Total k-clique mutual-visibility of graphs

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Abstract

Let G be a graph and $k \leq \omega(G)$. A set $A \subseteq V(G)$ is a total k-clique mutual-visibility set if for each two distinct k-cliques X and Y of G, there exists a shortest X,Y-path P such that $V(P) \cap A \subseteq V(X) \cup V(Y)$. The order of a largest total k-clique mutual-visibility set of G is the total k-clique mutual-visibility number $\mu_t^k(G)$. In this paper, we propose this concept as an extension of the total mutual-visibility number of G. The total k-clique mutual-visibility number is determined for several families of graphs including C_n, K_n, and $\Gamma(\mathbb{Z}_n)$. This invariant is studied under the generalized lexicographic product, the direct product, the corona product, and the edge corona product. Using the results on μ_t^k of direct products, the invariant is determined for unitary Cayley graph of $\mathbb{Z}_n$. Furthermore, results on μ_t^k of corona products are applied to prove that the decision problem for μ_t^k is NP-complete.

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1 Introduction

In this paper, all graphs considered are undirected and simple. If $u, v \in V(G)$, then $d(u, v)$ denotes the number of edges on a shortest u, v-path in G. A clique X of G is its complete subgraph, by abuse of language we may also consider a clique to be the vertex set of it. The number of vertices in a largest clique of G is the clique number $\omega(G)$ of G. If $|V(X)| = k$, then we say that X is a k-clique. By $\mathcal{Q}(G)$ we will denote the set of all cliques of G and by $\mathcal{Q}_k(G)$ the set of all k-cliques of G.

The distance $d(X, Y)$ between $X, Y \in \mathcal{Q}(G)$ is defined as $d(X, Y) = \min\{d(u, v) : u \in V(X), v \in V(Y)\}$. A path between $x \in X$ and $y \in Y$ of length $d(X, Y)$ is a *shortest X, Y-path*. Let $A \subseteq V(G)$ and $k \in [\omega(G)] = \{1, \dots, \omega(G)\}$. Then $X, Y \in \mathcal{Q}_k(G)$ are *k-clique A-visible* if there exists a shortest X, Y-path P such that no inner vertex of P belongs to A. The set A is a *total k-clique mutual-visibility set* if every $X, Y \in \mathcal{Q}_k(G)$ are k-clique A-visible. The order of a largest total k-clique mutual-visibility set of G is the *total k-clique mutual-visibility number* of G and it is denoted by $\mu_t^k(G)$. Moreover, A is a *μ_t^k-set* if it is a total k-clique mutual-visibility set of order $\mu_t^k(G)$. The above definitions can also be extended to $k > \omega(G)$, in which case we have $\mu_t^k(G) = |V(G)|$. Hence unless stated otherwise, we will assume in the rest that $k \le \omega(G)$.

If $k = 1$, then $\mu_t^1(G) = \mu_t(G)$, where $\mu_t(G)$ is the total mutual-visibility number of G introduced in [11] as a tool to better understand the mutual-visibility number of strong products of graphs. The latter concept was coined by Di Stefano in [14]. Soon after the concept was explored in several dozen articles, see, for example, [12, 20, 23–25, 30, 31]. We would like to highlight the paper [8] which extends the mutual-visibility in a different direction than we do in this paper. In [8] the k-distance mutual-visibility problem was investigated, where for a given threshold k we require that only pair of vertices of a given set are visible if they are at distance at most k. The total mutual-visibility number has also been the subject of wide interest, see [3–7, 9, 10, 24, 26, 34].

Network robustness denotes the capacity of a network to preserve its operational integrity despite the removal or failure of certain nodes or edges. In the context of social networks, this implies that individuals or groups can continue to interact and share information, even in the event of a member's departure or the loss of a connection. Accordingly, identifying the largest subset of individuals whose absence or reduced engagement causes minimal disruption to the overall structure equates to determining the maximum total mutual-visibility set within the graph representing the social network.

If, the emphasis is placed on interactions between groups rather than individuals, the corresponding objective shifts to finding the maximum total clique mutual-visibility set in the associated graph.

