# Improved Bounds on the $L(2,1)$-Number of Direct and Strong Products of Graphs

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***Abstract*— The frequency assignment problem is to assign a frequency which is a nonnegative integer to each radio transmitter so that interfering transmitters are assigned frequencies whose separation is not in a set of disallowed separations. This frequency assignment problem can be modelled with vertex labelings of graphs. An $L(2,1)$-labeling of a graph $G$ is a function $f$ from the vertex set $V(G)$ to the set of all nonnegative integers such that $|f(x)-f(y)| \geq 2$ if $d(x,y)=1$ and $|f(x)-f(y)| \geq 1$ if $d(x,y)=2$, where $d(x,y)$ denotes the distance between $x$ and $y$ in $G$. The $L(2,1)$-labeling number $\lambda(G)$ of $G$ is the smallest number $k$ such that $G$ has an $L(2,1)$-labeling with $\max\{f(v) : v \in V(G)\} = k$. This paper considers the graph formed by the direct product and the strong product of two graphs and gets better bounds than those of [14] with refined approaches.**

***Index Terms*— channel assignment, $L(2,1)$-labeling, graph direct product, graph strong product**

## I. Introduction

THE frequency assignment problem is to assign a frequency which is a nonnegative integer to each radio transmitter so that interfering transmitters are assigned frequencies whose separation is not in a set of disallowed separations. Hale [10] formulated this into a graph vertex coloring problem.

In a private communication with Griggs, Roberts proposed a variation of the channel assignment problem in which "close" transmitters must receive different channels and "very close" transmitters must receive channels that are at least two channels apart. To translate the problem into the language of graph theory, the transmitters are represented by the vertices of a graph; two vertices are "very close" if they are adjacent and "close" if they are of distance 2 in the graph. Motivated by this problem, Griggs and Yeh [9] proposed the following labeling on a simple graph. An $L(2,1)$-*labeling* of a graph $G$ is a function $f$ from the vertex set $V(G)$ to the set of all nonnegative integers such that $|f(x)-f(y)| \geq 2$ if $d(x,y)=1$ and $|f(x)-f(y)| \geq 1$ if $d(x,y)=2$, where $d(x,y)$ denotes the distance between $x$ and $y$ in $G$. A $k$-$L(2,1)$-*labeling* is an $L(2,1)$-labeling such that no label is greater than $k$. The $L(2,1)$-*labeling number* of $G$, denoted by $\lambda(G)$, is the smallest number $k$ such that $G$ has a $k$-$L(2,1)$-labeling.

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From then on, a large number of articles have been published devoted to the study of the frequency assignment problem and its connections to graph labelings, in particular, to the class of $L(2,1)$-labelings and its generalizations: Over 100 references on the subject are provided in a very comprehensive survey [3]. In addition to graph theory and combinatorial techniques, other interesting approaches in studying these labelings include: neural networks [7], [15], genetic algorithms [18], and simulated annealing [5], [19]. Most of these papers are considering the values of $\lambda$ on particular classes of graphs.

From the algorithmic point of view it is not surprising that it is NP-complete to decide whether a given graph $G$ allows an $L(2,1)$-labeling of span at most $n$ [9]. Hence good lower and upper bounds for $\lambda$ are clearly welcome. For instance, if $G$ is a diameter 2 graph, then $\lambda(G) \leq \Delta^2$. The upper bound is attainable by *Moore graphs* (diameter 2 graph with order $\Delta^2+1$), see [9]. (Such graphs exist only if $\Delta = 2,3,7$, and possibly 57.)

The above considerations in particular motivated Griggs and Yeh [9] to conjecture that for any graph $G$ with the maximum degree $\Delta \geq 2$ the best upper bound on $\lambda(G)$ is $\Delta^2$. (Note that this is not true for $\Delta = 1$. For example, $\Delta(K_2)=1$ but $\lambda(K_2)=2$.) They provided an upper bound $\Delta^2+2\Delta$ for general graphs with maximum degree $\Delta$. Chang and Kuo [4] improved the bound to $\Delta^2+\Delta$ while Král' and Škrekovski [17] further reduced the bound to $\Delta^2+\Delta-1$. Furthermore, in 2005, Gonçalves [8] announced the bound $\Delta^2+\Delta-2$. At the present moment, a journal paper containing a proof of the claimed bound is still to be published.

Graph products play an important role in connecting various useful networks and they also serve as natural tools for different concepts in many areas of research. To justify the first assertion we mention that the diagonal mesh with respect to multiprocessor network is representable as the direct product of two odd cycles [22] while for the other assertion we recall that one of the central concepts of information theory, the Shannon capacity, is most naturally expressed with the strong product of graphs, cf. [23].

In [14], upper bounds and some explicit labelings for the direct product and the strong of graphs and proved which in particular implies that the $L$(2,1)-labeling number of the product graph is bounded by the square of its maximum degree. Hence Griggs and Yeh's conjecture holds in both cases (with some minor exception). The main purpose of this paper is to improve the upper bounds obtained in [14]. The main tool for this purpose is a more refined analysis of neighboorhoods in product graphs than the analysis in [14].

In the next section a heuristic labeling algorithm is presented that forms the basis for these considerations while in Sections 3 and 4 direct products and strong products of graphs are considered, respectively. IMprovements with respect to the previously known upper bounds are explicitly computed.

