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Partial cubes as subdivision graphs and as generalized Petersen graphs ☆

Sandi Klavžar[a,*], Alenka Lipovec[b]

[a] Department of Mathematics, University of Maribor, Koroška cesta 160, 2000 Maribor, Slovenia
[b] Department of Education, University of Maribor, Koroška cesta 160, 2000 Maribor, Slovenia

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Abstract

Isometric subgraphs of hypercubes are known as partial cubes. The subdivision graph of a graph G is obtained from G by subdividing every edge of G. It is proved that for a connected graph G its subdivision graph is a partial cube if and only if every block of G is either a cycle or a complete graph. Regular partial cubes are also considered. In particular, it is shown that among the generalized Petersen graphs $P(10,3)$ and $P(2n,1)$, $n \geqslant 2$, are the only (regular) partial cubes.

Keywords: Partial cube; Subdivision graph; Regular graph; Generalized Petersen graphs

1. Introduction

Partial cubes are isometric subgraphs of hypercubes. They were first introduced in computer science [12] and have later found several other applications, for instance in mathematical chemistry and biology, cf. [5,9,15].

Clearly, partial cubes are bipartite. If G is an arbitrary graph, a simple way to modify it to a bipartite graph is to subdivide every edge of G by a single vertex. Such a graph is called a *subdivision graph* of G and denoted $S(G)$. A natural question appears for which graphs G their subdivision graphs are partial cubes.

Subdivision graphs were studied before in different contexts, see, for instance, [1,3,19]. A more general construction is to replace edges of G by paths, thus

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* Corresponding author. Tel.: +386-2-22-93-604; fax: +386-1-25-172-81.
E-mail addresses: sandi.klavzar@uni-lj.si (S. Klavžar), alenka.lipovec@uni-mb.si (A. Lipovec).

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obtaining a *subdivided graph* of G. In the context of partial cubes, subdivided graphs of wheels turned out to be useful in settling (in negative) a conjecture of Chepoi and Tardif asserting that a bipartite graph is a partial cube if and only if all of its intervals are convex [2]. Subdivided graphs of wheels were also studied in [6] where it was proved that except in three particular cases they are the so-called l_1-graphs—a class of graphs that properly contains partial cubes.

The main result of this paper asserts that $S(G)$ is a partial cube if and only if every block of G is either a cycle or a complete graph. We also consider the problem of classifying regular partial cubes and show in particular that $P(10,3)$ and $P(2n,1)$, $n\geqslant 2$, are the only (regular) partial cubes among the generalized Petersen graphs.

All graphs considered in this paper are connected and simple. The vertex set of the n-cube Q_n consists of all n-tuples $b_1b_2\dots b_n$ with $b_i\in\{0,1\}$, where two vertices are adjacent if the corresponding tuples differ in precisely one place. The vertices of Q_n can also be understood as characteristic functions of subsets of an n-set. Then, two such subsets are adjacent if their symmetric difference consists of a single element.

The *Cartesian product* $G\,\Box\,H$ of graphs G and H is the graph with vertex set $V(G)\times V(H)$ in which the vertex (a,x) is adjacent to the vertex (b,y) whenever $ab\in E(G)$ and $x=y$, or $a=b$ and $xy\in E(H)$. Note that the Cartesian product of n copies of K_2 is the n-cube Q_n.

For a graph G, the *distance* $d_G(u,v)$, or briefly $d(u,v)$, between vertices u and v is defined as the number of edges on a shortest u,v-path. A subgraph H of G is called *isometric* if $d_H(u,v)=d_G(u,v)$ for all $u,v\in V(H)$. A subgraph H of G is *convex*, if for any $u,v\in V(H)$, all shortest u,v-paths belong to H.

A maximal connected subgraph without a cutvertex is called a *block*. Thus, every block of a connected graph is either a maximal 2-connected subgraph or a bridge (with its ends), cf. [7].

