the following hold.*

(i) If n is even, then

$$\mathrm{sg_e}(K_{n,m}) = \begin{cases} n+1; & n = m, \\ n; & n \geq m+1. \end{cases}$$

(ii) If n is odd, then

$$\mathrm{sg_e}(K_{n,m}) = \begin{cases} n+2; & n = m, \\ n+1; & n = m+1, \\ n; & n \geq m+2. \end{cases}$$

In the rest of the section we assume that $n \geq m \geq 2$ and that the bipartition of $K_{n,m}$ is (X,Y), where $X = \{x_0, \dots, x_{n-1}\}$ and $Y = \{y_0, \dots, y_{m-1}\}$.

Lemma 2.2 *If U is a strong edge geodetic set of $K_{n,m}$, then $X \subseteq U$ or $Y \subseteq U$.*

Proof. Let U be a strong edge geodetic set of the graph $K_{n,m}$. Suppose on the contrary that there exist vertices $x_i \in U^C \cap X$ and $y_j \in U^C \cap Y$. Because $\mathrm{diam}(K_{n,m}) = 2$ and $x_i y_j$ is an edge of $K_{n,m}$, none of the shortest paths with endpoints from U can cover the edge $x_i y_j$, that is, U cannot be a strong edge geodetic set. $\square$

Lemma 2.3 *If U is a strong edge geodetic set of $K_{n,m}$ and $Y \subseteq U$, then $|U| \geq n+1$.*

Proof. Suppose U is a strong edge geodetic set of $K_{n,m}$, where $U = Y \cup X'$ with $X' \subset X$ and $|X'| = k$, $0 \leq k \leq n$. Consider an arbitrary vertex $y_j \in Y$. There are exactly $n-k$ edges between y_j and $X \setminus X'$. Because the shortest paths that cover these edges have both of their endpoints in Y, it has to hold $m-1 \geq n-k$. This in turn implies that $|U| = |Y| + |X'| = m + k \geq n+1$. $\square$

Corollary 2.4 *If $n \geq m \geq 2$, then $\mathrm{sg_e}(K_{n,m}) \geq n$. Moreover, if $m = n$, then $\mathrm{sg_e}(K_{n,n}) \geq n+1$.*

Proof. If $m = n$, then the second assertion of the corollary follows immediately from Lemmas 2.2 and 2.3. Suppose now that $n > m$ and let U be a smallest strong edge geodetic set of $K_{n,m}$, so that $|U| = \mathrm{sg_e}(K_{n,m})$. Then $X \subseteq U$ or $Y \subseteq U$ by Lemma 2.2. If $X \subseteq U$, then $\mathrm{sg_e}(K_{n,m}) = |U| \geq |X| = n$. And if $Y \subseteq U$, then $\mathrm{sg_e}(K_{n,m}) \geq n+1$ follows by Lemma 2.3. $\square$

We have thus established the lower bound for the case when n is even. For n odd we proceed as follows.

Lemma 2.5 *Let U be a strong edge geodetic set of $K_{n,m}$. If n is odd and $X \subseteq U$, then $|U| \geq \frac{2n}{n+1} + m$.*

Proof. Let U be a strong edge geodetic set of $K_{n,m}$, where $U = X \cup Y'$ with $Y' \subseteq Y$ and $|Y'| = k$. For each edge xy, where $x \in X$ and $y \in Y'$, we put the shortest path xy to $\widehat{U}$. The edges between vertices from X and $Y \setminus Y'$ must be covered by the shortest paths of length 2 with both of their endpoints in X. For each pair of vertices from X we can put only one shortest path to $\widehat{U}$, so we can only put $\binom{n}{2}$ shortest paths to $\widehat{U}$ to cover the $n \cdot (m-k)$ edges between the vertices from X and the vertices from $Y \setminus Y'$. Moreover, because the degree of every vertex from $Y \setminus Y'$ is n, which we have assumed to be odd, each vertex from $Y \setminus Y'$ must be the central vertex of at least $(n+1)/2$ shortest paths from $\widehat{U}$. Since U is a strong edge geodetic set this implies that $\binom{n}{2} \geq (m-k) \cdot \frac{n+1}{2}$. This inequality rewrites to $k \geq m - n(n-1)/(n+1)$ which in turn implies that $|U| = n + k \geq n + m - n(n-1)/(n+1) = \frac{2n}{n+1} + m$. $\square$