## II. A LABELING ALGORITHM

A subset $X$ of $V(G)$ is called an *i-stable set* (or *i-independent set*) if the distance between any two vertices in $X$ is greater than $i$. An 1-stable (independent) set is a usual independent set. A $maximal$ 2-stable subset $X$ of a set $Y$ is a 2-stable subset of $Y$ such that $X$ is not a proper subset of any 2-stable subset of $Y$.

Chang and Kuo [4] proposed the following algorithm to obtain an $L$(2,1)-labeling and the maximum value of that labeling on a given graph.

**Algorithm 2.1.**

**Input:** A graph $G = (V, E)$.

**Output:** The value $k$ is the maximum label.

**Idea:** In each step $i$, find a maximal 2-stable set from the unlabeled vertices that are distance at least two away from those vertices labeled in the previous step. Then label all the vertices in that 2-stable with $i$ in current stage. The label $i$ starts from 0 and then increases by 1 in each step. The maximum label $k$ is the final value of $i$.

**Initialization:** Set $X_{-1} = \emptyset$; $V = V(G)$; $i = 0$.

**Iteration:**

1) Determine $Y_i$ and $X_i$.
   - $Y_i = \{x \in V : x$ is unlabeled and $d(x, y) \geq 2$ for all $y \in X_{i-1}\}$.
   - $X_i$ a maximal 2-stable subset of $Y_i$.
   - If $Y_i = \emptyset$ then set $X_i = \emptyset$.
2) Label the vertices in $X_i$ (if there is any) by $i$.
3) $V \leftarrow V - X_i$.
4) If $V \neq \emptyset$, then $i \leftarrow i + 1$, go to Step 1.
5) Record the current $i$ as $k$ (which is the maximum label). Stop.

Thus $k$ is an upper bound on $\lambda(G)$. We would like to find a bound in terms of the maximum degree $\Delta(G)$ of $G$ analogous to the bound in terms of the chromatic number $\chi(G)$.

Let $x$ be a vertex with the largest label $k$ obtained by Algorithm 2.1. Set

$I_1 = \{i : 0 \leq i \leq k-1$ and $d(x, y) = 1$ for some $y \in X_i\}$,

$I_2 = \{i : 0 \leq i \leq k-1$ and $d(x, y) \leq 2$ for some $y \in X_i\}$,

and

$I_3 = \{i : 0 \leq i \leq k-1$ and $d(x, y) \geq 3$ for all $y \in X_i\}$.

It is clear that $|I_2| + |I_3| = k$.

For any $i \in I_3$, $x \notin Y_i$; otherwise $X_i \cup \{x\}$ is a 2-stable subset of $Y_i$, which contradicts the choice of $X_i$. That is, $d(x, y) = 1$ for some vertex $y$ in $X_{i-1}$; i.e., $i - 1 \in I_1$. So, $|I_3| \leq |I_1|$. Hence $k \leq |I_2| + |I_3| \leq |I_2| + |I_1|$.

In order to find $k$, it suffices to estimate $B = |I_1| + |I_2|$ in terms of $\Delta(G)$. We will investigate the value $B$ with respect to a particular graph. The notations which have been introduced in this section will also be used in the following sections.

## III. THE DIRECT PRODUCT OF GRAPHS

In this section, we obtain an upper bound for the $L(2, 1)$-labeling number of the direct product of two graphs in terms of the maximum degrees of the graphs involved.

The *direct product* $G \times H$ of two graphs $G$ and $H$ is the graph with vertex set $V(G) \times V(H)$, in which the vertex $(v, w)$ is adjacent to the vertex $(v', w')$ if and only if $v$ is adjacent to $v'$ and $w$ is adjacent to $w'$. See Figure 1 for an example.

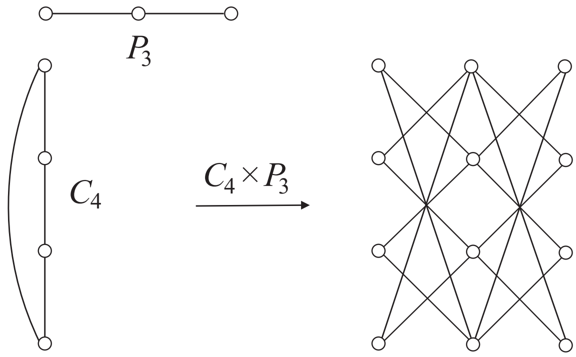


Fig. 1. Direct product $C_4 \times P_3$.

Suppose $G$ and $H$ are graphs with $\Delta(G) = 0$ or $\Delta(H) = 0$. Then, by the definition of the direct product, $G \times H$ contains no edges. Therefore we assume in the rest of this section that $\Delta(G) \geq 1$ and $\Delta(H) \geq 1$. Here is the main result of this section.

**Theorem 3.1:** Let $\Delta$, $\Delta_1$, and $\Delta_2$ be maximum degrees of $G \times H$, $G$, and $H$, respectively. Then

$$\lambda(G \times H) \leq \Delta^2 + \Delta - (\Delta_1 + \Delta_2)(\Delta_1 - 1)(\Delta_2 - 1).$$

*Proof:* Let $x = (u, v)$ in $V(G) \times V(H)$. Then $\deg_{G \times H}(x) = \deg_G(u)\deg_H(v)$. Denote $d = \deg_{G \times H}(x)$, $d_1 = \deg_G(u)$, $d_2 = \deg_H(v)$, $\Delta_1 = \Delta(G)$ and $\Delta_2 = \Delta(H)$. Hence $d = d_1 d_2$ and $\Delta = \Delta(G \times H) = \Delta_1 \Delta_2$.