For an edge uv of a graph G, let W_{uv} be the set of vertices of G that are closer to u than to v. In a bipartite graph G the sets W_{uv} and W_{vu} form a partition of $V(G)$. Djoković [8] proved that a graph G is a partial cube if and only if it is bipartite and if for any edge ab of G the subgraph W_{ab} is convex.

Two edges $e=xy$ and $f=uv$ of G are in the Djoković–Winkler [8,20] relation Θ if $d_G(x,u)+d_G(y,v)\neq d_G(x,v)+d_G(y,u)$. In the case of bipartite graphs relation Θ reduces to:

Lemma 1.1. *Let G be a bipartite graph and $e=uv$, $f=xy$ be two edges of G with $e\Theta f$. Then the notation can be chosen such that* $d(u,x)=d(v,y)=d(u,y)-1=d(v,x)-1$.

Clearly, Θ is reflexive and symmetric. We now list several basic and well-known properties of Θ to be used in the sequel, cf. [13], for Lemma 1.5 see [11].

Lemma 1.2. *Let P be a shortest path in a graph G. Then no two distinct edges of P are in relation Θ.*

Lemma 1.3. *Suppose that a walk P connects the endpoints of an edge e but does not contain it. Then P contains an edge f with $e\Theta f$.*

Lemma 1.4. *Let G be a partial cube, C an isometric cycle of G, and e an edge of C. Then the Θ class of e intersects C in exactly two edges: e and its antipodal (opposite) edge on C.*

Lemma 1.5. *Let e and f be edges from different blocks of a graph G. Then e is not in relation Θ with f.*

Let Θ^* be the transitive closure of Θ. Winkler [20] proved the following characterization of partial cubes:

Theorem 1.6. *A bipartite graph is a partial cube if and only if $\Theta = \Theta^*$.*

Let G be a connected graph. A *proper cover* of G consists of two isometric subgraphs G_1 and G_2 with nonempty intersection and union G. The *expansion* of G with respect to G_1, G_2 is the graph H obtained by the following procedures:

(i) Replacement of each vertex $v \in G_1 \cap G_2$ by vertices v_1, v_2 and insertion of the edge v_1v_2.
(ii) Insertion of edges between v_1 and the neighbors of v in $G_1 \setminus G_2$ as well as between v_2 and the neighbors of v in $G_2 \setminus G_1$.
(iii) Insertion of the edges v_1u_1 and v_2u_2 whenever $v, u \in G_1 \cap G_2$ are adjacent in G.

Chepoi [4] proved:

Theorem 1.7. *A graph is a partial cube if and only if it can be obtained from the one vertex graph by a sequence of expansions.*

2. Partial cubes as subdivision graphs

If T is a tree, then $S(T)$ is also a tree and thus a partial cube. Similarly, subdivision graphs of cycles are partial cubes since $S(C_n) = C_{2n}$. Moreover, we have:

Proposition 2.1. *For any $n \geqslant 1$, $S(K_n)$ is a partial cube.*

Proof. As $S(K_1) = K_1$ and $S(K_2) = P_3$, the assertion is true for $n = 1, 2$. Let $n \geqslant 2$ and consider the subset representation of Q_n. Let G be the subgraph of Q_n induced by the subsets on at most two elements. Let G_1 be the subgraph of G induced by the subsets on at most one element and G_2 the subgraph of G obtained from G be removing the empty set. Clearly, G_1 is isometric in G and it is also easy to check that G_2 is isometric in G. Thus G_1 and G_2 form a proper cover of G and since G is also a partial cube, the expansion H of G with respect to G_1 and G_2 is a partial cube by Theorem 1.7. Finally, observe that H is isomorphic to $S(K_{n+1})$. □

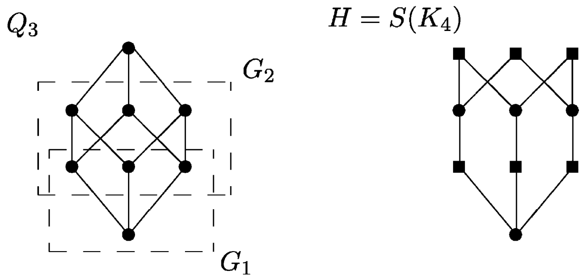

Fig. 1. Expanding a subgraph of Q_3 to $S(K_4)$.