Corollary 2.6 *If $n \geq 3$ is odd, then $\mathrm{sg_e}(K_{n,n}) \geq n+2$ and $\mathrm{sg_e}(K_{n,n-1}) \geq n+1$.*

Proof. Let U be a smallest strong edge geodetic set of $K_{n,n}$. By Lemma 2.5,

$$|U| \geq n + \frac{2n}{n+1}\,.$$

As $|U|$ is an integer and $2n/(n+1) > 1$ for $n \geq 2$ we get $|U| = \mathrm{sg_e}(K_{n,n}) \geq n+2$.

Let now U be a smallest strong edge geodetic set of $K_{n,n-1}$. By Lemma 2.2 we have $X \subseteq U$ or $Y \subseteq U$. In the latter case, Lemma 2.3 gives $\mathrm{sg_e}(K_{n,n-1}) \geq n+1$. Assume second that $X \subseteq U$. Then Lemma 2.5 gives

$$|U| \geq \frac{2n}{n+1} + (n-1) = n + \frac{n-1}{n+1}\,.$$

Since $\frac{n-1}{n+1} > 0$ for $n \geq 2$ and since $|U|$ is an integer, also in this case we get $\mathrm{sg_e}(K_{n,n-1}) \geq n+1$. □

So far, we have proved the lower bound for all the cases of Theorem 2.1. In the following we will construct in each case a strong edge geodetic set of the required cardinality.

Case 1: n is even.
We first consider $K_{n,n}$ and prove that

$$\mathrm{sg_e}(K_{n,n}) \leq n+1\,. \tag{1}$$

We claim that $U = X \cup \{y_{n-1}\}$ is a strong edge geodetic set of $K_{n,n}$. For every $0 \leq i \leq n-1$, add the shortest path x_iy_{n-1} to $\widehat{U}$ to cover the edge x_iy_{n-1}. Then all the other edges must be covered by shortest paths of the form $x_iy_jx_k$, where $i \neq k$. To do so, we use edge-colorings of K_n. It is well-known that $\chi'(K_n) = n-1$ for even n. Let $V(K_n) = \{0, 1, \ldots, n-1\}$. Then an edge-coloring c of K_n using $n-1$ colors can be defined as follows: if $i, j \in \{0, 1, \ldots, n-2\}$, $i \neq j$, then let $c(ij) = (i+j) \mod (n-1)$, and for for $i \in \{0, 1, \ldots, n-2\}$ let $c(i(n-1)) = 2i \mod (n-1)$.

In the covering of $K_{n,n}$ that we are constructing, we put the shortest path $x_iy_jx_k$ to $\widehat{U}$ if and only if $c(ik) = j$. See Fig. 1, where this construction is illustrated for the case $n = 6$ and the edges incident to y_2. In K_6, we have $c(02) = c(15) = c(34) = 2$, hence the paths $x_0y_2x_2$, $x_1y_2x_5$, and $x_3y_2x_4$ belong to $\widehat{U}$.

Using this construction, a pair of vertices x_i and x_k is never used twice, and for each vertex $y \in Y \setminus \{y_{n-1}\}$ the shortest paths in $\widehat{U}$ have pairwise different endpoints. Since in c every color is used exactly $n/2$ times, the shortest paths from $\widehat{U}$ passing through y_j cover all the edges incident with y_j. This proves (1).