For any vertex $u'$ in $G$ with distance 2 from $u$, there must be a path $u'u''u$ of length two between $u'$ and $u$ in $G$. REcall that the degree of $v$ in $H$ is $d_2$, i.e., $v$ has $d_2$ adjacent vertices in $H$. Then by the definition of a direct product $G \times H$, there must be $d_2$ internally-disjoint paths of length two between $(u', v)$ and $(u, v)$ in $G \times H$. Hence for any vertex in $G$ with distance 2 from $u$, there must be corresponding $d_2$ vertices in $G \times H$ with distance 2 from $x = (u, v)$ which are coincided in $G \times H$; on the other hand, whenever there is not such a vertex in $G$ with distance 2 from $u$ in $G$, there will never exist such corresponding $d_2$ vertices with distance 2 from $x = (u, v)$ which are coincided in $G \times H$. In the former case, since such $d_2$ vertices with distance 2 from $x = (u, v)$ are coincided in $G \times H$ and hence they can only be counted once, we have to subtract $d_2 - 1$ from the value $d(\Delta - 1)$ ( the number $d(\Delta - 1)$ is best possible); in the latter case, since there do not exist such $d_2$ vertices with distance 2 from $x = (u, v)$ which are coincided in $G \times H$ at all and hence they must be counted zero, we have to subtract $d_2$ from the value $d(\Delta - 1)$. Let the number of vertices in $G$ with distance 2 from $u$ be $t$, then $t \in [0, d_1(\Delta_1 - 1)]$. The minimum number we have to subtract from the value $d(\Delta - 1)$ in this sense occurs when

$t = d_1(\Delta_1 - 1)$ and we can get that in this sense the number of vertices with distance 2 from $x = (u, v)$ in $G \times H$ will decrease at least $d_1(\Delta_1 - 1)(d_2 - 1)$ from the value $d(\Delta - 1)$. (We should notice that the bound $d(\Delta - 1)$ includes the case $d_1(\Delta_1 - 1)d_2$.) See Figure 2 for the illustration of the above argument. In the figure $d(v)$ denotes the degree of $v$ in $H$, that is, $d(v) = d_2$.

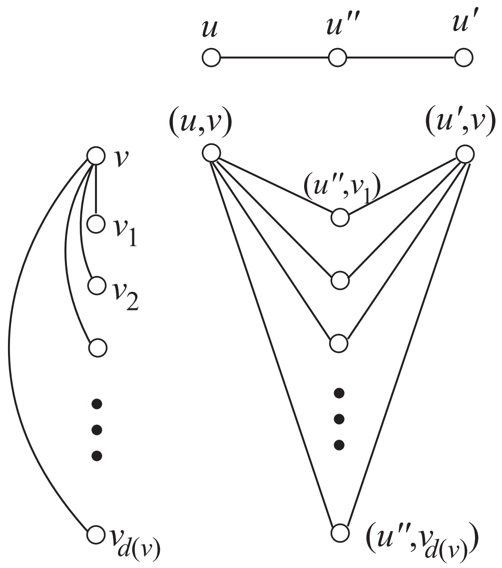


Fig. 2. Situation from the proof of Theorem 3.1.

By the commutativity of the direct product we also infer that the number of vertices of distance 2 from $x = (u, v)$ in $G \times H$ will still decrease $d_2(\Delta_2 - 1)(d_1 - 1)$ from the value $d(\Delta - 1)$. Hence the number of vertices with distance 2 from $x = (u, v)$ in $G \times H$ will decrease $d_1(\Delta_1 - 1)(d_2 - 1) + d_2(\Delta_2 - 1)(d_1 - 1)$ from the value $d(\Delta - 1)$ altogether.

Hence for the vertex $x$, the number of vertices with distance 1 from $x$ is not greater than $\Delta$, and the number of vertices with distance 2 from $x$ is not greater than $d(\Delta - 1) - d_1(\Delta_1 - 1)(d_2 - 1) - d_2(\Delta_2 - 1)(d_1 - 1) = d_1 d_2(\Delta_1\Delta_2 - 1) - d_1(\Delta_1 - 1)(d_2 - 1) - d_2(\Delta_2 - 1)(d_1 - 1)$.