The article is organized as follows. At the end of this section, we provide additional necessary definitions. In Section 2 we compute the total k-clique mutual-visibility number μ_t^k for several families of graphs. In Section 3 we determine μ_t^k for generalized lexicographic products where the inserted graphs are edgeless, and for the comaximal graph of $\mathbb{Z}_n$. In the subsequent section we determine μ_t^k for the direct products of complete multiparite graphs. Using these results, we investigate the total 1-clique mutual-visibility number for the unitary Cayley graph of $\mathbb{Z}_n$. In Section 5, we obtain the exact value of μ_t^k for the corona and the edge corona product of two graphs. Using these results, we prove that the decision problem for μ_t^k is NP-complete.

As already indicated, for a positive integer t, the set $\{1,\ldots,t\}$ is denoted by $[t]$. The complement of a graph G is denoted by $\overline{G}$. $K_{n_1,\ldots,n_m}$ is the complete multipartite graph with m parts of respective cardinalities n_i, $i \in [m]$. A *dominating set* of G is a set of vertices S such that each vertex of $V(G)\setminus S$ is adjacent to at least one vertex from S. The minimum cardinality among dominating sets of G is the *dominating number* of G and denoted by $\gamma(G)$. A vertex of G adjacent to all the other vertices is a *dominating vertex*. We will also use the notation $u \sim_G v$ saying that u and v are adjacent vertices in G, and $G \cong H$ saying that G and H are isomorphic. An edgeless graph G is a graph with no edges; if its order is n, then $G \cong \overline{K}_n$.

2 Some basic graph classes

In this section, we determine μ_t^k for several families of graphs. Before we move on to standard graph classes, we have the following result.

Proposition 2.1. *For a connected graph G, the following statements hold.*

(i) $\mu_t^k(G) = |V(G)|$ *if and only if* $d(X,Y) \leqslant 1$ *for each* $X,Y \in \mathcal{Q}_k(G)$.

(ii) If $\gamma(G) = 1$ *and there exist* $X,Y \in \mathcal{Q}_k(G)$ *with* $d(X,Y) > 1$*, then* $\mu_t^k(G) = |V(G)| - 1$.

(iii) If $\mu_t^k(G) = |V(G)| - 1$*, then* $d(X,Y) \leqslant 2$ *for each* $X,Y \in \mathcal{Q}_k(G)$*, and there exist* $X',Y' \in \mathcal{Q}_k(G)$ *with* $d(X',Y') = 2$.

Proof. (i) Obvious.

(ii) Since $\gamma(G) = 1$, for each $X,Y \in \mathcal{Q}_k(G)$ we have $d(X,Y) \leqslant 2$. Let s be a dominating vertex of G. Then $A = G \setminus \{s\}$ is a total k-clique mutual-visibility set for

G. By hypothesis, there exist $X, Y \in \mathcal{Q}_k(G)$ with $d(X,Y) > 12$, hence $d(X,Y) = 2$. Moreover, s is the inner vertex of some shortest X,Y-path. Therefore A is a μ_t^k-set, and so $\mu_t^k(G) = |V(G)| - 1$.

(iii) Let $\mu_t^k(G) = |V(G)| - 1$ and let $A = G \setminus \{u\}$ be a μ_t^k-set. Let $X, Y \in \mathcal{Q}_k(G)$. Then $d(X,Y) \leq 2$, for otherwise we would have $\mu_t^k(G) = |V(G)| - 2$. In addition, (i) implies that exist $X', Y' \in \mathcal{Q}_k(G)$ with $d(X', Y') = 2$. □

By Proposition 2.1(i) we have $\mu_t^k(K_n) = n$. This is a special case of the following result which can be deduced without difficulties from Proposition 2.1, hence we omit its proof.