Consider now $K_{n,m}$, where $m \leq n-1$ (and n is even). We need to show that $\mathrm{sg_e}(K_{n,m}) \leq n$. For this sake we claim that X is a strong edge geodetic set. Indeed, use the above edge-coloring c of K_n and for each $y_i \in Y$, $i \in [m]$, put all the shortest paths $x_jy_ix_k$ to $\widehat{U}$ for which $c(jk) = i$. By the above argument, X is indeed a strong edge geodetic set and hence $\mathrm{sg_e}(K_{n,m}) \leq n$ in this subcase.

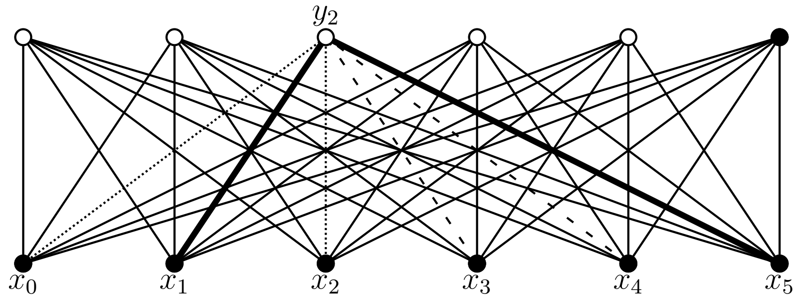

Figure 1: Shortest paths from $\widehat{U}$ that cover edges incident to y_2.

Case 2: n is odd.
We first consider $K_{n,n}$ and prove that $\mathrm{sg_e}(K_{n,n}) \leq n+2$. For this purpose consider the set $U = X \cup \{y_{n-2}, y_{n-1}\}$. The subgraph of $K_{n,n}$ induced by the set of vertices $V(K_{n,n}) \setminus \{x_{n-1}, y_{n-1}\}$ is isomorphic to $K_{n-1,n-1}$. As $n-1$ is even, we can cover its edges by the paths as described in Case 1 to derive (1). Recall that for this covering, the vertices $x_0, \ldots, x_{n-1}$ and y_{n-2} are used. To cover the edges $y_{n-1}x_i$, $0 \leq i \leq n-1$, add the shortest paths $y_{n-1}x_i$ to $\widehat{U}$. Finally, to cover the remaining yet uncovered edges, that is, the edges $x_{n-1}y_i$, where $i \in \{0, \ldots, n-2\}$, put the shortest paths $x_{n-1}y_ix_i$ to $\widehat{U}$.

We next show that $\mathrm{sg_e}(K_{n,n-1}) \leq n+1$. In this subcase set $U = X \cup \{y_{n-2}\}$. Then as in the above subcase, cover the edges of the subgraph of $K_{n,n-1}$ induced by the set of vertices $V(K_{n,n-1}) \setminus \{x_{n-1}\}$ as described in Case 1 to derive (1). After that, to cover the edges $x_{n-1}y_i$, where $i \in \{0, \ldots, n-2\}$ we add to $\widehat{U}$ the shortest paths $x_{n-1}y_ix_i$.

Consider finally $K_{n,m}$, where $m \leq n-2$. In this case, X is a strong edge geodetic set. For this sake note that by the second subcase of Case 1 we know that $\{x_0, \ldots, x_{n-2}\}$ is a strong edge geodetic set of the subgraph of $K_{n,m}$ induced by the set $V(K_{n,m}) \setminus \{x_{n-1}\}$. To cover the remaining not yet covered edges $x_{n-1}y_i$, where $i \in \{0, \ldots, m-1\}$ we add to $\widehat{U}$ the shortest paths $x_{n-1}y_ix_i$. From here it is clear that X is a strong edge geodetic set of $K_{n.n}$ and we conclude that is this subcase $\mathrm{sg_e}(K_{n,m}) \leq n$.

We have thus established all the upper bounds which completes the proof of Theorem 2.1.

