Some new bounds and exact results on the independence number of Cartesian product graphs

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Abstract

The independence number of Cartesian product graphs is considered. An upper bound is presented that covers all previously known upper bounds. A construction is described that produces a maximal independent set of a Cartesian product graph and turns out to be a reasonably good lower bound for the independence number. The construction defines an invariant of Cartesian product graphs that is compared with its independence number. Several exact independence numbers of products of bipartite graphs are also obtained.

1 Introduction

The independence number $\alpha(G)$ of a graph G is one of the most important graph invariants, and, as usual, the problem of determining it is NP-hard. On the other hand, Sabidussi-Vizing's theorem [15, 16] asserts that every connected graph has a unique prime factor decomposition with respect to the Cartesian product. Moreover, the decomposition can be found in almost linear time [2], cf. also [6, 17]; for more information on the decomposition and related algorithms see [10]. It is therefore reasonable to investigate the independence number of Cartesian product graphs as a function of their factors. The first results in this direction are due to Vizing [16], while recently several studies on the topic appeared [4, 7, 13, 14], see also [5].

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(For related and conceptually similar studies of the independence number of the direct product of graphs see [11, 12].) The independence number of Cartesian product graphs is also the key notion in the investigations of the so-called chromatic difference sequences of graphs [1, 18, 19] as well as it is the key notion for the ultimate independence ratio of a graph; for a graph G the latter is defined as $\lim_{k\to\infty} \alpha(G^k)/|V(G)|$, cf. [8, 9, 20]. For instance, the independence number of powers of the Petersen graph are computed in [1].

In the next section a conceptually simple upper bound on the independence number of the Cartesian product is given that covers all previously known upper bounds. Four such upper bounds are obtained as corollaries. We follow with a section containing a construction, called a diagonal procedure, that always produces a maximal independent set of a Cartesian product graph. Exact independence numbers of the Cartesian product of a large class of caterpillars are also obtained. In Section 4 we concentrate on bipartite graphs. Three typical approaches are described that yield lower bounds for the independence number: a bipartite approach, a greedy approach, and an alternative approach. A large class of graphs is described for which the bipartite approach is optimal. More exact independence numbers are also obtained and bipartite graphs that attain Vizing's upper bound are treated. We follow with a section where the independence number of products is compared with the invariant induced by the construction presented in Section 3. We conclude with some open problems.

The size of a largest (or maximum) independent set of vertices of a graph G is called the *independence number* of G and denoted $\alpha(G)$. An independent set S of G with $|S| = \alpha(G)$ is called an α-set of G. An independent set is called *maximal* if it is not contained in a larger independent set. Let $\overline{\alpha}(G) = \max\{\alpha(G \setminus S)\}$, where the maximum is taken over all α-sets S of G. With $\alpha_k(G)$ we will denote the size of a largest k-colorable subgraph of G, that is, the size of its largest k-independent set. The size of a largest independent set of edges of a graph G is called the *matching number* of G and denoted $\tau(G)$.

The *Cartesian product* $G \Box H$ of graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$ is the graph with vertex set $V(G) \times V(H)$ where vertex (a, x) is adjacent to vertex (b, y) whenever $ab \in E(G)$ and $x = y$, or $a = b$ and $xy \in E(H)$. G^n denotes the Cartesian product of n copies of G. For a fixed vertex a of G, the vertices $\{(a, x) \mid x \in V(H)\}$ induce a subgraph of $G \Box H$ isomorphic to H. We call it an *H-layer* and denote it by H^a. Analogously we define G-layers. The Cartesian product is commutative, associative and K_1 is a unit. Also, $G \Box H$ is connected if and only if both G and H are connected. (For more information on the Cartesian product of graphs see [10].) We may therefore assume that all the graphs considered are connected, as well as finite undirected graphs without loops or multiple edges.