Proposition 2.2. *Let* $1 \leq m_1 \leqslant \cdots \leqslant m_n$. *If* $k > 1$, *then* $\mu_t^k(K_{m_1,\ldots,m_n}) = m_1 + \cdots + m_n$, *and if* $k = 1$, *then*

$$\mu_t^1(K_{m_1,\ldots,m_n}) = \begin{cases} m_1 + \cdots + m_n; & m_n = 1, \\ m_1 + \cdots + m_n - 1; & m_{n-1} = 1, m_n > 1, \\ m_1 + \cdots + m_n - 2; & \text{otherwise.} \end{cases}$$

By [11, Corollary 3.6] we know that if G is complete multipartite graph, then $\mu(G) = \mu_t(G)$. Hence Proposition 2.2 also reports the mutual-visibility number of complete multipartite graphs.

The *fan graph* $F_{1,n}$, $n \geq 3$, is the graph obtained from the disjoint union of P_n and K_1 by adding all the edges between the vertex of K_1 and every vertex of P_n. The *wheel graph* $W_{1,n}$ is obtained in an analogous way from the disjoint union of C_n and K_1. The proof of the next result is tedious but straightforward and hence omitted.

Theorem 2.3. *The following equalities hold.*

$$\mu_t^k(P_r) = \begin{cases} 2; & k = 1, r \geqslant 2, \\ r; & k = 2, 2 \leqslant r \leqslant 3, \\ 4; & k = 2, r > 3, \\ r; & k \geqslant 3. \end{cases} \qquad \mu_t^k(C_s) = \begin{cases} 3; & k = 1, s = 3, \\ 2; & k = 1, s = 4, \\ 0; & k = 1, s > 4, \\ s; & k = 2, 3 \leqslant s \leqslant 5, \\ 3; & k = 2, s = 6, \\ 0; & k = 2, s > 6, \\ s; & k \geqslant 3. \end{cases}$$

$$\mu_t^k(F_{1,n}) = \begin{cases} n; & k = 1, \\ n; & k = 2, n \geqslant 5, \\ n+1; & k = 2, n < 5, \\ n+1; & k \geqslant 3. \end{cases} \qquad \mu_t^k(W_{1,n}) = \begin{cases} n; & k = 1, \\ n; & k = 2, n \geqslant 6, \\ n+1; & k = 2, n < 6, \\ n+1; & k \geqslant 3. \end{cases}$$

The *helm graph* H_n, $n \geq 3$, is the graph with $2n+1$ vertices obtained from $W_{1,n}$ by attaching a pendant edge to each vertex of the n-cycle of $W_{1,n}$.

Theorem 2.4. *If* $n \geq 3$*, then*

$$\mu_t^k(H_n) = \begin{cases} n+1; & k = 1, 3 \leqslant n \leqslant 5, \\ n; & k = 1, n \geqslant 6, \\ 7; & k = 2, n = 3, \\ 2n; & k = 2, n > 3, \\ 2n+1; & k \geqslant 3. \end{cases}$$

Proof. Let $V(H_n) = \{u_0, u_1, , \ldots, u_n, v_1, \ldots, v_n\}$, where $u_1, \ldots, u_n$ are the consecutive vertices of the n-cycle of $W_{1,n}$ and $v_1, \ldots, v_n$ are the corresponding pendant vertices.

First assume that $k = 1$ and $3 \leqslant n \leqslant 5$. Then $A = \{u_0, v_1, \ldots, v_n\}$ is a total mutual-visibility set. Since for any i, at most two vertices among u_0, u_i, v_i lie in a total mutual-visibility set, A is a μ_t^1-set. Hence $\mu_t^1(H_n) = n+1$ in this situation. If $k = 1$ and $n \geqslant 6$, then we can similarly verify that $A = \{v_1, \ldots, v_n\}$ is a μ_t^1-set. The cases $k = 2$ and $n = 3$, and $k \geqslant 3$ are covered by Proposition 2.1. The last case to consider is $k = 2$ and $n > 3$, in which case we can verify that $\{u_1, \ldots, u_n, v_1, \ldots, v_n\}$ is a μ_t^2-set. □

3 Generalized lexicographic products

Let G be a (connected) graph with the vertex set $V(G) = \{v_1, \ldots, v_n\}$, $n \geq 2$, and let $H_1, \ldots, H_n$ be pairwise disjoint graphs. The *generalized lexicographic product* $G[H_1, \ldots, H_n]$ is the graph formed by replacing each vertex v_i of G by the graph H_i, and then joining each vertex of H_i to each vertex of H_j whenever $v_i \sim_G v_j$. Since $G[K_1, \ldots, K_1] \cong G$, we may assume in the rest of this section that at least one H_i is of order at least 2.