3 Complete mutipartite graphs

Using Theorem 2.1 we determine in this section the strong edge geodetic number of complete multipartite graphs. To do so, we first need the following:

Lemma 3.1 *If $2 \le m \le n$ and ${\rm sg_e}'(K_{n,m})$ denotes the cardinality of a smallest strong edge geodetic set U of $K_{n,m}$ such that $Y \subseteq U$, then*

$$
{\rm sg_e}'(K_{n,m}) = \begin{cases} n+1; & m \text{ even}, \\ n+2; & m \text{ odd}. \end{cases}
$$

Proof. Let U be a strong edge geodetic set of $K_{n,m}$ with $Y \subseteq U$ and let $X' = U \cap X$. By Lemma 2.3 we have $|U| \ge n+1$. It follows that $|X'| \ge n-m+1$ and $|X \setminus X'| \le m-1$.

To cover the edges between Y and X' we of course use the edges themselves. The edges between Y and $X \setminus X'$ must be covered by paths of length 2 which implies that every such a path must have endpoints in Y. Let G be the subgraph of $K_{m,n}$ induced by $V(K_{m,n}) \setminus X'$. Suppose m is even. Then selecting X' to be of cardinality $n-m+1$, we have $G \cong K_{m,m-1}$ and Theorem 2.1(i) and its proof imply that we can cover the edges from G by paths between the vertices of X. Hence ${\rm sg_e}'(K_{n,m}) = n+1$ when n is even. Assume next that m is odd. Then Theorem 2.1(ii) implies that if $|X'| = n-m+1$, then the edges of $G = K_{m,m-1}$ cannot be covered by paths between vertices from Y alone. On the other hand, if $|X'| = n-m+2$, then $G = K_{m,m-2}$ and the same theorem says that we can cover the edges of G only using the vertices from Y. In conclusion, ${\rm sg_e}'(K_{n,m}) = n+2$ when n is odd. $\square$

Theorem 3.2 *If $k \ge 2$ and $2 \le n_1 \le n_2 \le \cdots \le n_k$, then the following hold.*

(i) If n_1 is even, then

$$
{\rm sg_e}(K_{n_1,\ldots,n_k}) = \begin{cases} \sum\limits_{j=2}^{k} n_j + 1; & n_2 \in \{n_1, n_1+1\}, \\ \sum\limits_{j=2}^{k} n_j; & \text{otherwise}, \end{cases}
$$

(ii) If n_1 is odd, then

$$
{\rm sg_e}(K_{n_1,\ldots,n_k}) = \begin{cases} \sum\limits_{j=2}^{k} n_j + 2; & n_2 = n_1, \\ \sum\limits_{j=2}^{k} n_j; & \text{otherwise}, \end{cases}
$$

Proof. Let $k \ge 2$ and $2 \le n_1 \le n_2 \le \cdots \le n_k$, and set $G = K_{n_1,\ldots,n_k}$ for the rest of the proof. Let X_i, $i \in [k]$, be the partition sets of G, where $|X_i| = n_i$. Let U be an arbitrary (smallest) strong edge geodetic set of G. If $i \ne j$, then we see that $X_i \subseteq U$ or $X_j \subseteq j$. If follows that U contains $k-1$ of the partite sets.

To prove the upper bounds, let W be a set of vertices constructed as follows. First, W contains $\cup_{i=2}^k X_i$. Then it remains to cover the edges between X_1 and each of the X_i, $i \geq 2$. More precisely, we need to cover the edges in induced subgraphs K_{n_1,n_i}, $i \geq 2$, where the partite sets of cardinality n_i are already included. Assume first that $n_2 \geq n_1+2$. Then by Theorem 2.1 we infer that W is a strong edge geodetic set. Assume second that $n_2 \geq n_1 + 1$. If n_1 is odd, then n_2 is even and by the same theorem W is also a strong edge geodetic set. On the other hand, if n_1 is even, then Theorem 2.1 implies that W together with one vertex from X_1 is a strong edge geodetic set. Assume finally that $n_2 = n_1$. If n_1 is odd, then Theorem 2.1 implies that W together with two vertices from X_1 is a strong edge geodetic set, while if n_1 is even, we need one more vertex next to W by Theorem 2.1(i).

