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# Strong products of $\chi$-critical graphs

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*Summary*. Let $G[H]$ be the lexicographic product and let $G \boxtimes H$ be the strong product of the graphs $G$ and $H$. It is proved that, if $G$ is a $\chi$-critical graph, then, for any graph $H$,

$$\chi(G[H]) \leqslant \chi(H)(\chi(G) - 1) + \left\lceil \frac{\chi(H)}{\alpha(G)} \right\rceil.$$

This upper bound is used to calculate several chromatic numbers of strong products. It is shown in particular that for $k \geqslant 2$, $\chi(C_5 \boxtimes C_5 \boxtimes C_{2k+1}) = 10 + \lceil 5/k \rceil$, and for $k \geqslant 2$ and $n \geqslant 1$, $\chi(\overline{C_{2k+1}} \boxtimes K_n) = kn + \lceil n/2 \rceil$. That the general upper bound cannot be improved for graphs which are not $\chi$-critical is demonstrated by two infinite series of graphs. The paper is concluded with an application to graph retracts: if for some graph $H$ with at least one edge $\chi(G[H]) = \chi(G)\chi(H)$, then no $\chi$-critical subgraph $G'$ of $G$, $G' \neq K_n$, is a retract of $G$.

## 1. Introduction and definitions

In the last few years graph products became again a very flourishing topic in graph theory. The revival of interest seems to be mostly due to the algorithmic point of view. In particular, algorithms for decomposing a graph with respect to a given product and for isometrically embedding a graph into a (Cartesian) product of graphs were proposed [1, 3, 4, 7, 19, 20]. Furthermore, retracts of graph products, the reconstruction of products and some other properties of products were investigated [2, 9–13].

It turned out that both the Cartesian product and the strong product admit a polynomial algorithm for decomposing a given connected graph into its factors. Here, we are interested in studying those parameters of strong products of graphs whose determination is in general NP-complete, for example, the chromatic number of a graph. On the other hand, it is not surprising that information about the chromatic number of strong products helps to understand retracts of strong

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products [9, 11, 12]. Finally, the chromatic number of the strong product of graphs seems to be more unpredictable than the chromatic number of the Cartesian, the categorical, and the lexicographic product.

All graphs considered in this paper will be undirected, finite, and simple graphs, i.e., graphs without loops or multiple edges.

An *$n$-colouring* of a graph $G$ is a function $f$ from $V(G)$ onto a set $X$ with $|X| = n$, such that $xy \in E(G)$ implies $f(x) \neq f(y)$. The smallest number $n$ for which an $n$-colouring exists is the *chromatic number* $\chi(G)$ of $G$. $G$ is called $\chi$-critical if $\chi(G - v) < \chi(G)$ for every $v \in V(G)$. Every nontrivial graph contains a $\chi$-critical subgraph with the same chromatic number.

The size of a largest complete subgraph of a graph $G$ will be denoted by $\omega(G)$ and the size of a largest independent set by $\alpha(G)$. Clearly $\omega(G) \leqslant \chi(G)$ and $\omega(G) = \alpha(\bar{G})$.

The *strong product* $G \boxtimes H$ of graphs $G$ and $H$ is the graph with vertex set $V(G) \times V(H)$ and $(a, x)(b, y) \in E(G \boxtimes H)$ whenever $ab \in E(G)$ and $x = y$, or $a = b$ and $xy \in E(H)$, or $ab \in E(G)$ and $xy \in E(H)$. The *lexicographic product* $G[H]$ of graphs $G$ and $H$ is the graph with vertex set $V(G) \times V(H)$ and $(a, x)(b, y) \in E(G[H])$ whenever $ab \in E(G)$, or $a = b$ and $xy \in E(H)$.

In the next section we recall some known results and prepare a few observations for the rest of the paper. In Section 3 we prove an upper bound for lexicographic (strong) products in which one factor is $\chi$-critical and construct an infinite sequence of graphs which shows that the general upper bound cannot be improved for graphs which are not $\chi$-critical. In Section 4 we use the upper bound for $\chi$-critical graphs to calculate several chromatic numbers of strong products. We finally give an application to graph retracts.