The proof of Proposition 2.1 is illustrated in Fig. 1. The vertices of $S(K_4)$ that are obtained by subdividing edges are shown as filled squares.

In the rest of this section we are going to prove that these are precisely the cases when $S(G)$ is a partial cube. More precisely, we will prove

Theorem 2.2. *Let G be a connected graph. Then $S(G)$ is a partial cube if and only if every block of G is either a cycle or a complete graph.*

Let u be a vertex and e an edge of a graph G. We will denote the vertex of $S(G)$ corresponding to u by $\bar{u}$, and by $\bar{e}$ the vertex of $S(G)$ that is obtained by a subdivision of e.

The following straightforward lemma will be implicitly used in the rest of the section:

Lemma 2.3. *Let e and f be edges of a graph G and let u and v be endpoints of e and f, respectively, such that $d(u,v)$ is minimal. Then*

(i) $d_{S(G)}(\bar{u},\bar{v}) = 2d_G(u,v)$,
(ii) $d_{S(G)}(\bar{e},\bar{f}) = 2d_G(u,v) + 2$,
(iii) $d_{S(G)}(\bar{u},\bar{f}) = 2d_G(u,v) + 1$.

Lemma 2.4. *Let G be a 2-connected graph that is not a cycle. Then G contains two isometric cycles $C_1 = u_1u_2\ldots u_ku_{k+1}\ldots u_nu_1$ and $C_2 = u_1u_2\ldots u_kv_{k+1}\ldots v_mu_1$, where $n \geqslant m > k \geqslant 2$ and $v_i \neq u_j$ for $i,j \geqslant k+1$; see Fig. 2.*

Proof. Let C_1 be a shortest cycle of G. Clearly, C_1 is isometric. As G is not a cycle, there exists a vertex $u \in C_1$ of degree at least 3. Let $u' \notin C_1$ be a neighbor of u. Note that such a vertex exists because C_1 is isometric and thus chordless. As u is not a cut vertex, there is a path P between u' and a vertex of C_1. Let v be the endpoint of P different from u. We may select u and v such that $d(u,v)$ is as small as possible among such pairs. Fixing a pair u,v, let P be as short as possible. We claim that the cycle $C_2 : u \to u' \to \cdots P \cdots \to v \to \cdots C_1 \cdots \to u$ is an isometric cycle. Indeed, there is no shortcut between a vertex of $C_2 \cap C_1$ and a vertex of P because of the way u, v and P are selected. Likewise, there is no shortcut between two vertices of P. □

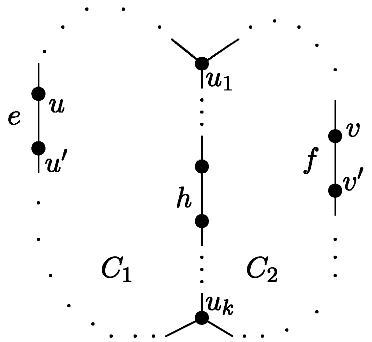

Fig. 2. Two isometric cycles with at least one edge in common.

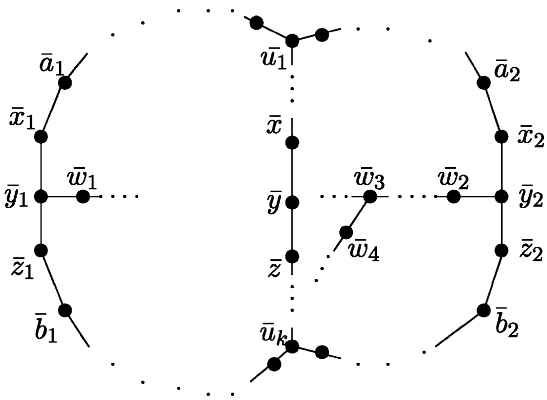

Fig. 3. Cycle $C_1 \cup C_2$ is not isometric.