Hence $|I_1| \le d$ and

$$\begin{aligned} |I_2| &\le d + d(\Delta - 1) - d_1(\Delta_1 - 1)(d_2 - 1) \\ &\quad - d_2(\Delta_2 - 1)(d_1 - 1) \\ &= d\Delta - d_1(\Delta_1 - 1)(d_2 - 1) - \\ &\quad d_2(\Delta_2 - 1)(d_1 - 1). \end{aligned}$$

Then

$$\begin{aligned} B &= |I_1| + |I_2| \\ &\le d + d\Delta - d_1(\Delta_1 - 1)(d_2 - 1) - \\ &\quad d_2(\Delta_2 - 1)(d_1 - 1) \\ &= d(\Delta + 1) - d_1(\Delta_1 - 1)(d_2 - 1) \\ &\quad - d_2(\Delta_2 - 1)(d_1 - 1) \\ &= d_1 d_2(\Delta_1\Delta_2 + 1) - d_1(\Delta_1 - 1)(d_2 - 1) - \\ &\quad d_2(\Delta_2 - 1)(d_1 - 1). \end{aligned}$$

Define

$$\begin{aligned} f(s,t) &= st(\Delta_1\Delta_2 + 1) - s(\Delta_1 - 1)(t - 1) - \\ &\quad t(\Delta_2 - 1)(s - 1). \end{aligned}$$

Then $f(s, t)$ has the absolute maximum at $(\Delta_1, \Delta_2)$ on $[0, \Delta_1] \times [0, \Delta_2]$ and its value is

$$f(\Delta_1, \Delta_2) = \Delta^2 + \Delta - (\Delta_1 + \Delta_2)(\Delta_1 - 1)(\Delta_2 - 1).$$

Then

$$\begin{aligned} \lambda(G \times H) &\le k \le B \\ &\le \Delta^2 + \Delta - (\Delta_1 + \Delta_2)(\Delta_1 - 1)(\Delta_2 - 1) \end{aligned}$$

and the proof is complete. ■

**Corollary 3.2:** Let $\Delta$ be the maximum degree of $G \times H$. Then $\lambda(G \times H) \le \Delta^2$ except if one of $\Delta(G)$ and $\Delta(H)$ is 1.

*Proof:* Suppose $\Delta_1 \ge 2$ and $\Delta_2 \ge 2$. Then

$$\begin{aligned} &(\Delta_1 + \Delta_2)(\Delta_1 - 1)(\Delta_2 - 1) - \Delta_1\Delta_2 \\ &= (\Delta_1 + \Delta_2 - 1)((\Delta_1 - 1)(\Delta_2 - 1) - 1) \ge 0. \end{aligned}$$

This implies

$$\begin{aligned} \lambda(G \times H) &\le k \le B \\ &\le \Delta^2 + \Delta_1\Delta_2 - (\Delta_1 + \Delta_2)(\Delta_1 - 1)(\Delta_2 - 1) \\ &\le \Delta^2. \end{aligned}$$

Therefore the result follows. ■

Note that when $\Delta_1$ and $\Delta_2$ are sufficiently large, $\Delta^2$ is a good aproximation for $(\Delta_1 - 1)(\Delta_2 - 1)$. Hence in such cases a good aproximation for the bound of Theorem 3.1 is given by

$$\Delta^2 + \Delta - (\Delta_1 + \Delta_2)\Delta = \Delta^2 - (\Delta_1 + \Delta_2 - 1)\Delta.$$

In [14] it is proved that

$$\begin{aligned} \lambda(G \times H) \quad &\le \quad \Delta^2 - \max\{(\Delta_1 - 1)^2(\Delta_2 - 1) - \Delta_1 \\ &\quad -\Delta_2 + 1, (\Delta_1 - 1)(\Delta_2 - 1)^2 - \Delta_1 - \Delta_2 + 1\}. \end{aligned} \tag{1}$$

We conclude the section by demonstrating that the upper bound of Theorem 3.1 is an improvement of (1).

Because

$$\begin{aligned} &\Delta^2 - \max\{(\Delta_1 - 1)^2(\Delta_2 - 1) - \Delta_1 - \Delta_2 + 1, \\ &(\Delta_1 - 1)(\Delta_2 - 1)^2 - \Delta_1 - \Delta_2 + 1\} \\ &\quad - (\Delta^2 + \Delta_1\Delta_2 - (\Delta_1 + \Delta_2)(\Delta_1 - 1)(\Delta_2 - 1)) \\ &= \min\{-(\Delta_1 - 1)^2(\Delta_2 - 1) + \Delta_1 + \Delta_2 - 1, \\ &\quad - (\Delta_1 - 1)(\Delta_2 - 1)^2 + \Delta_1 + \Delta_2 - 1\} - \Delta_1\Delta_2 \\ &\quad + (\Delta_1 + \Delta_2)(\Delta_1 - 1)(\Delta_2 - 1) \\ &= \min\{-(\Delta_1 - 1)^2(\Delta_2 - 1) + \Delta_1 + \Delta_2 - 1, \\ &\quad - (\Delta_1 - 1)(\Delta_2 - 1)^2 + \Delta_1 + \Delta_2 - 1\} \\ &\quad + (\Delta_1 + \Delta_2 - 1)((\Delta_1 - 1)(\Delta_2 - 1) - 1) \\ &= \min\{-(\Delta_1 - 1)^2(\Delta_2 - 1) + \\ &(\Delta_1 + \Delta_2 - 1)(\Delta_1 - 1)(\Delta_2 - 1), \\ &\quad - (\Delta_1 - 1)(\Delta_2 - 1)^2 \\ &\quad + (\Delta_1 + \Delta_2 - 1)(\Delta_1 - 1)(\Delta_2 - 1)\} \\ &= \min\{\Delta_2(\Delta_1 - 1)(\Delta_2 - 1), \Delta_1(\Delta_1 - 1)(\Delta_2 - 1)\}, \end{aligned}$$

we have thus reduced (1) by

$$\min\{\Delta_2(\Delta_1-1)(\Delta_2-1), \Delta_1(\Delta_1-1)(\Delta_2-1)\}.$$

## IV. THE STRONG PRODUCT OF GRAPHS

The *strong product* $G \boxtimes H$ of graphs $G$ and $H$ is the graph with vertex set $V(G) \times V(H)$, in which the vertex $(v, w)$ is adjacent to the vertex $(v', w')$ if and only if $v = v'$ and $w$ is adjacent to $w'$, or $w = w'$ and $v$ is adjacent to $v'$, or $v$ is adjacent to $v'$ and $w$ is adjacent to $w'$. See Figure 3 for an example.