A tree is a called a *caterpillar* if a path remains after the removal of all its pendant vertices. This path is called the *spine* of the caterpillar. Let $u_1, u_2, \ldots u_k$ be the consecutive vertices of the spine of a caterpillar C and suppose that u_i is adjacent to x_i (pendant) vertices not in the spine. Then we will write $C = C(x_1, x_2, \ldots, x_k)$. Note that $K_{1,k} = C(k)$.

2 Upper bounds

The main result of this section is the following upper bound on the independence number of Cartesian product graphs. The result as such is rather straightforward, however—a bit surprisingly—it implies all previously known upper bounds, which we demonstrate after. For a graph G and $X \subseteq V(G)$, let $\langle X \rangle$ denote the subgraph induced by X.

Theorem 2.1 *Let H be a graph and let $\{V_1, V_2, \ldots, V_k\}$ be a partition of $V(H)$. Then for any graph G,*

$$\alpha(G \Box H) \leq \sum_{i=1}^{k} \alpha(G \Box \langle V_i \rangle) .$$

Proof. Set $H_i = \langle V_i \rangle$, $1 \leq i \leq k$, and let I be an α-set of $G \Box H$. Then $I_i = I \cap (G \Box H_i)$ is an independent set of $G \Box H_i$ and hence $|I_i| \leq \alpha(G \Box H_i)$. Therefore,

$$\alpha(G \Box H) = |I| = \sum_{i=1}^{k} |I_i| \leq \sum_{i=1}^{k} \alpha(G \Box H_i) .$$

□

The simplest partition of the vertex set of a graph is the one formed by its vertices. Then for any vertex u of G, $\alpha(G \Box \langle \{u\} \rangle) = \alpha(G \Box K_1) = \alpha(G)$, hence we get:

Corollary 2.2 ([16, Vizing, 1963]) *For any graphs G and H,*

$$\alpha(G \Box H) \leq \min\{\alpha(G)\, |V(H)|,\ \alpha(H)\, |V(G)|\} .$$

This bound was also (independently) observed in [9] in order to show that the limit in the definition of the ultimate independence ratio of a graph always exists.

Another special case of Theorem 2.1 is the following. Recall from [3] that for any graph G, $\alpha(G \Box K_k) = \alpha_k(G)$. Then we have:

Corollary 2.3 *Let H be a graph, and let $H_1, H_2, \ldots, H_k$ be a clique vertex-cover of H, where $|H_i| = q_i$. Then for any graph G,*

$$\alpha(G \Box H) \leq \sum_{i=1}^{k} \alpha_{q_i}(G).$$

Corollary 2.3 is from [5, Theorem 3.1], although its formulation there is slightly different. The special case of clique vertex-covers is when we cover H with $\tau(H)$ edges and $|H| - 2\tau(H)$ vertices. Then we obtain the following corollary from [7], where it is stated for the case when H is bipartite.

Corollary 2.4 *For any graph G and H,*

$$\alpha(G \Box H) \leq \tau(H)\, \alpha_2(G) + (|H| - 2\tau(H))\, \alpha(G).$$

Recently Martin, Powell, and Rall [14, Theorem 4.4] obtained an upper bound on the Cartesian product of two caterpillars using the following special partition. Let $G = C(x_1, \ldots, x_k)$. Then they cover G with (i) the stars containing vertices of the spine with at least two pendant vertex, (ii) the edges containing vertices of the spine with precisely one pendant vertex, and (iii) the remaining paths of G. Using such a cover they showed that for any $H = C(y_1, \ldots, y_s)$, $y_i \geq 1$,

$$\alpha(G \Box H) \leq \alpha(H) \sum_{i \in I} x_i + |I|\, \overline{\alpha}(H) + |J|\, |H| + \Xi(G, H),$$

where I and J are the sets of the spine vertices of G with at least two pendant vertices and with one pendant vertex, respectively, and $\Xi(G, H)$ takes care for the spine vertices with no pendant vertices. For our purposes the following special case of their result will be useful:

Corollary 2.5 *Let $G = C(x_1, \ldots, x_k)$, $x_i \geq 2$, and $H = C(y_1, \ldots, y_s)$, $y_i \geq 1$. Then*

$$\alpha(G \Box H) \leq \alpha(G)\, \alpha(H) + \tau(G)\, \overline{\alpha}(H).$$

Proof. As $x_i \geq 2$, $|I| = k$, $J = \emptyset$, and $\Xi(G, H) = 0$. In addition, $x_i \geq 2$ clearly implies that $\sum_{i \in I} x_i = \alpha(G)$. Finally, note that $|I| = \tau(G)$. □

3 Lower bounds from the diagonal procedure

We are going to describe a procedure that yields a maximal independent set of the Cartesian product of two graphs. It seems that in many instances, especially if the factor graphs are bipartite, the procedure returns the independence number of the product.

Let G and H be graphs. Set $G_1 = G$, $H_1 = H$, and let A_1 and B_1 be *maximal* independent sets of G_1 and H_1 respectively. Continue by setting $G_2 = G \setminus A_1$, $H_2 = H \setminus B_1$, and selecting maximal independent sets A_2 and B_2 of G_2 and H_2. We continue by the procedure, until we arrive at graphs G_k, H_k and sets A_k, B_k such that $V(G_k) = A_k$ or $V(H_k) = B_k$. The set $\cup_{i=1}^{k}(A_i \times B_i)$ obtained in the above procedure is clearly an independent set of $G \square H$ and will be called a *maximal diagonal set* of $G \square H$. The procedure itself will be called a *diagonal procedure.*

Let $G = C(3,0,0,3,2,0,0,1)$ and $H = C(5,4,5,5)$. It was shown in [14] that $\alpha(G \square H) \geq 217$. Consider now the following maximal diagonal set of $G \square H$: take maximum independent sets of G and H (of size 11 and 19), then in the remaining graphs again such sets (of size 4 and 2) and finally do again in the remaining graphs. The obtained independent set is schematically show in Fig. 1. In this way we improve the bound 217 to:

$$\alpha(G \square H) \geq 11 \cdot 19 + 4 \cdot 2 + 2 \cdot 2 = 221\,.$$

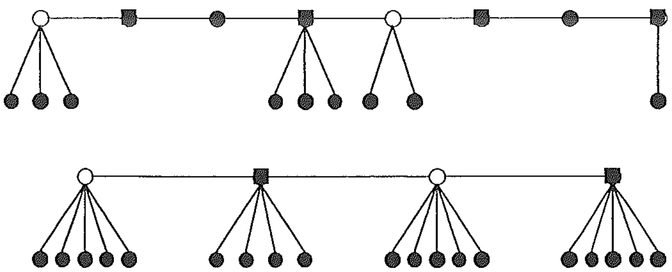

Figure 1: Maximal diagonal set in $C(3,0,0,3,2,0,0,1) \square C(5,4,5,5)$ of size 221

Note that the diagonal procedure is nondeterministic as in any step we may select any maximal independent sets of the remaining graphs. However, no matter how we make the selection, we have:

Proposition 3.1 *A maximal diagonal set of $G \square H$ is a maximal independent set of $G \square H$.*

Proof. We have already observed that a maximal diagonal set is an independent set of $G \square H$. It remains to show that it is maximal.