We now determine the total k-clique mutual visibility number for generalized lexicographic products $G[H_1, \ldots, H_n]$, where each H_i is edgeless. We consider two cases, first when $\mu_t^k(G) \neq |V(G)|$, and second when $\mu_t^k(G) = |V(G)|$.

Theorem 3.1. *Let* $k \geq 1$*. If* G *is a connected graph of order* $n > 1$ *with* $\mu_t^k(G) \neq n$*, and* H_i*,* $i \in [n]$ *are pairwise disjoint edgeless graphs, then*

$$\mu_t^k(G[H_1, \ldots, H_n]) = \sum_{i=1}^{n} |V(H_i)| - n + \mu_t^k(G).$$

Proof. Set $\widehat{G} = G[H_1, \ldots, H_n]$ for the rest of the proof. For $i \in [n]$, let h_i be an arbitrary fixed vertex of H_i. Then the subgraph G' of $\widehat{G}$ induced by the vertices $h_1, \ldots, h_n$ is isomorphic to G. Let $A_{G'}$ be a μ_t^k-set of G' and set

$$A = \bigcup_{i=1}^{n} (V(H_i) \setminus \{h_i\}) \bigcup A_{G'} .$$

We claim that A is a total k-clique mutual-visibility set for $\widehat{G}$.

Let $X, Y \in \mathcal{Q}_k(\widehat{G})$. If $V(X) \cup V(Y) \subseteq V(G')$, then clearly X and Y are k-clique A-visible. So without loss of generality, we assume that $V(X) \nsubseteq V(G')$. Note that $|V(X) \cap V(H_i)| \leqslant 1$ and also $|V(Y) \cap V(H_i)| \leqslant 1$, for each $i \in [n]$. Let $V(X) = \{h'_{i_1}, \ldots, h'_{i_k}\}$ and $V(Y) = \{h'_{j_1}, \ldots, h'_{j_k}\}$, where $h'_{i_r} \in H_{i_r}$ and $h'_{j_r} \in H_{j_r}$, for $1 \leqslant r \leqslant k$. Consider $X', Y' \in \mathcal{Q}_k(\widehat{G})$ with vertex-sets $V(X') = \{h_{i_1}, \ldots, h_{i_k}\}$ and $V(Y') = \{h_{j_1}, \ldots, h_{j_k}\}$, where $h_{i_l}, h_{j_l} \in V(G')$, $l \in [k]$. We distinguish the following two cases.

Case 1: $V(X') \cap V(Y') = \emptyset$.
Since X' and Y' are k-clique $A_{G'}$-visible, there exists a shortest X', Y'-path P in G' such that $V(P) \cap A_{G'} \subseteq X' \cup Y'$. Assume that P is of the form $h_{i_l} \sim g_1 \sim \cdots \sim g_r \sim h_{j_{l'}}$, where $h_{i_l} \in V(X')$, $h_{j_{l'}} \in V(Y')$ and $g_1, \ldots, g_r \in G'$. Since $h'_{i_l} \in V(X)$ is adjacent to g_1 and $h'_{j_{l'}} \in V(Y)$ is adjacent to g_r, we see that $h'_{i_l} \sim g_1 \sim \cdots \sim g_r \sim h'_{j_{l'}}$ is a path between $X, Y \in \mathcal{Q}_k(\widehat{G})$ of length, say r. We claim that this is a shortest path between X and Y. Assume on the contrary that there exists a path $h'_{i_s} \sim a_1 \sim \cdots \sim a_m \sim h'_{j_{s'}}$ of length less than r between X and Y, where $h'_{i_s} \in V(X)$, $h'_{j_{s'}} \in V(Y)$ and $a_i \in \widehat{G}$, for each $i \in [m]$. Now by replacing each $a_i \in V(H_i) \setminus V(G')$ with h_i, we find the path $h_{i_s} \sim b_1 \sim \cdots \sim b_m \sim h_{j_{s'}}$ of length less than r between X' and Y' in G', such that $b_i \in V(G')$, $i \in [m]$, and this is impossible. Therefore X and Y are k-clique A-visible.