It remains to prove the lower bounds. For this sake let U be a strong edge geodetic set of G. If $\cup_{i=2}^k X_i \subseteq U$, then the above argument also implies that $|U|$ is at least as stated in the theorem. Assume second that U does not contain all the vertices from X_i, where $i > 1$. To cover the edges between X_1 and X_i, we must use the paths from the corresponding induced subgraph K_{n_i,n_1}. By Lemma 3.1, we need $n_i + 1$ vertices from this subgraph if n_i is even, and $n_i + 2$ vertices from this subgraph if n_i is odd. By a simple case analysis we see that in every case ($n_i = n_1$, $n_i = n_1+1$, or $n_i \geq n_1+2$) we have $|U| \geq |W|$, where W is the strong edge geodetic set from the previous paragraph. $\square$

4 Graph with large strong edge geodetic sets

In this section we first characterize graphs G with ${\rm sg_e}(G) = n(G)$. After that we consider graphs G with ${\rm sg_e}(G) = n(G) - 1$ and determine ${\rm sg_e}(P_n \,\square\, K_m)$. In particular, ${\rm sg_e}(P_2 \,\square\, K_m) = 2m - 1$, which corrects a result from [25].

Let G be a graph and $uv \in E(G)$. Then we say that a vertex v is a *dominant neighbor* of u if $N[u] \subseteq N[v]$. Vertices u and v of a graph G are *twins* if $N[u] = N[v]$. Note that twins are necessarily adjacent and that if u and v are twins, then u is a dominant neighbor of v and v is a dominant neighbor of u.

The following lemma seems to be of independent interest.

Lemma 4.1 *Let G be a graph and $U \subseteq V(G)$ a strong edge geodetic set. If v is a dominant neighbor of u, then $u \in U$. In particular, if u and v are twin vertices, then $u \in U$ and $v \in U$.*

Proof. Let $uv \in E(G)$ and $N[u] \subseteq N[v]$. If P is a shortest path in G which contains the edge uv, then one of the end-points of P must be u, for otherwise P would not be shortest. If further u and v are twins, then also $N[v] \subseteq N[u]$ and thus also $v \in U$. $\square$

Proposition 4.2 *Let G be a graph. Then $\mathrm{sg_e}(G) = n(G)$ if and only if every vertex of G has a dominant neighbor.*

Proof. If every vertex of G has a dominant neighbor, then every vertex lies in every strong edge geodetic set by Lemma 4.1. Hence $\mathrm{sg_e}(G) = n(G)$.

Assume now that a vertex $u \in V(G)$ does not admit a dominant neighbor. We claim that $U = V(G) \setminus \{u\}$ is a strong edge geodetic set of G. Let v be an arbitrary neighbor of u. Since $N[u] \not\subseteq N[v]$, there exists a vertex $w \in N[u] \setminus N[v]$. To cover the edge uv, put the shortest path wuv to $\widehat{U}$. Proceed analogously for every neigbor v' of u, where if the edge $v'u$ has been already covered before, do nothing. In this way all edges incident with u are covered. Let next xy be an arbitrary edge from $E(G)$ where $\{x,y\} \cap \{u\} = \emptyset$. Then add to $\widehat{U}$ the shortest path xy. Clearly, the paths added so far to $\widehat{U}$ cover all the edges of G and we conclude that $\mathrm{sg_e}(G) < n(G)$. $\square$

Proposition 4.2 implies several results from [25] as for instance [25, Theorem 8] which asserts that if a graph G contains at least two universal vertices, then $\mathrm{sg_e}(G) = n(G)$.