Lemma 2.4 will be used in the proof of Lemma 2.6. We wish to point out that it is essential that the numbers n, m, k satisfy the inequalities $n \geqslant m > k \geqslant 2$, hence the shortest of the three paths between u_1 and u_k is the common part of C_1 and C_2, cf. Fig. 2.

Lemma 2.5. *Let G be a partial cube, and let C_1 and C_2 be cycles of G as in Lemma* 2.4. *Let $e = uu'$ be an edge of $C_1 \setminus C_2$, let $f = vv'$ be an edge of $C_2 \setminus C_1$, and let h be an edge of $C_1 \cap C_2$ such that $e \Theta f \Theta h$. If $d(u,u_1) < d(u',u_1)$ and $d(v,u_1) < d(v',u_1)$, then $d(u,v) = d(u',v') = d(u,v') - 1 = d(u',v) - 1$.*

Proof. Since $d(u,u_1) < d(u',u_1)$, we have $u_1 \in W_{uu'}$. Similarly, $d(v,u_1) < d(v',u_1)$ implies $u_1 \in W_{vv'}$, see Fig. 2. It follows that $W_{uu'} = W_{vv'}$. Indeed, otherwise $W_{uu'} = W_{v'v}$ would hold, which is not possible since $u_1 \in W_{uu'}$ but $u_1 \notin W_{v'v}$. As G is a partial cube, Djoković's characterization implies that the sets $W_{uu'}$ and $W_{vv'}$ are convex. Hence $d(u',v) = d(u,v) + 1$ and $d(u,v') = d(u',v') + 1$. By using Lemma 1.1 the proof is complete. □

Lemma 2.6. *Let G be 2-connected graph that is not a cycle. If $S(G)$ is a partial cube and C_1 and C_2 cycles of G as in Lemma* 2.4, *then $C_1 \cup C_2$ induces a K_4.*

Proof. Let $S(G)$ be a partial cube and let $\bar{C}_1$ and $\bar{C}_2$ be the cycles of $S(G)$ that correspond to the cycles C_1 and C_2 of G. Consider the middle two edges of $\bar{C}_1 \cap \bar{C}_2$. More precisely, if k is even then these are the edges of the path $\bar{u}_{k/2} \to \bar{e} \to \bar{u}_{k/2+1}$, and if k is odd then these are the edges of the path $\bar{e} \to \bar{u}_{(k+1)/2} \to \bar{f}$. We denote the corresponding vertices of $S(G)$ with $\bar{x}$, $\bar{y}$, and $\bar{z}$.

By Lemma 1.4 there are edges $\bar{y}_1\bar{z}_1$ on $\bar{C}_1$ and $\bar{y}_2\bar{z}_2$ of $\bar{C}_2$ such that $\bar{x}\bar{y} \Theta \bar{y}_1\bar{z}_1$ and $\bar{x}\bar{y} \Theta \bar{y}_2\bar{z}_2$. Note that these are antipodal (opposite) edges in the corresponding cycles. Similarly, $\bar{y}\bar{z}$ is in relation Θ to $\bar{x}_1\bar{y}_1$ on $\bar{C}_1$ and to $\bar{x}_2\bar{y}_2$ on $\bar{C}_2$, cf. Fig. 3. As $S(G)$ is a partial cube we infer that $\bar{y}_1\bar{z}_1 \Theta \bar{y}_2\bar{z}_2$ and $\bar{x}_1\bar{y}_1 \Theta \bar{x}_2\bar{y}_2$. Let $d(\bar{x}_1,\bar{x}_2) = n$, where $n \geqslant 2$. By Lemma 1.1 we have $d(\bar{x}_1,\bar{x}_2) = d(\bar{y}_1,\bar{y}_2) = d(\bar{z}_1,\bar{z}_2)$. Hence Lemma 2.5 implies $d(\bar{x}_1,\bar{y}_2) = d(\bar{y}_1,\bar{x}_2) = d(\bar{y}_1,\bar{z}_2) = d(\bar{z}_1,\bar{y}_2) = n + 1$.