Case 2: $V(X') \cap V(Y') \neq \emptyset$.
Let $h_i \in V(X') \cap V(Y')$. Then there exist h'_i and h''_i in $V(H_i)$ such that $h'_i \in V(X)$ and $h''_i \in V(Y)$. If $h_i = h'_i = h''_i$, then $d(X, Y) = 0$, and so they are k-clique A-visible. Otherwise, we may assume that $h'_i \neq h_i$. Since h'_i is adjacent to any vertex that h''_i is adjacent to, we have $d(X, Y) \leqslant 1$, and so they are k-clique A-visible.

We have thus proved that $\mu_t^k(G[H_1, \ldots, H_n]) \geq \sum_{i=1}^n |V(H_i)| - n + \mu_t^k(G)$. To prove the reverse inequality, suppose on the contrary that there exists a k-clique mutual-visibility set A of $\widehat{G}$ of cardinality larger than $\sum_{i=1}^n |V(H_i)| - n + \mu_t^k(G)$. Then by the pigeonhole principle we infer that $A \cap V(H_i) = V(H_i)$ holds for more than $\mu_t^k(G)$ indices i. But then restricting to the subgraph G' as described at the beginning of the proof we would get a k-clique mutual-visibility set of G strictly larger than $\mu_t^k(G)$, a contradiction.

□

Theorem 3.2. *Let $k \geq 1$. If G is a connected graph of order $n > 1$ with $\mu_t^k(G) = n$, and H_i, $i \in [n]$, are pairwise disjoint edgeless graphs, then*

$$\mu_t^k(G[H_1, \dots, H_n]) = \begin{cases} \sum_{i=1}^n |V(H_i)|; & k > 1, \\ \sum_{i=1}^n |V(H_i)| - 1; & k = 1, f = 1, \\ \sum_{i=1}^n |V(H_i)| - 2; & k = 1, f > 1, \end{cases}$$

where f is the number of subgraphs H_i with $|V(H_i)| > 1$.

Proof. Set $\widehat{G} = G[H_1, \dots, H_n]$. Since $\mu_t^k(G) = |V(G)|$, by Proposition 2.1 we have $d(X, Y) \leqslant 1$ for each $X, Y \in \mathcal{Q}_k(G)$.

Assume first that $k > 1$ and let $X, Y \in \mathcal{Q}_k(\widehat{G})$. Let $V(X) = \{h'_{i_1}, \dots, h'_{i_k}\}$ and $V(Y) = \{h'_{j_1}, \dots, h'_{j_k}\}$, where $h'_{i_r} \in H_{i_r}$ and $h'_{j_r} \in H_{j_r}$, for $r \in [k]$. Let h_i, $i \in [n]$, be an arbitrary fixed vertex of H_i, and let G' be the subgraph of $\widehat{G}$ induced by the vertices $h_1, \dots, h_n$. Consider the k-cliques X' and Y' with vertex sets $V(X') = \{h_{i_1}, \dots, h_{i_k}\}$ and $V(Y') = \{h_{j_1}, \dots, h_{j_k}\}$, where $h_{i_l}, h_{j_l} \in V(G')$, $l \in [k]$. If $V(X') \cap V(Y') = \emptyset$, then since $d(X', Y') \leqslant 1$, there exist $h_{i_l} \in V(X')$ and $h_{j_s} \in V(Y')$ such that $d(h_{i_l}, h_{j_s}) \leqslant 1$. Thus we have $d(h'_{i_l}, h'_{j_s}) \leqslant 1$ which implies that $d(X, Y) \leqslant 1$. If $V(X') \cap V(Y') \neq \emptyset$, then again $d(X, Y) \leqslant 1$. So for $k > 1$, by Proposition 2.1, we have $\mu_t^k(G[H_1, \dots, H_n]) = \sum_{i=1}^n |V(H_i)|$.