## 2. Preliminaries

The upper bound $\chi(G \boxtimes H) \leqslant \chi(G)\chi(H)$ on the chromatic number of the strong product is well-known and easy to prove. It is attained for any $G$ and $H$ with $\chi(G) = \omega(G)$ and $\chi(H) = \omega(H)$. Indeed,

$$\chi(G)\chi(H) = \omega(G)\omega(H) = \omega(G \boxtimes H) \leqslant \chi(G \boxtimes H) \leqslant \chi(G)\chi(H).$$

Perfect graphs form an especially important class for which the upper bound is reached. This is in particular true for bipartite graphs ([16, Theorem 5]) and complete $n$-partite graphs ([16, Theorem 4]).

Next we consider the case in which one of the factors has chromatic number equal to its clique number, but not the other factor.

PROPOSITION 1. *Let $\chi(H) = \omega(H) = n$. Then for any graph G*

$$\chi(G \boxtimes H) = \chi(G \boxtimes K_n).$$

A short proof (using retracts) of this proposition will be given in the last section. In [10] Jha showed that $\chi(G \boxtimes K_n) \geqslant \chi(G) + n$, generalizing a result of Vesztergombi [17] for $n = 2$. Both authors asked for a better lower bound. However, the answer was given already by Stahl [16], although in a different context. He proved

THEOREM 2 [16, Corollary, p. 189]. *If G has at least one edge then*

$$\chi(G \boxtimes K_n) \geqslant \chi(G) + 2n - 2.$$

A short proof of this theorem and some of its consequences are given in [12].

We conclude this section with a well-known inequality and a simple observation. Since we will need these two facts several times let us state them as a lemma.

LEMMA 3. *For any graph G, the following hold:*

(*i*) $\chi(G) \geqslant \dfrac{|V(G)|}{\alpha(G)}$.

(*ii*) $\alpha(G \boxtimes K_n) = \alpha(G)$.

## 3. An upper bound for $\chi$-critical graphs

The strong product is a subgraph of the lexicographic product. Therefore, any upper bounds for the chromatic number of the lexicographic product is also an upper bound for the strong product and any lower bound for the strong product is a lower bound for the lexicographic product. The following upper bound which extends the Corollary of Theorem 5 from [6], will have several consequences and applications.

THEOREM 4. *If G is a $\chi$-critical graph then, for any graph H,*

$$\chi(G[H]) \leqslant \chi(H)(\chi(G) - 1) + \left\lceil \frac{\chi(H)}{\alpha(G)} \right\rceil.$$

*Proof.* Let $\chi(G) = n$. Let $\chi(H) = m$ and for a given $m$-colouring of $H$ let $\{C_0, C_1, \ldots, C_{m-1}\}$ be the corresponding colour classes. Let $\alpha(G) = k$ and let $S = \{a_0, a_1, \ldots, a_{k-1}\}$ be an independent set of $G$.

As $G$ is $\chi$-critical, the graph $G - a$ can be coloured with $n - 1$ colours for any $a \in V(G)$. In particular, let $f_i\colon V(G) - a_i \to \{1, 2, \ldots, n-1\}$ be an $(n-1)$-colouring of the graph $G - a_i$, $i = 0, 1, \ldots, k-1$.

For $j = 0, 1, \ldots, m-1$ write $j = p_j k + q_j$, where $0 \leqslant q_j < k$ and set

$$f(a, x) = \begin{cases} f_{q_j}(a) + jn, & x \in C_j, \quad a \neq a_{q_j}; \\ p_j n, & x \in C_j, \quad a = a_{q_j}. \end{cases}$$

We are going to show that $f$ is a colouring of $G[H]$. Let $f(a, x) = f(b, y)$, $(a, x) \neq (b, y)$, where $x \in C_i$ and $y \in C_j$. Note first that if $a = a_{q_i}$ and $b \neq a_{q_j}$ then $f(a, x)$ is divisible by $n$ and $f(b, y)$ is not, hence $f(a, x) \neq f(b, y)$. We now distinguish two cases.

*Case* 1. $a = a_{q_i}$, $b = a_{q_j}$. Then $p_i n = p_j n$ and hence $p_i = p_j$. It follows that, for some $s$, $0 \leqslant s < \lceil m/k \rceil$, both $i$ and $j$ belong to the set

$$\{sk, sk + 1, \ldots, sk + k - 1\}.$$

If $i \neq j$ then $q_i \neq q_j$ and hence $a_{q_i} \neq a_{q_j}$. It follows that $(a, x)(b, y) \notin E(G[H])$. Assume next $i = j$. Then $q_i = q_j$ and therefore $a = b$, $x \neq y$ but $x$ and $y$ are in the same colour class of $H$. It follows again $(a, x)(b, y) \notin E(G[H])$.