Let $X = \cup_{i=1}^{k}(A_i \times B_i)$ be a maximal diagonal set of $G \square H$ and assume without loss of generality that $V(G_k) = A_k$. Let (u, v) be a vertex of $G \square H$ not in X, where $u \in A_i$ and $v \in B_j$. Clearly, $i \neq j$. Assume $i > j$. Since A_j is maximal in G_j, u is adjacent to at least one vertex of A_j, say x. It follows that (u, v) is adjacent to $(x, v) \in X$. Analogously we find a neighbor of (u, v) in X when $i < j$. Finally, consider a vertex (if it exists) (u, v) with $u \in A_i$ and $v \in V(G_k) \setminus B_k$. Then as B_i is maximal in H_i, u is adjacent to at least one vertex of H_i, say y. But then (u, v) is adjacent to $(u, y) \in X$. □

Proposition 3.1 is proved in [13] for the case when in the diagonal procedure the sets A_i and B_i are selected as largest independent sets. However, as pointed out in [14], selecting maximal instead of maximum independent sets can eventually give larger independent sets in product graphs. Set

$$\lambda(G\Box H) = \max \{ \sum_i |A_i|\,|B_i| \},$$

where the maximum is taken over all possible selections of A_i's and B_i's in the diagonal procedure. Then, by the above, we can state:

Theorem 3.2 *For any graphs G and H, $\alpha(G\Box H) \geq \lambda(G\Box H)$.* □

The following consequence of Theorem 3.2 is immediate.

Corollary 3.3 ([16, Vizing, 1963]) *For any graphs G and H,*

$$\alpha(G\Box H) \geq \alpha(G)\,\alpha(H) + \min\{|V(G)| - \alpha(G), |V(H)| - \alpha(H)\}.$$

To conclude this section we give an example showing that using the presented bounds one can exactly compute the independence number.

Proposition 3.4 *Let $G = C(x_1, \ldots, x_{2k}), x_i \geq 2$, and $H = C(y_1, \ldots, y_{2s})$, $y_i \geq 1$. Then*

$$\alpha(G\Box H) = (\sum_{i=1}^{2k} x_i)\,(\sum_{i=1}^{2s} y_i) + 2ks.$$

Proof. Construct a maximal diagonal set of $G\Box H$ where in each step we take maximum independent sets of G and H. In this way we obtain an independent set of $G\Box H$ of size

$$(\sum_{i=1}^{2k} x_i)\,(\sum_{i=1}^{2s} y_i) + ks + ks.$$

On the other hand, by Corollary 2.5,

$$\alpha(G\Box H) \leq \alpha(G)\,\alpha(H) + \tau(G)\,\overline{\alpha}(H) = (\sum_{i=1}^{2k} x_i)\,(\sum_{i=1}^{2s} y_i) + (2k)\,s.$$

□

Note that Corollary 2.4 is not strong enough for Proposition 3.4: it would only give us: $\alpha(G\Box H) \leq (\sum_{i=1}^{2k} x_i)\,(\sum_{i=1}^{2s} y_i) + 4ks$.

4 Exact results on bipartite graphs

In this section we consider products of bipartite graphs and obtain some exact independence numbers. The main idea is to find good upper and lower bounds and then to exclude all possibilities but one by a closer inspection of the structure of largest independence sets. First a notation: Let $V(H) = \{x_1, x_2, \ldots, x_n\}$ and let $S \subseteq V(G\Box H)$. Let $X_i = S \cap G^{x_i}$. Then we will write $S = \langle X_1, X_2, \ldots, X_n \rangle$.

Let us begin with a relatively simple example that indicates the main ideas that can be used to obtain exact independence numbers. Let $G = C(2,1,0,0,0,0,0,1,2)$ and $H = C(2) = P_3$. In [14] an independent set of size 23 is constructed. We show here that $\alpha(G\Box H)$ is indeed 23. First, using Corollary 2.4, we find that $\alpha(G\Box H) \leq 24$. Suppose that we have an independent set of size 24. Then by [7, Lemma 4.1], there is an α-set of $G\Box H$ of the form $\langle A, B, A \rangle$. Setting $|A| = a$ and $|B| = b$ we have $2a+b = 24$. The solutions of this Diophant equation are $a = 8+t$, $b = 8-2t$, $t \in \mathbb{Z}$. Since $a + b \leq 15$ and $\alpha(G) = 9$, the only possible pair for (a, b) is $(9,6)$. But since the bipartition of G is unique (and of size $8 + 7$), also this pair is not possible.