Assume second that $k = 1$. In the first subcase let $f = 1$ and let $|V(H_i)| > 1$, for some $j \in [n]$. Then $\cup_{i=1}^n H_i \setminus \{h_j\}$, where $V(H_j) = \{h_j\}$, $1 \leqslant j \neq i \leqslant n$, is a μ_t^1-set for $\widehat{G}$. And if $f > 1$, then $\cup_{i=1}^n H_i \setminus \{h_r, h_s\}$, where $h_r \in V(H_r)$, $h_s \in V(H_s)$ and $1 \leqslant r \neq s \leqslant n$, is a μ_t^1-set for $\widehat{G}$. □

3.1 Comaximal graphs of $\mathbb{Z}_n$

Let $\mathbb{Z}_n$ be the ring of integers modulo n. As a consequence of the findings of the first part of the section, we next compute the total k-clique mutual-visibility number for the comaximal graph of $\mathbb{Z}_n$.

Let R be a commutative ring with nonzero identity. We denote the set of all unit elements and zero divisors of R by $U(R)$ and $Z(R)$, respectively. Also by $Z^*(R)$ we denote the set $Z(R) \setminus \{0\}$. Sharma and Bhatwadekar [33] defined the *comaximal graph* of a commutative ring R as a simple graph whose vertices are the elements of R, and two distinct vertices a and b are adjacent if $aR + bR = R$, where cR is the ideal generated by $c \in R$. Let $\Gamma(R)$ be an induced subgraph of the comaximal graph with nonunit elements of R as vertices. The properties of the graph $\Gamma(R)$ were studied in [27,28,35].

For two integers r and s, the notation (r, s) stands for the greatest common divisor of r and s. Also we denote the elements of the ring $\mathbb{Z}_n$, where $n > 1$, by $0, 1, 2, \dots, n-1$.

For every nonzero element a in $\mathbb{Z}_n$, if $(a, n) = 1$, then a is a unit element; otherwise, $(a, n) \neq 1$, and so a is a zero divisor. Therefore, $|U(\mathbb{Z}_n)| = \phi(n)$ and $|Z(\mathbb{Z}_n)| = n - \phi(n)$, where ϕ is the Euler's totient function.

An integer d is said to be a *proper divisor* of n if $1 < d < n$ and $d \mid n$. Now let $d_1, \ldots, d_r$ be the distinct proper divisors of n. For $i \in [r]$, set

$$A_{d_i} := \{x \in \mathbb{Z}_n : \ (x, n) = d_i\}.$$

The sets $A_{d_1}, \ldots, A_{d_r}$ are pairwise disjoint. Further,

$$Z^*(\mathbb{Z}_n) = A_{d_1} \cup \cdots \cup A_{d_r}$$

and

$$V(\Gamma(\mathbb{Z}_n)) = \{0\} \cup A_{d_1} \cup \cdots \cup A_{d_r}.$$

Lemma 3.3. [36, Proposition 2.1] *If $i \in [r]$, then $|A_{d_i}| = \phi(\frac{n}{d_i})$.*

In the rest, the induced subgraph of $\Gamma(\mathbb{Z}_n)$ on the set A_{d_i}, $i \in [r]$, is denoted by $\Gamma(A_{d_i})$.

Lemma 3.4. [1, Lemma 3.2] *The following statements hold.*

(i) Two distinct vertices x and y are adjacent in $\Gamma(\mathbb{Z}_n)$ if and only if $(x, y) \in U(\mathbb{Z}_n)$.

(ii) If $i \in [r]$, then $\Gamma(A_{d_i})$ is isomorphic to $\overline{K}_{\phi(\frac{n}{d_i})}$.

(iii) For $1 \leqslant i \neq j \leqslant r$, a vertex of A_{d_i} is adjacent to a vertex of A_{d_j} if and only if $(d_i, d_j) = 1$.

Now, we introduce a graph G_n, which plays an important role in the structure of $\Gamma(\mathbb{Z}_n)$. The graph G_n has vertex set $\{d_1, \ldots, d_r\}$, where d_i, $i \in [r]$, is a proper divisor of n, and two distinct vertices d_i and d_j are adjacent if $(d_i, d_j) = 1$.