*Case* 2. $a \neq q_{q_i}$, $b \neq a_{q_j}$. Then $f_{q_i}(a) + in = f_{q_j}(b) + jn$. If we assume w.l.o.g. $f_{q_i}(a) \leqslant f_{q_j}(b)$ then we have $(i - j)n = f_{q_j}(b) - f_{q_i}(a)$, where $0 \leqslant f_{q_j}(b) - f_{q_i}(a) < n - 1$. Since $n \geqslant 2$ the equality $i = j$ follows and hence $f_{q_i}(a) = f_{q_i}(b)$. But then $ab \notin E(G)$ and $x, y$ are in the same colour class. We conclude $(a, x)(b, y) \notin E(G[H])$.

We have seen that $f$ is indeed a colouring of $G[H]$. It is easy to see that the set onto which $f$ maps vertices of $G[H]$ has $(m(n-1) + \lceil m/k \rceil)$ elements. This completes the proof. □

COROLLARY 5. *If $G$ is a $\chi$-critical graph, then for any graph $H$,*

$$\chi(G \boxtimes H) \leqslant \chi(H)(\chi(G) - 1) + \left\lceil \frac{\chi(H)}{\alpha(G)} \right\rceil.$$

We note here that in [14] M. Rosenfeld characterized those graphs $G$ (called *universal*) for which $\alpha(G \boxtimes H) = \alpha(G)\alpha(H)$, for every graph $H$. With this terminology, Corollary 5 states that no graph is universal for the chromatic version of this problem.

The natural question now is whether the upper bound can be improved also in the case when $\omega(G) < \chi(G)$, yet $G$ is not $\chi$-critical. We need the following lemma which is a generalization of Theorem 5 in [5].

LEMMA 6. *Let* $k \geqslant 1$ *and let*

$$\alpha(G) < \frac{k|V(G)|}{k\chi(G) - 1}$$

*hold for a graph* $G$. *Then* $\chi(G \boxtimes K_k) = \chi(G)k$.

*Proof.* Using Lemma 3 (i) and (ii) we infer

$$\chi(G \boxtimes K_k) \geqslant \frac{k|V(G)|}{\alpha(G)}.$$

Hence, $\chi(G \boxtimes K_k) > k\chi(G) - 1$, which yields the desired result since $\chi(G \boxtimes K_k) \leqslant \chi(G)k$. □

THEOREM 7. *For* $k = 2, 3$ *there is an infinite sequence of graphs* $G_n^k$ *such that* $\omega(G_n^k) < \chi(G_n^k)$ *and*

$$\chi(G_n^k \boxtimes K_k) = \chi(G_n^k)k.$$

*Proof.* Let $T_n$, $n = 1, 2, 3, \ldots$ be any sequence of trees, where $|T_n| = n$.

Let $G_n^2$ be a graph which we get from the tree $T_n$ in which we replace every vertex $u \in V(T_n)$ with a copy $G_u$ of the graph $G_8$ from Fig. 1. If $uv \in E(T_n)$ then we select a vertex of $G_u$ and a vertex of $G_v$ and join them with an edge (i.e., we have a tree-like structure of graphs $G_8$). Since $\alpha(G_8) = 3$, we get $\alpha(G_n^2) \leqslant 3n$. It is also straightforward to verify that $\chi(G_n^2) = 3$. Hence as $|V(G_n^2)| = 8n$ and $k = 2$ the condition of Lemma 6 is fulfilled ($15n < 16n$). It follows $\chi(G_n^2 \boxtimes K_2) = \chi(G_n^2)2$.

For $k = 3$ consider the graph $G_{11}$ of Fig. 1. Let $S$ be an independent set of vertices of $G_{11}$ and let $w \in V(G_{11})$ be the vertex of degree 2. If $w \notin S$ then the other 10 vertices lie on two disjoint 5 cycles. As $\alpha(C_5) = 2$ it follows $|S| \leqslant 4$. The case when $w \in S$ can be analysed similarly by a simple case distinction. It follows that $\alpha(G_{11}) = 4$.

Let $G_n^3$ be a graph which we get from the tree $T_n$ in the same way as we constructed $G_n^2$ but instead of $G_8$ we use $G_{11}$. Then $\alpha(G_n^3) \leqslant 4n$ and $\chi(G_n^3) = 3$. Again the condition of Lemma 6 is fulfilled and hence $\chi(G_n^3 \boxtimes K_3) = \chi(G_n^3)3$.