For bipartite graphs there are three typical instances of maximal diagonal sets. Let G and H be connected bipartite graph with bipartitions $V(G) = V_1 + V_2$ and $V(H) = W_1 + W_2$, where $|V_1| \geq |V_2|$ and $|W_1| \geq |W_2|$. As G and H are connected, V_1 and W_1 are maximal independent sets of G and H, respectively. Then the three typical approaches are the following.

(a) *Bipartite approach*: Returns $(V_1 \times W_1) \cup (V_2 \times W_2)$.

(b) *Greedy approach*: Returns an independent set obtained by the diagonal procedure during which largest independent sets are selected in each step.

(c) *Alternative approach*: Returns an independent set I obtained by the diagonal procedure, where I is returned neither with the bipartite approach nor with the greedy approach.

That the alternative approach can indeed give larger independent sets than the other two approaches was noticed in [14] and—justifiably—called it a "counter-intuitive aspect of the Cartesian product." They considered $C(5,5,5,2,5)\Box P_5$ and showed that the approaches (a), (b), (c) give independent sets of size 71, 72, 73, respectively. In fact, the alternative approach in this case is optimal, which is a particular instance of the following result.

Theorem 4.1 *Let $G = C(5,5,5,2,5)$. Then*

$$\alpha(G\Box P_{2k+1}) = \begin{cases} 26k+21; & 1 \leq k \leq 4, \\ 27k+17; & k \geq 5. \end{cases}$$

Proof. The bipartite approach gives an independent set of $G\Box H$ of size $|V_1|\,|W_1| + |V_2|\,|W_2|$.

For the converse we apply Corollary 2.4 and the classical König-Gallai result asserting that $\alpha(H) + \tau(H) = |V(H)|$ holds for bipartite H. Then,

$$\begin{aligned}\alpha(G\Box H) &\le \tau(H)\,\alpha_2(G) + (|H| - 2\tau(H))\,\alpha(G)\\ &= |W_2|\,(|V_1| + |V_2|) + (|W_1| - |W_2|)\,|V_1|\\ &= |V_1|\,|W_1| + |V_2|\,|W_2|\,.\end{aligned}$$

□

We conclude this section with some remarks on the bipartite graphs that attain the bound of Corollary 2.2. Hell, Yu, and Zhou [9] (cf. also [4]) proved that for graphs G and H this bound is attained if and only if there is a homomorphism from G to $\mathrm{Ind}(H)$ or a homomorphism from H to $\mathrm{Ind}(G)$. Here $\mathrm{Ind}(H)$ is the *independence graphs* of H: its vertices are the α-sets of H, two vertices being adjacent whenever the corresponding α-sets are disjoint. For bipartite graphs this yields:

Corollary 4.3 ([4]) *Let G and H be connected bipartite graphs. Then*

$$\alpha(G\Box H) = \min\{\alpha(G)\,|V(H)|,\ \alpha(H)\,|V(G)|\}$$

if and only if $|V(G)| = 2\alpha(G)$ or $|V(H)| = 2\alpha(H)$.

Proof. As G is bipartite, $\alpha(G) \ge |V(G)|/2$, and G_α can have at most one edge. Moreover, $\mathrm{Ind}(G)$ has an edge if and only if the bipartition $V_1 + V_2$ of G fulfills the conditions $|V_1| = |V_2| = \alpha(G)$. □

Similarly we also deduce:

Corollary 4.4 *Let G be a connected bipartite graphs. Then for any $n \ge 1$, $\alpha(G^n) = \alpha(G)\,|V(G)|^{n-1}$ if and only if $|V(G)| = 2\alpha(G)$.*

The last corollary in particular implies Corollary 3.3 of [7] and Corollaries 2.4 and 2.5 of [18]. Analogous result to Corollary 4.4 has been proved by Zhou [19, Theorem 2.2] for powers of Cayley graphs of Abelian groups.