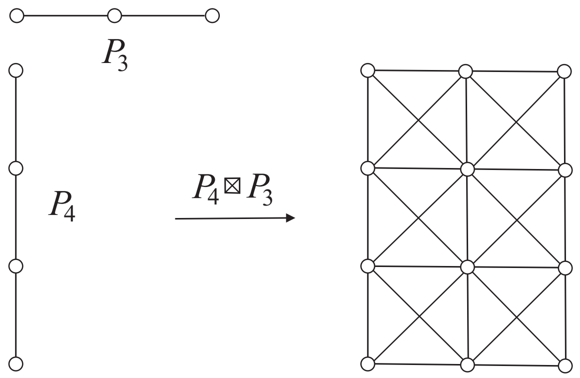


Fig. 3. Strong product $P_4 \boxtimes P_3$

By the definition of the strong product $G \boxtimes H$ of two graphs $G$ and $H$, if $\Delta(G) = 0$ or $\Delta(H) = 0$, then $G \boxtimes H$ consists of disjoint copies of $H$ or $G$. Thus $\lambda(G \boxtimes H) = \lambda(H)$ or $\lambda(G \boxtimes H) = \lambda(G)$. Therefore we assume $\Delta(G) \geq 1$ and $\Delta(H) \geq 1$.

In [13] and [16] the $\lambda$-numbers of the strong product of cycles are considered. In this section, we obtain a general upper bound for the $\lambda$-number of strong products in terms of maximum degrees of the factor graphs (and the product).

**Theorem 4.1:** Let $\Delta$, $\Delta_1$, and $\Delta_2$ be the maximum degree of $G \boxtimes H$, $G$, and $H$, respectively. Then

$$\lambda(G \boxtimes H) \leq \Delta^2 + \Delta - (\Delta_1 + \Delta_2 + 4)\Delta_1\Delta_2.$$

*Proof:* Let $x = (u, v)$ in $V(G) \times V(H)$. Then $\deg_{G\boxtimes H}(x) = \deg_G(u) + \deg_H(v) + \deg_G(u)\deg_H(v)$. Denote $d = \deg_{G\boxtimes H}(x)$, $d_1 = \deg_G(u)$, $d_2 = \deg_H(v)$, $\Delta_1 = \Delta(G)$ and $\Delta_2 = \Delta(H)$. Hence $d = d_1 + d_2 + d_1 d_2$ and $\Delta = \Delta(G \boxtimes H) = \Delta_1 + \Delta_2 + \Delta_1\Delta_2$.

For any vertex $u'$ in $G$ with distance 2 from $u$, there must be a path $u'u''u$ of length two between $u'$ and $u$ in $G$; but the degree of $v$ in $H$ is $d_2$, i.e., $v$ has $d_2$ adjacent vertices in $H$, by the definition of a strong product $G \boxtimes H$, there must be $d_2 + 1$ internally-disjoint paths of length two between $(u', v)$ and $(u, v)$. Hence for any vertex in $G$ with distance 2 from $u$, there must be corresponding $d_2 + 1$ vertices with distance 2 from $x = (u, v)$ which are coincided in $G \boxtimes H$; on the contrary whenever there is not such a vertex in $G$ with distance 2 from $u$ in $G$, there will never exist such corresponding $d_2+1$ vertices with distance 2 from $x = (u, v)$ which are coincided in $G \boxtimes H$. In the former case, since such $d_2 + 1$ vertices with distance 2 from $x = (u, v)$ are coincided in $G \boxtimes H$ and hence they can only be counted once, we have to subtract $d_2 + 1 - 1$ from the value $d(\Delta - 1)$ ( the number $d(\Delta - 1)$ is the best possible); on the latter case, since there do not exist such $d_2 + 1$ vertices with distance 2 from $x = (u, v)$ which are coincided in $G \boxtimes H$ at all and hence they must be counted zero, we have to subtract $d_2 + 1$ from the value $d(\Delta - 1)$. Let the number of vertices in $G$ with distance 2 from $u$ be $t$, then $t \in [0, d_1(\Delta_1 - 1)]$. The minimum number we have to subtract from the value $d(\Delta - 1)$ in this sense occurs when $t = d_1(\Delta_1 - 1)$ and we can get that in this sense the number of vertices with distance 2 from $x = (u, v)$ in $G \boxtimes H$ will decrease at least $d_1(\Delta_1 - 1)(d_2 + 1 - 1) = d_1(\Delta_1 - 1)d_2$ from the value $d(\Delta - 1)$. (We should notice that the bound $d(\Delta - 1)$ includes the case $d_1(\Delta_1 - 1)d_2$.) See Figure 4 for an example of the discussion in this paragraph, where $d(v)$ denotes the degree of $v$ in $H$, i.e., $d(v) = d_2$.