A vertex u of a graph G is *simplicial* if $N(u)$ induces a clique of G. If u is a simplicial vertex and v its arbitrary neighbor, then $N[u] \subseteq N[v]$. Denoting by $s(G)$ the number of simplicial vertices of G Lemma 4.1 thus implies:

Corollary 4.3 *If G is a graph, then $\mathrm{sg_e}(G) \geq s(G)$.*

Since in K_n every vertex is simplicial, Corollary 4.3 implies that $\mathrm{sg_e}(K_n) = n$. We can also deduce this fact from Lemma 4.1 by observing that each pair of vertices of K_n are twins.

Lemma 4.1 implies also the following.

Corollary 4.4 *If a graph G contains a universal vertex, then $\mathrm{sg_e}(G) \geq n(G) - 1$. Moreover, if there is only one universal vertex, then $\mathrm{sg_e}(G) = n(G) - 1$.*

Proof. Let w be a universal vertex of G. Then w is a dominant neighbor of every vertex $u \in V(G) \setminus \{w\}$, hence Lemma 4.1 implies that $V(G) \setminus \{w\} \subseteq U$ for every strong edge geodetic set U of G. Thus $\mathrm{sg_e}(G) \geq n(G) - 1$. In the case when w is a unique universal vertex of G, then with the same arguments as we had in the last part of the proof of Proposition 4.2 we infer that $V(G) \setminus \{w\}$ is a strong edge geodetic set. Hence $\mathrm{sg_e}(G) \leq n(G) - 1$ when G has a unique universal vertex, so that in this case $\mathrm{sg_e}(G) = n(G) - 1$. $\square$

The second assertion of Corollary 4.4 was earlier presented as [25, Theorem 5]. Moreover, in [25, Theorem 5] it was also claimed that $\mathrm{sg_e}(P_2 \,\square\, K_m) = 2m - 2$. If can be checked that the result is not true and that instead the Cartesian products $P_2 \,\square\, K_m$ also belong to the family of graphs G for which $\mathrm{sg_e}(G) \geq n(G) - 1$. More generally, we have the following result.

Theorem 4.5 *If $m \geq 3$ and $n \geq 2$, then*

$$\operatorname{sg_e}(P_n \,\square\, K_m) = \begin{cases} mk; & n = k^2, \\ mk + (m-1); & n = k^2 + h, 1 \leq h \leq k, \\ mk + m; & n = k^2 + h, k+1 \leq h \leq 2k. \end{cases}$$

Proof. Set $V(K_m) = [m]$ and $V(P_n) = [n]$ where $i(i+1) \in E(P_n)$ for $i \in [n-1]$. If $y \in V(K_m)$, then we will denote by P_n^y the subgraph of $P_n \,\square\, K_m$ induced by the vertices (i, y), $i \in [n]$. P_n^y is also called a P_n-layer of $P_n \,\square\, K_m$ and is isomorphic to P_n.

Consider first the case $n = k^2$, where $k \in \mathbb{N}$. In this case we claim that $U_1 = \bigcup_{i=1}^{k} \bigcup_{j=1}^{m} \{(i^2, j)\}$ is a strong edge geodetic set of $P_n \,\square\, K_m$. To cover all the edges of $P_n \,\square\, K_m$ we proceed as follows. For every pair $y_1, y_2 \in V(K_m)$, $y_1 < y_2$, we put the following shortest paths to $\widehat{U_1}$.

- For every $i \in [k]$, put to $\widehat{U_1}$ the unique shortest path (of length 1) between the vertices (i^2, y_1) and (i^2, y_2).
- For every $2 \leq i \leq k$, and for every $l \in [i-1]$, put to $\widehat{U_1}$ the shortest path between the vertices (l^2, y_1) and (i^2, y_2) that contains the edge $((i-1)^2 + l, y_1)((i-1)^2 + l, y_2)$, and the shortest path between the vertices (l^2, y_2) and (i^2, y_1) that passes through the edge $(i(i-1) + l, y_1)(i(i-1) + l, y_2)$.
- For every $j \in [m]$, put to $\widehat{U_1}$ the unique shortest path between the vertices $(1, j)$ and (n, j).