Let $n = p_1^{\alpha_1} \cdots p_s^{\alpha_s}$ be the prime factorization of n, where $s, \alpha_1, \ldots, \alpha_s$ are positive integers and $p_1, \ldots, p_s$ are distinct prime numbers. Every divisor of n is of the form $p_1^{\beta_1} \cdots p_s^{\beta_s}$, for some integers $\beta_1, \ldots, \beta_s$, where $0 \leqslant \beta_i \leqslant \alpha_i$ for each $i \in [s]$. Hence the number of proper divisors of n is equal to $\prod_{i=1}^{s}(n_i + 1) - 2$. Therefore we have

$$r = |V(G_n)| = \prod_{i=1}^{s}(n_i + 1) - 2. \tag{1}$$

Let $\Gamma^*(\mathbb{Z}_n) = \Gamma(\mathbb{Z}_n) \setminus \{0\}$. Consider the graph G_n and replace each vertex d_i of G_n by $\Gamma[A_{d_i}]$. In view of Lemma 3.3, we have

$$\Gamma^*(\mathbb{Z}_n) = G_n[\overline{K}_{\phi(\frac{n}{d_1})}, \ldots, \overline{K}_{\phi(\frac{n}{d_r})}].$$

Theorem 3.5. *If $n > 1$ and G_n is connected graph with $\mu_t^k(G_n) \neq r$, where r is as in* (1)*, then*

$$\mu_t^k(\Gamma^*(\mathbb{Z}_n)) = n - r - 1 - \phi(n) + \mu_t^k(G_n).$$

Proof. As established above, we have

$$\Gamma^*(\mathbb{Z}_n) = G_n[\overline{K}_{\phi(\frac{n}{d_1})}, \overline{K}_{\phi(\frac{n}{d_2})}, \dots, \overline{K}_{\phi(\frac{n}{d_r})}].$$

Since the vertex-set of $\Gamma^*(\mathbb{Z}_n)$ consists of nonzero and nonunit elements of $\mathbb{Z}_n$, we have $|V(\Gamma^*(\mathbb{Z}_n))| = n - \phi(n) - 1$. Now the result follows from Theorem 3.1. □

4 Direct product

The direct product of G and H, denoted by $G \times H$, is the graph with vertex set $V(G) \times V(H)$ in which (u_1, v_1) and (u_2, v_2) are adjacent if u_1 and u_2 are adjacent in G and v_1 and v_2 are adjacent in H. Note that if at least one of the graphs G or H has a cycle of odd length, then $G \times H$ is connected. See [17] for more information on this product. This section initially focuses on analyzing the total k-clique mutual-visibility number of direct products of complete mutipartite graphs. The findings are then employed to evaluate this parameter in specific unitary Cayley graphs.

4.1 Total k-clique mutual-visibility number under direct products

In the below theorem, we determine the total 1-clique mutual-visibility number for the direct product of a complete m-partite graph with $m > 2$ and a complete bipartite graph.

Theorem 4.1. *If $G = K_{n_1,\dots,n_m}$, $m > 2$, and $H = K_{r_1,r_2}$, $r_1, r_2 \geq 1$, then*

$$\mu_t^1(G \times H) = |V(G)| \cdot |V(H)| - 6.$$

Proof. Let G_i, $i \in [m]$, be the mutipartition part of G of order n_i, and let H_j, $j \in [2]$ be the bipartition part of H of order r_j.

Let S be an arbitrary total 1-clique mutual-visibility set for $G \times H$. We claim that $|S| \leq |V(G)| \cdot |V(H)| - 6$. Consider three vertices $u_i \in V(G_i) \times V(H_1)$, $i \in [3]$. Since S is total mutual-visibility set and $d(u_1, u_2) = 2$, there exists a vertex $w_{12} \in V(G) \times V(H_2)$ such that $w_{12} \notin S$. Similarly, there exist vertices $w_{13}, w_{23} \in V(G) \times V(H_2)$ which also do not lie in S. Observe that w_{12}, w_{13}, and w_{23} are pairwise different. Analogously, by considering three vertices $u'_i \in V(G_i) \times V(H_2)$, $i \in [3]$, we find three vertices from

$V(G)\times V(H_1)$ which do not belong to S. We can conclude that $|S| \le |V(G)|\cdot|V(H)|-6$ and hence $\mu_t^1(G\times H) \le |V(G)|\cdot|V(H)|-6$.