Let P be a shortest $\bar{y}_1,\bar{y}_2$-path. By the above distances none of the vertices $\bar{x}_1$, $\bar{x}_2$, $\bar{z}_1$, and $\bar{z}_2$ lies on P. It follows that the degrees of $\bar{y}_1$ and $\bar{y}_2$ are at least 3. Hence, by the definition of $S(G)$, the degrees of $\bar{x}_1$, $\bar{x}_2$, $\bar{z}_1$, and $\bar{z}_2$ are 2.

Let $\bar{w}_1$ be the neighbor of $\bar{y}_1$ and $\bar{w}_2$ the neighbor of $\bar{y}_2$ on P, let $\bar{a}_1$ be the other neighbor of $\bar{x}_1$ (on $\bar{C}_1$), $\bar{b}_1$ the other neighbor of $\bar{z}_1$, $\bar{a}_2$ the other neighbor of $\bar{x}_2$ (on $\bar{C}_2$), and $\bar{b}_2$ the other neighbor of $\bar{z}_2$, cf. Fig. 3.

Suppose that $\bar{w}_2\bar{y}_2\Theta\bar{x}_1\bar{a}_1$, $\bar{w}_2\bar{y}_2\Theta\bar{z}_1\bar{b}_1$, $\bar{w}_1\bar{y}_1\Theta\bar{x}_2\bar{a}_2$, and $\bar{w}_1\bar{y}_1\Theta\bar{z}_2\bar{b}_2$. Then, by the transitivity of Θ, the first two conditions imply $\bar{z}_1\bar{b}_1\Theta\bar{a}_1\bar{x}_1$. As C_1 and C_2 are isometric, this is only possible if $\bar{a}_1=\bar{x}$ and $\bar{b}_1=\bar{z}$. Analogously, the second two conditions imply that $\bar{x}=\bar{a}_2$ and $\bar{z}=\bar{b}_2$ and $\bar{w}_1=\bar{w}_2$. But then the vertices $\bar{x}_1$, $\bar{y}_1$, $\bar{z}_1$, $\bar{x}$, $\bar{y}$, $\bar{z}$, $\bar{x}_2$, $\bar{y}_2$, $\bar{z}_2$, and $\bar{w}_1$ induce an $S(K_4)$.

Assume now that one of the four conditions of the previous paragraph is not fulfilled. We may without loss of generality assume that $\bar{w}_2\bar{y}_2$ is not in relation Θ with $\bar{z}_1\bar{b}_1$. Now we have $d(\bar{z}_1,\bar{y}_2)=n+1$, $d(\bar{z}_1,\bar{w}_2)=n$. If a shortest $\bar{z}_1,\bar{z}_2$-path would pass $\bar{y}_1$, then, since $d(\bar{y}_1,\bar{z}_2)=n+1$, we would have $d(\bar{z}_1,\bar{z}_2)=n+2$, which is not the case. Since z_1 has degree 2 it follows that any shortest $\bar{z}_1,\bar{z}_2$-path passes $\bar{b}_1$, hence $d(\bar{b}_1,\bar{y}_2)\leqslant n$. Moreover, $d(\bar{b}_1,\bar{y}_2)=n$ for otherwise $d(\bar{z}_1,\bar{y}_2)\leqslant n$. Now, since $\bar{w}_2\bar{y}_2$ is not in relation Θ with $\bar{z}_1\bar{b}_1$ we must have $d(\bar{b}_1,\bar{w}_2)=n-1$. Let Q be a shortest $\bar{b}_1,\bar{w}_2$-path. Let $\bar{w}_3$ be the last common vertex of P and Q traversing from $\bar{w}_2$ and set $d(\bar{w}_3,\bar{y}_2)=p$. Let $\bar{w}_4$ be the neighbor of $\bar{w}_3$ on Q that is not on P. Then $\bar{w}_3\bar{w}_4$ is in relation Θ with $\bar{y}_1\bar{z}_1$. Indeed, $d(\bar{y}_1,\bar{w}_3)=n-p$, $d(\bar{z}_1,\bar{w}_4)=n-p$, $d(\bar{w}_3,\bar{z}_1)=n-p+1$, and $d(\bar{w}_4,\bar{y}_1)=n-p+1$. Transitivity of Θ thus implies that $\bar{w}_3\bar{w}_4\Theta\bar{y}_2\bar{z}_2$. It follows that $d(\bar{w}_3,\bar{y}_2)=d(\bar{w}_4,\bar{z}_2)$. Note that $\bar{w}_4$ is of degree 2. Therefore, a geodesic between $\bar{w}_4$ and $\bar{z}_2$ must pass the neighbor of $\bar{w}_4$ different from $\bar{w}_3$. However, this is not possible as we would get that $d(\bar{z}_1,\bar{z}_2)<n$, a contradiction. □