5 Relation between $\alpha(G\Box H)$ and $\lambda(G\Box H)$

Recall that for any graphs G and H, $\alpha(G\Box H) \ge \lambda(G\Box H)$, where $\lambda(G\Box H)$ is the size of a largest possible set obtained by the diagonal procedure. To see that we can have strict inequality consider the products $C_{2k+1}\Box C_{2k+1}$, $k \ge 2$. Recall [7] that $\alpha(C_{2k+1}\Box C_{2k+1}) = k(2k+1)$ and let I be an α-set of

$C_{2k+1} \square C_{2k+1}$. In view of Corollary 2.2, every layer of the product contains k vertices from I. But a maximal diagonal set would be, in the best case, of size $k \cdot k + k \cdot k + 1 \cdot 1$, so an α-set cannot be realized by the diagonal procedure.

In fact, $\alpha(G \square H) = \lambda(G \square H)$ need not hold even for bipartite graphs G and H, although it was more difficult to find a counterexample. For this sake we consider the following product: $C(2,0,1,0,0,2,0,0,0,3) \square C(4,2)$. In [14] it was proved that its independence number in between 75 and 81. We can compute the exact value and, moreover, show that this is an example with $\lambda < \alpha$.

Theorem 5.1 *Let $G = C(2,0,1,0,0,2,0,0,0,3)$ and $H = C(4,2)$. Then*

$$78 = \lambda(G \square H) < \alpha(G \square H) = 79 .$$

Proof. Let x and y be the vertices of the spine of H, where x has four pendant vertices. Note that there is an α-set I of $G \square H$ of the form $\langle A, A, A, A, X, Y, B, B \rangle$, where A and B correspond to the pendant vertices of x and y. Let $|A| = a$, $|B| = b$, $|X| = x$, and $|Y| = y$. We are first going to show that $\alpha(G \square H) \leq 79$.

Suppose that $a \leq 11, b \leq 10$. Since $a + x \leq |V(G)| = 18$ and $b + y \leq |V(G)| = 18$, we have

$$|I| = 4a + 2b + x + y \leq 4a + 2b + (18 - a) + (18 - b) = 3a + b + 36 \leq 79 .$$

The case $a \leq 10, b \leq 11$ is treated analogously. Suppose next $a = b = 11$. Then, as the bipartition of G is unique (and is of size 10+8), we have $x \leq 6$ and $y \leq 6$. But then $|I| \leq 78$. If $a = 11$ and $b = 12$, then $x \leq 6$, $y \leq 5$, and hence $|I| \leq 79$.

For the case $a = 12$ and $b = 11$ some more arguments are needed. Let $u_1, \ldots, u_{10}$ be the vertices of the spine of G, where u_1 has two pendant vertices. Note first that an independent set of size 12 of G contains all the pendant vertices as well as u_2, u_7, and u_9. The only selection that we can make is between u_4 and u_5. On the other hand, any independent set of G of size 11 contains the pendant vertices of u_1, u_6, and u_{10}. Moreover, such a set contains at least one of the vertices u_2 and u_9. We now consider two subcases.

Suppose $u_2 \in B$. Then $u_2 \notin X, Y$, and so there are 10 vertices that can be selected for X and Y: all the u_i's except u_2, and the pendant vertex of u_2. Since $u_9 \in A$, it follows that u_9 must belong to Y. Moreover, u_9, u_7, u_5, u_3 must all belong to Y if we wish that $|X| + |Y| = 10$. But then $u_9, u_7, u_5, u_3 \notin B$ in which case $|Y| < 11$. It follows that $|X| + |Y| \leq 9$ whence $|I| \leq 79$.