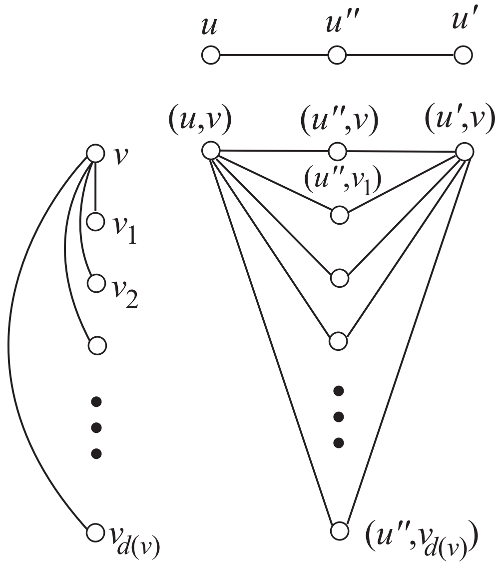


Fig. 4. Situation from the proof of Theorem 4.1

For $H$, we can analyze similarly and get that the number of vertices with distance 2 from $x = (u, v)$ in $G \boxtimes H$ will still decrease $d_2(\Delta_2 - 1)(d_1 + 1 - 1) = d_2(\Delta_2 - 1)d_1$ from the value $d(\Delta - 1)$. Hence the number of vertices with distance 2 from $x = (u, v)$ in $G \boxtimes H$ will decrease $d_1(\Delta_1 - 1)d_2 + d_2(\Delta_2 - 1)d_1 = (\Delta_1 + \Delta_2 - 2)d_1 d_2$ from the value $d(\Delta - 1)$ altogether. By the above analysis, the number $d(\Delta - 1) - (\Delta_1 + \Delta_2 - 2)d_1 d_2$ is now the best possible for the number of vertices with distance 2 from $x = (u, v)$ in $G \boxtimes H$.

Moreover, by the definition of the strong product we can again analyze as follows:

Denote $\varepsilon$, the number of edges of the subgraph $F$ induced by the neighbors of $x$. The edges of the subgraph $F$ induced by the neighbors of $x$ can be divided into the following three cases.

Case 1: Edges between $(u', v)$ and $(u, v')$, where $(u', u) \in E(G)$ and $(v', v) \in E(H)$. There are totally $d_1$ neighbors $(u', v)$ (where $u'$ is adjacent to $u$ in $G$) of $x = (u, v)$ and totally $d_2$ neighbors $(u, v')$ (where $v'$ is adjacent to $v$ in $H$) of $x = (u, v)$. Hence the number of edges of the subgraph $F$

induced by the neighbors of $x$ is at least $d_1d_2$. See Figure 5 for an example.

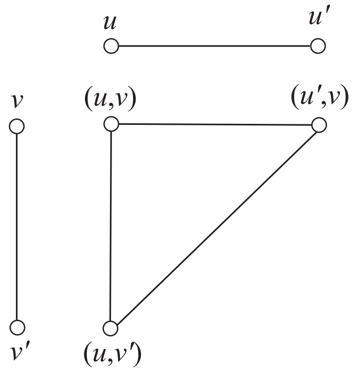


Fig. 5. Situation from the first case

Case 2: Edges between $(u', v')$ and $(u, v')$, where $(u', u) \in E(G)$ and $(v', v) \in E(H)$. There are totally $d_1d_2$ neighbors $(u', v')$ (where $u'$ is adjacent to $u$ in $G$ and $v'$ is adjacent to $v$ in $H$) of $x = (u, v)$. Hence the number of edges of the subgraph $F$ induced by the neighbors of $x$ should again add least $d_1d_2$ apart from the edges in case 1. See Figure 6 for an example.

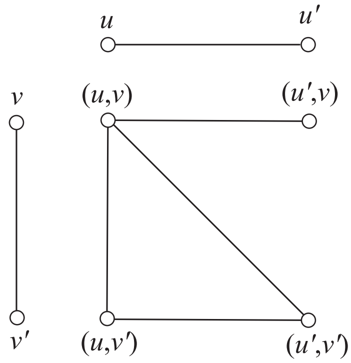


Fig. 6. Situation from the second case

Case 3: Edges between $(u', v')$ and $(u', v)$, where $(u', u) \in E(G)$ and $(v', v) \in E(H)$. There are totally $d_1d_2$ neighbors $(u', v')$ (where $u'$ is adjacent to $u$ in $G$ and $v'$ is adjacent to $v$ in $H$) of $x = (u, v)$. Hence the number of edges of the subgraph $F$ induced by the neighbors of $x$ should again add least $d_1d_2$ apart from the sum of the edges in case 1 and case 2. See Figure 7 for an example.

The present upper bound on the number of vertices at distance two from $x$ is $d(\Delta-1)-(\Delta_1+\Delta_2-2)d_1d_2$. For each edge in $F$, this upper bound is decreased by 2. Hence, by the analysis of the above three cases, the number of vertices with distance 2 from $x = (u, v)$ in $G \boxtimes H$ will still need to decrease by at least $6d_1d_2$. (the number $d(\Delta-1)-(\Delta_1+\Delta_2-2)d_1d_2$ is now the best possible for the number of vertices with distance 2 from $x = (u, v)$ in $G \boxtimes H$.)