Observe finally that $\omega(G_n^2) = \omega(G_n^3) = 2$. □

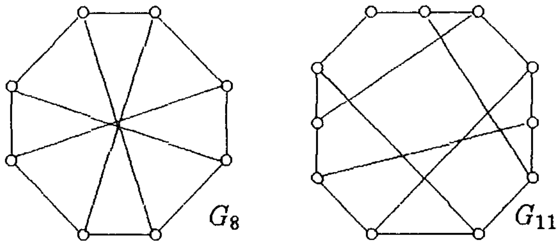


Figure 1. The graphs $G_8$ and $G_{11}$.

## 4. Some chromatic numbers

In this section we are going to apply Corollary 5 to get several chromatic numbers of strong products.

In [16] Stahl proved that $\chi(C_{2k+1} \boxtimes K_n) = 2n + \lceil n/k \rceil$. Besides odd cycles the second prototype of nonperfect graphs are the complements of odd cycles. We have

COROLLARY 8. *For $k \geq 2$ and $n \geq 1$, $\chi(\overline{C_{2k+1}} \boxtimes K_n) = kn + \lceil n/2 \rceil$.*

*Proof.* As $\chi(\overline{C_{2k+1}}) = k + 1$ it follows from Corollary 5 that

$$\chi(\overline{C_{2k+1}} \boxtimes K_n) \leq (k+1)n - n + \left\lceil \frac{n}{2} \right\rceil = kn + \left\lceil \frac{n}{2} \right\rceil.$$

On the other hand, using Lemma 3 (i) and (ii) and the fact $\alpha(\overline{C_{2k+1}}) = 2$, we infer

$$\chi(\overline{C_{2k+1}} \boxtimes K_n) \geq \frac{|\overline{C_{2k+1}} \boxtimes K_n|}{\alpha(\overline{C_{2k+1}} \boxtimes K_n)} = \frac{(2k+1)n}{\alpha(\overline{C_{2k+1}})} = kn + \frac{n}{2}. \qquad \square$$

It is known that $\chi(C_{2s+1} \boxtimes C_{2k+1}) = 5$ for $s \geq 2$, $k \geq 2$, see [17]. Next we consider products of three odd cycles.

THEOREM 9. *For $k \geq 2$, $\chi(C_5 \boxtimes C_5 \boxtimes C_{2k+1}) = 10 + \lceil 5/k \rceil$.*

*Proof.* Note first that $\alpha(C_{2k+1}) = k$ and $\alpha(C_5 \boxtimes C_5) = 5$. If we apply the following result from [15, p. 142]

$$\alpha(C_5 \boxtimes C_5 \boxtimes C_{2k+1}) = \alpha(C_5 \boxtimes C_5)\alpha(C_{2k+1}),$$

we obtain from Lemma 3 (i)

$$\chi(C_5 \boxtimes C_5 \boxtimes C_{2k+1}) \geqslant \frac{5 \cdot 5 \cdot (2k+1)}{5 \cdot k} = 10 + \frac{5}{k}.$$

On the other hand, if we write $\chi(C_5 \boxtimes C_5 \boxtimes C_{2k+1}) = \chi(C_{2k+1} \boxtimes (C_5 \boxtimes C_5))$ and use Corollary 5, we get

$$\chi(C_{2k+1} \boxtimes (C_5 \boxtimes C_5)) \leqslant 15 - 5 + \left\lceil \frac{5}{k} \right\rceil. \qquad \square$$

With the same technique as in Theorem 9 one can also show that for $2 \leqslant t \leqslant s \leqslant k$,

$$\left\lceil \frac{2(2s+1)(2t+1)}{st + \lfloor t/2 \rfloor} \right\rceil \leqslant \chi(C_{2t+1} \boxtimes C_{2s+1} \boxtimes C_{2k+1}) \leqslant 10 + \left\lceil \frac{5}{k} \right\rceil.$$

However, the exact result for the chromatic number of three odd cycles remains open.

The *join* $G_1 + G_2$ of graphs $G_1$ and $G_2$ is the graph which we obtain from the disjoint union of $G_1$ and $G_2$ in which we join every vertex of $G_1$ with every vertex of $G_2$.