The shortest paths from $\widehat{U_1}$ cover all the edges of $P_n \,\square\, K_m$, hence we can conclude that $\operatorname{sg_e}(P_n \,\square\, K_m) \leq mk$ when $n = k^2$.

Assume next that $n = k^2 + h$, where $h \in [k]$. The we claim that the set $U_2 = U_1 \cup \bigcup_{j=2}^{m} \{(n, j)\}$ is a strong edge geodetic set of $P_n \,\square\, K_m$.

- First put all the shortest paths from $\widehat{U_1}$ to $\widehat{U_2}$.
- For every pair $y_1, y_2 \in V(K_m)$, where $y_1 < y_2$, and for every $i \in [h]$, put to $\widehat{U_2}$ the shortest path between the vertices (i^2, y_1) and (n, y_2) that contains the edge $(k^2 + i, y_1)(k^2 + i, y_2)$.
- For every $y \in \{2, \ldots, m\}$, put to $\widehat{U_2}$ the unique shortest path between the vertices $(1, y)$ and (n, y). Note that all the edges from P_n^1 are already covered by the shortest path from $\widehat{U_2}$ between vertices $(h^2, 1)$ and $(n, 2)$.

Since the shortest paths from $\widehat{U_2}$ cover all the edges of $P_n \,\square\, K_m$, we can conclude that $\mathrm{sg_e}(P_n \,\square\, K_m) \le mk + (m-1)$, when $n = k^2 + h$ and $h \in [k]$.

Assume finally that $n = k^2 + h$, where $k + 1 \le h \le 2k$. In this case we claim that $U_3 = U_1 \cup \bigcup_{j=1}^{m} \{(n, j)\}$ is a strong edge geodetic set of $P_n \,\square\, K_m$ and proceed as follows.

- Put all the shortest paths from $\widehat{U_1}$ to $\widehat{U_3}$.
- For every pair $y_1, y_2 \in V(K_m)$, where $y_1 < y_2$, do the following. For every $i \in [k]$, put to $\widehat{U_3}$ the shortest path between vertices (i^2, y_1) and (n, y_2) that contains the edge $(k^2 + i, y_1)(k^2 + i, y_2)$. Moreover, for every $i \in [h - k]$ also add the shortest path between the vertices (n, y_1) and (i^2, y_2) that contains the edge $(k(k + 1) + i, y_1)(k(k + 1) + i, y_2)$.
- For every $j \in [m]$, put to $\widehat{U_3}$ the unique shortest path between the vertices $(1, j)$ and (n, j).

Since the shortest paths from $\widehat{U_3}$ cover all the edges of $P_n \,\square\, K_m$, we get the upper bound $\mathrm{sg_e}(P_n \,\square\, K_m) \le mk + m$ when $n = k^2 + h$ with $k + 1 \le h \le 2k$.

In the second part of the proof we need to demonstrate that the obtained upper bounds are sharp, that is, there exist no smaller strong edge geodetic sets as the one constructed above. Let U be a arbitrary strong edge geodetic set of $P_n \,\square\, K_m$.

Assume first that $n = k^2$ for some $k \in \mathbb{N}$. Then we need to show that $|U| \ge mk$. If for every vertex $y \in K_m$, the set U has at least k vertices in the P_n^y-layer, then clearly $|U| \ge mk$. Assume therefore that for some $y_i \in V(K_m)$, the $P_n^{y_i}$- layer contains $k - l$, $l \ge 1$, vertices from U. Since $|V(P_n^{y_i}) \cap U| = k - l$, for every vertex $y \in V(K_m)$, $y \ne y_i$, the strong edge geodetic set U has to have at least x vertices from P_n^y, where $(k - l)x \ge k^2$, in order to cover all the edges between $P_n^{y_i}$ and P_n^y. Because $x \ge k^2/(k - l) = k + kl/(k - l) \ge k + kl/k = k + l$, we get

$$|U| \ge k - l + (m - 1)(k + l) = mk + (m - 2)l \ge mk + 1\,,$$

where the last assertion follows since $m \ge 3$. We conclude that in any case, $|U| \ge mk$ and $l \ge 1$.