Hence for the vertex $x$, the number of vertices with distance 1 from $x$ is not greater than $\Delta$. The number of vertices with

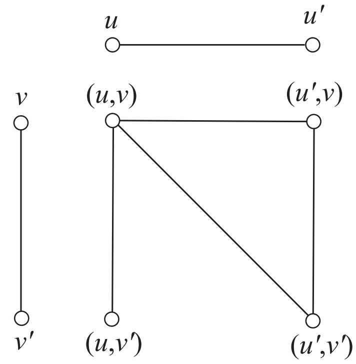


Fig. 7. Situation from the third case

distance 2 from $x$ is not greater than

$$\begin{aligned} & d(\Delta-1) - (\Delta_1+\Delta_2-2)d_1d_2 - 6d_1d_2 \\ & \quad = (d_1+d_2+d_1d_2)(\Delta_1+\Delta_2+\Delta_1\Delta_2-1) - \\ & \qquad (\Delta_1+\Delta_2+4)d_1d_2. \end{aligned}$$

Hence $|I_1| \le d$ and

$$\begin{aligned} |I_2| &\le d + d(\Delta-1) - (\Delta_1+\Delta_2+4)d_1d_2 \\ &= d\Delta - (\Delta_1+\Delta_2+4)d_1d_2. \end{aligned}$$

Then

$$\begin{aligned} B &= |I_1| + |I_2| \\ &\le d + d\Delta - (\Delta_1+\Delta_2+4)d_1d_2 \\ &= d(\Delta+1) - (\Delta_1+\Delta_2+4)d_1d_2 \\ &= (d_1+d_2+d_1d_2)(\Delta_1+\Delta_2+\Delta_1\Delta_2+1) - \\ & \quad (\Delta_1+\Delta_2+4)d_1d_2. \end{aligned}$$

Define

$$\begin{aligned} f(s,t) &= (s+t+st)(\Delta_1+\Delta_2+\Delta_1\Delta_2+1) - \\ & \quad (\Delta_1+\Delta_2+4)st. \end{aligned}$$

On $[0, \Delta_1] \times [0, \Delta_2]$ the function $f(s, t)$ has the absolute maximum at $(\Delta_1, \Delta_2)$ and the value $f(\Delta_1, \Delta_2)$ is equal

$$\begin{aligned} & (\Delta_1+\Delta_2+\Delta_1\Delta_2)(\Delta_1+\Delta_2+\Delta_1\Delta_2+1) - \\ & \quad (\Delta_1+\Delta_2+4)\Delta_1\Delta_2 \\ & = \Delta(\Delta+1) - (\Delta_1+\Delta_2+4)\Delta_1\Delta_2 \\ & = \Delta^2 + \Delta - (\Delta_1+\Delta_2+4)\Delta_1\Delta_2 \\ & = \Delta^2 + (\Delta_1+\Delta_2+\Delta_1\Delta_2) - (\Delta_1+\Delta_2+4)\Delta_1\Delta_2. \end{aligned}$$

Then

$$\begin{aligned} \lambda(G \boxtimes H) &\le k \le B \\ &\le \Delta^2 + \Delta - (\Delta_1+\Delta_2+4)\Delta_1\Delta_2 \end{aligned}$$

and we are done. ■

**Corollary 4.2:** Let $\Delta$ be the maximum degree of $G \boxtimes H$. Then $\lambda(G \boxtimes H) \le \Delta^2 - 3$ if both $\Delta(G)$ and $\Delta(H)$ are 1, otherwise $\lambda(G \boxtimes H) \le \Delta^2 - 9$.

*Proof:* Case 1. If both $\Delta_1$ and $\Delta_2$ are 1, then the connected components of $G \boxtimes H$ are $K_1$, $K_2$, and $K_4$, hence

$$\lambda(G \boxtimes H) = \lambda(K_4) = 6 = 9 - 3 = \Delta^2 - 3.$$

Case 2. Suppose at least one of $\Delta_1$ and $\Delta_2$ is greater than 1. Then

$$\begin{aligned}&(\Delta_1+\Delta_2+4-1)(\Delta_1\Delta_2-1)=\\&(\Delta_1+\Delta_2+4)\Delta_1\Delta_2-(\Delta_1+\Delta_2+\Delta_1\Delta_2+4)+1\\&\geq 6\times 1=6.\end{aligned}$$

This implies

$$(\Delta_1+\Delta_2+4+\Delta_1\Delta_2)-(\Delta_1+\Delta_2+4)\Delta_1\Delta_2\leq -5.$$

Hence

$$\begin{aligned}\lambda(G\boxtimes H)&\leq k\leq B\\&\leq \Delta^2+(\Delta_1+\Delta_2+\Delta_1\Delta_2)-\\&\quad(\Delta_1+\Delta_2+4)\Delta_1\Delta_2\\&= \Delta^2+(\Delta_1+\Delta_2+4+\Delta_1\Delta_2)-\\&\quad(\Delta_1+\Delta_2+4)\Delta_1\Delta_2-4\\&\leq \Delta^2-9\end{aligned}$$

and the proof is complete. ■

In [14] it is proved that

$$\lambda(G\boxtimes H)\leq \Delta^2+\Delta_1+\Delta_2-5\Delta_1\Delta_2.$$

Because

$$\begin{aligned}&\Delta^2+\Delta_1+\Delta_2-5\Delta_1\Delta_2-(\Delta^2+\Delta_1+\Delta_2+\Delta_1\Delta_2\\&-(\Delta_1+\Delta_2+4)\Delta_1\Delta_2)\\&=-6\Delta_1\Delta_2+(\Delta_1+\Delta_2+4)\Delta_1\Delta_2\\&=(\Delta_1+\Delta_2-2)\Delta_1\Delta_2,\end{aligned}$$

we reduce the bound by $(\Delta_1+\Delta_2-2)\Delta_1\Delta_2$.