Suppose $u_9 \in B$. Then $u_9 \notin X, Y$, and so there are 10 vertices that can be selected for X and Y: all the u_i's except u_9, and the pendant vertex of

u_2. Since $u_7 \in A$, it follows that u_7 must belong to Y, u_6 to X, and so on, until we get $u_2 \in X$, which is not possible. So again $|X| + |Y| \leq 9$ whence $|I| \leq 79$. This completes the case $a = 12$, $b = 11$.

Finally, let $a = 12$ and $b = 12$. Then $4a + 2b = 72$. Suppose first that $u_4 \in A, B$. Then, in X and Y, there are only 6 parallel vertices free, so we cannot get more than 78 independent vertices. And if $u_4 \in A$, $u_5 \in B$, we have 7 such vertices, so we get at most 79 independent vertices.

We have thus proved that $\alpha(G\Box H) \leq 79$. To see that $\alpha(G\Box H) \geq 79$ consider the α-set of $G\Box H$ that is schematically shown on Figure 2. More precisely, the four G layers of the figure present an A set, X set, Y set, and B set.

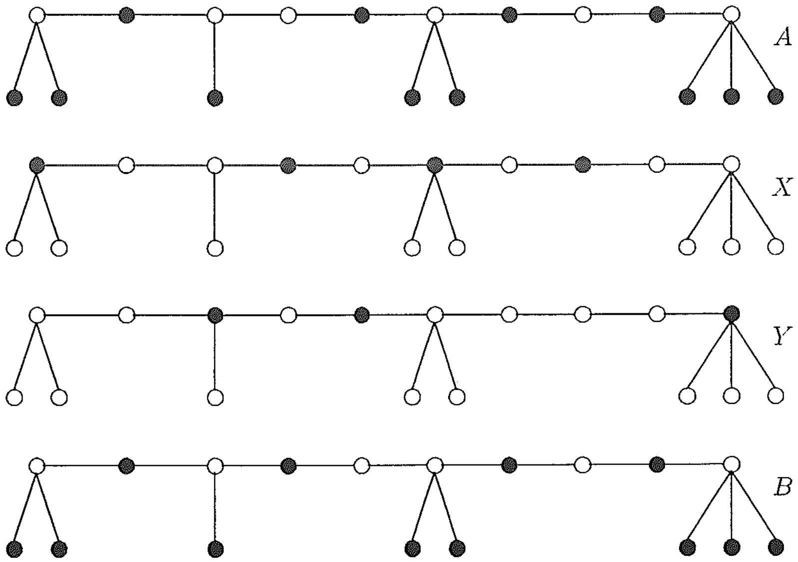

Figure 2: An independent set with 79 vertices

To see that $\lambda(G\Box H) < 79$, we first observe that in a λ-set we must have $A = B$. In addition, by the above considerations, $a = b \leq 11$ is not possible, hence $a = b = 12$. But then there are only 6 vertices left that can be used for X and Y, and we can get at most 78 independent vertices. Finally, it is easy to see that a diagonal set with $12 \cdot 6 + 5 \cdot 1 + 1 \cdot 1$ vertices can indeed be constructed, proving that $\lambda(G\Box H) = 78$. $\Box$

To Theorem 5.1 it is interesting to add that there exists an α-set of $G\Box H$ with $a = 12$, $b = 11$, $x = 3$, and $y = 6$.

6 Some problems

There are many questions and problems that one can pose based on the results presented in this paper. Here are some of them.

1. Characterize the graphs G and H for which $\alpha(G\Box H) = \lambda(G\Box H)$. In particular, for which bipartite graphs G and H we have $\alpha(G\Box H) = \lambda(G\Box H)$?

2. Is it true that for almost any bipartite graphs G and H, $\alpha(G\Box H) = \lambda(G\Box H)$?

3. Characterize bipartite graphs G and H for which $\alpha(G\Box H)$ can be realized by the bipartite approach (by the greedy approach). Note that Theorem 4.2 is a result in this direction.

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