Assume second that $n = k^2 + h$, where $1 \le h \le k$. Now we need to prove that $\mathrm{sg_e}(P_n \,\square\, K_m) \ge m(k + 1) - 1$. If for every vertex $y \in K_m$, the set U has at least $k + 1$ vertices in P_n^y, then clearly $|U| \ge m(k + 1)$ and we are done. Assume therefore that for some $y_i \in V(K_m)$, the set U has $(k + 1) - l$, $l \ge 1$, vertices from $P_n^{y_i}$. Since $|V(P_n^{y_i}) \cap U| \le k + 1 - l$, for every vertex $y \in V(K_m)$, $y \ne y_i$, the set U has to have at least x vertices from P_n^y, where $(k + 1 - l)x \ge k^2 + h$ has to hold in order to cover all the edges between $P_n^{y_i}$ and P_n^y. Because x is an integer we can compute as follows:

$$\begin{aligned}
x &\geq \left\lceil \frac{k^2+h}{k+1-l} \right\rceil = \left\lceil \frac{k(k+h/k)}{k+1-l} \right\rceil = \left\lceil \frac{k(k+1-l)+k(h/k-1+l)}{k+1-l} \right\rceil \\
&= k + \left\lceil \frac{k(h/k-1+l)}{k+1-l} \right\rceil .
\end{aligned}$$

Because $l \geq 1$ and therefore $1/(k+1-k) \geq 1/k$, we also have

$$x \geq k + \left\lceil \frac{k(h/k-1+l)}{k} \right\rceil = k + l - 1 + \left\lceil \frac{h}{k} \right\rceil .$$

Since $h \in [k]$, we have $\lceil h/k \rceil = 1$ and therefore $x \geq k + l$. Altogether,

$$\begin{aligned}
|U| &\geq k+1-l+(m-1)(k+l) = mk+(m-2)l+1 \\
&\geq mk+(m-2)+1 = m(k+1)-1
\end{aligned}$$

which we wanted to show.

The remaining case is when $n = k^2 + h$, where $k+1 \leq h \leq 2k$. Now we need to prove that $\mathrm{sg_e}(P_n \,\square\, K_m) \geq m(k+1)$. If for every vertex $y \in K_m$, the set U has at least $k+1$ vertices from P_n^y, then clearly $|U| \geq m(k+1)$. Assume therefore that for some $y_i \in V(K_m)$, the set U has $(k+1)-l$, $l \geq 1$, vertices from $P_n^{y_i}$. So $|V(P_n^{y_i}) \cap U| \leq k+1-l$, hence for every vertex $y \in V(K_m)$, $y \neq y_i$, the set U has to have at least x vertices from P_n^y, where $(k+1-l)x \geq k^2+h$ has to hold in order to cover all the vertical edges between $P_n^{y_i}$ and P_n^y. Because x is an integer and $l \geq 1$, we can similarly as in the previous case estimate that

$$x \geq k + l - 1 + \lceil h/k \rceil .$$

Because h is an integer between $k+1$ and $2k$, we have $\lceil h/k \rceil = 2$, and therefore $x \geq k+l+1$. Altogether we see that

$$\begin{aligned}
|U| &\geq k+1-l+(m-1)(k+l+1) = m(k+1)+(m-2)l \\
&\geq m(k+1)+1 ,
\end{aligned}$$

where the last assertion holds since $m \geq 3$ and $l \geq 1$. $\square$

The following special case of Theorem 4.5 has been reported earlier in [25, Theorem 14].

Corollary 4.6 *If $k \geq 2$ and $m \geq 3$, then $\mathrm{sg_e}(P_{k^2} \,\square\, K_m) = mk$.*

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