Proof of Theorem 2.2. Suppose first that each block of G is either a cycle or a complete graph. Then the blocks of $S(G)$ are partial cubes by Proposition 2.1. Now Lemma 1.5 implies that Θ is transitive and as $S(G)$ is bipartite, it is a partial cube by Theorem 1.6.

Conversely, let $S(G)$ be a partial cube. We may without loss of generality assume that G is 2-connected. If G is a cycle, we are done. So we may assume that this is not the case. Then G contains isometric cycles C_1 and C_2 as described in Lemma 2.4. Hence by Lemma 2.6 we have an induced K_4 in G. Let x_1, x_2, x_3, and x_4 be the vertices of this K_4. If G has 4 vertices, we are done. Otherwise, let y be another vertex of G adjacent to x_1. Then, as G is 2-connected, there is a path P between y and another vertex of K_4, say x_2. We may select P is such a way that $x_1\to y\to\cdots P\to\cdots x_2\to x_1$ is an isometric cycle. Therefore by Lemma 2.6 y is adjacent with x_2 and x_3. Similarly, we infer that y is also adjacent with x_4. Induction completes the proof. □

3. On regular partial cubes

For the (probably) most important subclass of partial cubes, median graphs, Mulder [16] proved that hypercubes are the only regular median graphs. Besides hypercubes,