## REFERENCES


[1] S. S. Adams, J. Cass and D. S. Troxell, "An Extension of the Channel-Assignment Problem: $L(2,1)$-Labelings of Generalized Petersen Graphs," *IEEE Trans. Circuits & Systems I: Regular Papers*, **53** (2006), 1101–1107.

[2] H. L. Bodlaender, T. Kloks, R. B. Tan and J. v. Leeuwen, "$\lambda$-coloring of graphs," *Lecture Notes in Computer Science*, 1770, Springer-Verlag Berlin, Heidelberg, (2000), 395–406.

[3] T. Calamoneri, "The $L(h,k)$-Labelling Problem: A Survey and Annotated Bibliography," *The Computer Journal*, **49**, (2006) 585–608.

[4] G. J. Chang and D. Kuo, "The $L(2,1)$-labeling on graphs," *SIAM J. Discrete Math.* **9**, (1996), 309–316.

[5] M. Duque-Anton, N. Kunz and B. Ruber, "Channel assignment for cellular radio using simulated annealing," *IEEE Trans. On Vehicular Technology*, **42**, (1993), 14–21.

[6] J. Fiala, T. Kloks and J. Kratochvíl, "Fixed-parameter complex of $\lambda$-labelings," *Lecture Notes in Computer Science*, 1665, Springer-Verlag Berlin, Heidelberg, (1999), 350–363.

[7] N. Funabiki and Y. Takefuji, "A neural network parallel algorithm for channel assignment problem in cellular radio networks," *IEEE Trans. On Vehicular Technology*, **41**, (1992), 430–437.

[8] D. Gonçalves, "On the $L(p,1)$-labelling of graphs," in 2005 European Conference on Combinatorics, Graph Theory and Applications (EuroComb '05), Stefan Felsner (ed.), Discrete Mathematics and Theoretical Computer Science Proceedings AE, pp. 81–86.

[9] J. R. Griggs and R. K. Yeh, "Labeling graphs with a condition at distance two," *SIAM J. Discrete Math.* **5**, (1992), 586–595.

[10] W. K. Hale, "Frequency assignment: Theory and application," *Proc. IEEE*, **68**, (1980), 1497–1514.

[11] W. Imrich and S. Klavžar, *Product Graphs: Structure and Recognition*, Wiley, New York, 2000.

[12] P.K. Jha, "Optimal $L$(2,1)-labelings of Cartesian products of cycles with an application to independent domination," *IEEE Trans. Circuits & Systems I: Fundamental Theory and Appl.* **47**, (2000), 1531–1534.

[13] P.K. Jha, "Optimal $L$(2,1)-labeling of strong product cycles," *IEEE Trans. Circuits & Systems I: Fundamental Theory and Appl.* **48**, (2001), 498–500.

[14] S. Klavžar and S. Špacapan, "The $\Delta^2$-conjecture for L(2,1)-labelings is true for direct and strong products of graphs," *IEEE Trans. Circuits and Systems II: Express Briefs* **53**, (2006), 274–277.

[15] N. Kunz, "Channel assignment for cellular radio using neural networks," *IEEE Trans. On Vehicular Technology*, **40**, (1992), 188–193.

[16] D. Korže and A. Vesel, "$L(2,1)$-labeling of strong products of cycles," *Information Processing Letters*, **94**, (2005), 183–190.

[17] D. Král' and R. Škrekovski, "A theorem about channel assignment problem," *SIAM J. Discrete Math.* **16**, (2003), 426–437.

[18] W.K. Lai and G.G. Coghill, "Channel assignment through evolutionary optimization," *IEEE Trans. On Vehicular Technology*, **45**, (1996), 91–96.

[19] R. Mathar, "Channel assignment in cellular radio networks," *IEEE Trans. On Vehicular Technology*, **42**, (1993), 647–656.

[20] F. S. Roberts, "T-colorings of graphs: Recent results and open problems," *Discrete Math.*, **93**, (1991), 229–245.

[21] Z. Shao and R. K. Yeh, "The $L(2,1)$-labeling and operations of graphs," *IEEE Trans. Circuits & Systems I: Regular Papers*, **52**, (2005), 668–671.

[22] K. W. Tang and S. A. Padubirdi, "Diagonal and toroidal mesh networks," *IEEE Trans. Comput.*, **43**, (1994), 815–826.

[23] A. Vesel and J. Žerovnik, "Improved lower bound on the Shannon capacity of $C_7$," *Inform. Proc. Lett.*, **81**, (2002), 277–282.