PROPOSITION 10. *Let* $G = G_1 + G_2$. *Then*

$$\chi(G \boxtimes K_n) = \chi(G_1 \boxtimes K_n) + \chi(G_2 \boxtimes K_n).$$

*Proof.* Clearly,

$$\chi(G \boxtimes K_n) \leqslant \chi(G_1 \boxtimes K_n) + \chi(G_2 \boxtimes K_n).$$

On the other hand, if $a \in V(G_1)$ and $b \in V(G_2)$ then $(a, x)(b, y)$ is an edge of $G \boxtimes K_n$. It follows that

$$\chi(G \boxtimes K_n) \geqslant \chi(G_1 \boxtimes K_n) + \chi(G_2 \boxtimes K_n). \qquad \square$$

For an application of Proposition 10 we consider an interesting class of graphs proposed by Witzany (personal communication). Let $G_n$ be the join of the complete graph $K_{n-3}$, $n \geqslant 3$ and the five cycle $C_5$. Witzany showed that $G_n$ is the smallest graph with $\chi(G) = n$ and $\omega(G) = n - 1$.

COROLLARY 11. *For* $n \geqslant 3$ *and* $m \geqslant 1$, $\chi(G_n \boxtimes K_m) = nm - m + \lceil m/2 \rceil$.

*Proof.* As $C_5 = \bar{C}_5$, it follows from Corollary 8 that

$$\chi(C_5 \boxtimes K_m) = 2m + \left\lceil \frac{m}{2} \right\rceil.$$

Thus we have, using Proposition 10, that

$$\chi(G \boxtimes K_m) = \chi(K_{n-3} \boxtimes K_m) + \chi(C_5 \boxtimes K_m)$$
$$= (n-3)m + 2m + \left\lceil \frac{m}{2} \right\rceil = nm - m + \left\lceil \frac{m}{2} \right\rceil.$$

This completes the proof. □

Note that the upper bound of Corollary 5 coincides once more with the result in Corollary 11.

## 5. An application to graph retracts

A subgraph $R$ of a graph $G$ is a *retract* of $G$ if there is a homomorphism (an edge-preserving map) $r$: $V(G) \to V(R)$ with $r(x) = x$, for all $x \in V(R)$. The map $r$ is called a *retraction*. It is not hard to see that if $R$ is a retract of $G$ then $\chi(R) = \chi(G)$ and $R$ is an isometric subgraph.

Before stating an application of Theorem 4 to graph retracts, we owe a proof of Proposition 1.

*Proof of Proposition 1.* As $\chi(H) = \omega(H) = n$, $K_n$ is a retract of $H$. It follows that $G \boxtimes K_n$ is a retract of $G \boxtimes H$. Hence the assertion follows. □

THEOREM 12. *Let* $G$ *be a graph. If for some graph* $H$ *with at least one edge* $\chi(G[H]) = \chi(G)\chi(H)$, *then no* $\chi$*-critical subgraph* $G'$ *of* $G$, $G' \neq K_n$, *is a retract of* $G$.

*Proof.* Let $H$ be a graph with at least one edge and assume that $\chi(G[H]) = \chi(G)\chi(H)$. Let $G'$ be a retract of $G$ and assume that $G'$ is $\chi$-critical. As $G'$ is a retract of $G$, $\chi(G') = \chi(G)$. Furthermore, $G'[H]$ is a retract of $G[H]$; hence we obtain

$$\chi(G'[H]) = \chi(G[H]) = \chi(G)\chi(H).$$

On the other hand, it follows from Theorem 4 that

$$\chi(G'[H]) \leqslant \chi(G')\chi(H) - \chi(H) + \left\lceil \frac{\chi(H)}{\alpha(G')} \right\rceil.$$

Since $\chi(G) = \chi(G')$ we have

$$\chi(H) \leqslant \left\lceil \frac{\chi(H)}{\alpha(G')} \right\rceil.$$

Furthermore, $\chi(H) \geqslant 2$ and therefore $\alpha(G') = 1$. Thus, $G'$ is isomorphic to a complete graph. □

As an example consider once more the graphs $G_n^k$ from Theorem 7. Since $\chi(G_n^k \boxtimes K_k) = \chi(G_n^k)k$, we also have $\chi(G_n^k[K_k]) = \chi(G_n^k)k$. Therefore, no 5-cycle is a retract in any of the graphs $G_n^k$ although it is an isometric and isochromatic subgraph.

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