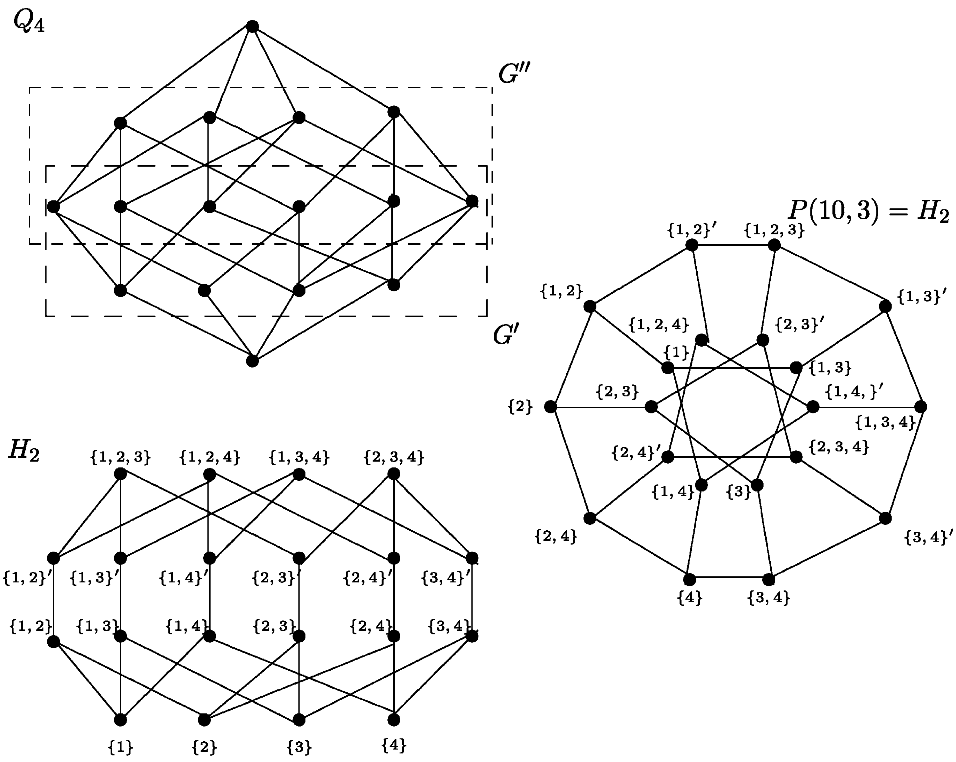

Fig. 4. Expanding the three middle levels of Q_4 to $P(10,3)$.

even cycles are regular partial cubes. Moreover, the Cartesian product of two (regular) partial cubes is a (regular) partial cube. In [14] it was asked if in this way one obtains all regular partial cubes. However, this is not the case.

First, an additional example of such a graph is due to Gedeonova [10], see also Fig. 2.4 of [13]. Also, J. Koolen (personal communication) pointed out that the middle-level graphs are also regular partial cubes. (For a given $(2n+1)$-cube its middle-levels graph is the subgraph induced by the middle two levels in its subset representation, cf. [18]. For instance, the middle-levels graph of Q_3 is C_6.)

The middle-levels graphs can be constructed also in the following way. Consider the $2n$-cube in its subset representation. Let G_n be the subgraph of Q_{2n} induced by the subsets on $n-1$, n, and $n+1$ elements. Let G' be the subgraph of G_n induced by the subsets on $n-1$ and n elements, and let G'' be the subgraph induced by the subsets on n and $n+1$ elements. Then it is straightforward to verify that G' and G'' form a proper cover of G_n. Thus we may expand G_n with respect to G' and G'' to obtain a partial cube H_n. Observe finally that H_n is the middle-levels graph of Q_{2n+1}.

The above construction gives an alternative argument for the fact that the middle-levels graphs are partial cubes. The construction for Q_4 is shown in Fig. 4. The graph

H_2 is isomorphic to the generalized Petersen graph $P(10,3)$, cf. also [17] where regular subgraphs of hypercubes are studied.

This observation raises the question whether there are more regular partial cubes among generalized Petersen graphs.

Proposition 3.1. *$P(2m,1)$, $m \geqslant 2$, and $P(10,3)$ are the only (regular) partial cubes among the generalized Petersen graphs $P(n,k)$.*

Proof. We have seen above that $P(10,3)$ is a partial cube. Moreover, $P(2m,1)$ is isomorphic to $C_{2m} \square K_2$, thus it is a (regular) partial cube.

It remains to show that in all the other cases $P(n,k)$ is not a partial cube. First note that a bipartite $P(n,k)$ is of the form $P(2k, 2\ell+1)$, $k,\ell \in \mathbb{N}$, $k \geqslant 2\ell+1$. Denote the vertices of the outer cycle of $P(2k,2\ell+1)$ with $1,2,\ldots,2k$ and the corresponding inner vertices by $1',2',\ldots,(2k)'$.

Assume $\ell > 1$. Considering the cycle $1 \to 2 \to \cdots \to 2\ell+2 \to (2\ell+2)' \to 1' \to 1$ we note that the edge $11'$ is in relation Θ with $(\ell+2)(\ell+3)$. Similarly, from the cycle $1 \to 2 \to \cdots \to \ell+3 \to (\ell+3)' \to (2k-\ell+2)' \to 2k-\ell+2 \to 2k-\ell+3 \to \cdots \to 1$ we infer that $(\ell+2)(\ell+3)$ is in relation Θ with $1(2k)$. Hence Θ is clearly not transitive.

It remains to consider the case $\ell = 1$. Note first that $11'\Theta 34$ and $34\Theta 66'$. Continuing in this way along the outer cycle we find that $2k \equiv 0 \pmod 5$, that is, $n = 10t$. However, in this case $2'5'$ must be in the same Θ-class as $11'$. As this is only possible if $t = 1$, the proof is complete. □

A classification of regular partial cubes remains a challenging open problem.

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