Let $V(G_i)=\{g_1^i,\dots,g_{n_i}^i\}$, $i\in[m]$, let $V(H_j)=\{h_1^j,\dots,h_{r_j}^j\}$, $j\in[2]$, and set

$$A = V(G\times H)\setminus\{(g_1^1,h_1^1),(g_1^2,h_1^1),(g_1^3,h_1^1),(g_1^1,h_1^2),(g_1^2,h_1^2),(g_1^3,h_1^2)\}.$$

We claim that A is a total 1-clique mutual-visibility set for $G\times H$. Let u,v be arbitrary vertices of $G\times H$. Since $\mathrm{diam}(G\times H)=3$, there are three typical cases to be considered. If $d(u,v)=1$, there is nothing to be proved. If $d(u,v)=3$, then there exists $i\in[m]$ such that, without loss of generality, $u\in V(G_i)\times V(H_1)$ and $v\in V(G_i)\times V(H_2)$. Then there exists two vertices (g_1^l,h_1^1) and $(g_1^{l'},h_1^2)$ in the set

$$\{(g_1^1,h_1^1),(g_1^2,h_1^1),(g_1^3,h_1^1),(g_1^1,h_1^2),(g_1^2,h_1^2),(g_1^3,h_1^2)\},$$

where $1\leqslant l\neq l'\neq i\leqslant 3$. Now $u\sim(g_1^l,h_1^2)\sim(g_1^{l'},h_1^1)\sim v$ is a shortest u,v-path, hence u and v are 1-clique A-visible. It remains to consider the situation when $d(u,v)=2$, for which we distinguish the following two cases.

Case 1: $u\in V(G_i)\times V(H_j)$, $v\in V(G_{i'})\times V(H_j)$, $i,i'\in[m]$, $i\neq i'$, $j\in[2]$.
Then there exists a vertex $(g_1^l,h_1^{l'})$ in the set

$$\{(g_1^1,h_1^1),(g_1^2,h_1^1),(g_1^3,h_1^1),(g_1^1,h_1^2),(g_1^2,h_1^2),(g_1^3,h_1^2)\},$$

where $l\in\{1,2,3\}\setminus\{i,i'\}$ and $1\leqslant l'\neq j\leqslant 2$. Now $u\sim(g_1^l,h_1^{l'})\sim v$ is a shortest u,v-path.

Case 2: $u,v\in V(G_i)\times V(H_j)$, $i\in[m]$, $j\in[2]$.
Then $u\sim(g_1^l,h_1^{l'})\sim v$ is a shortest u,v-path, where $1\leqslant l\neq i\leqslant 3$ and $1\leqslant l'\neq j\leqslant 2$.

We can conclude that $\mu_t^1(G\times H)\ge|V(G)|\cdot|V(H)|-6$ and henceforth $\mu_t^1(G\times H)=|V(G)|\cdot|V(H)|-6$. □

The smallest case covered by Theorem 4.1 is $K_{1,1,1}\times K_{1,1}\cong K_3\times K_2\cong C_6$. It is know from earlier (cf. [34]) and also easy to verify that $\mu_t^1(C_6)=0$ just as claimed by the theorem.

Theorem 4.2. *If $G=K_{n_1,\dots,n_m}$, $m>2$, and $H=K_{r_1,r_2,r_3}$, then*

$$\mu_t^1(G\times H)=|V(G)||V(H)|-6.$$

Proof. Let G_i, $i\in[m]$, be the mutipartition part of G of order n_i, and let H_j, $j\in[3]$, be the bipartition part of H of order r_j.

Let S be an arbitrary total 1-clique mutual-visibility set for $G\times H$. We claim that $|S|\le|V(G)|\cdot|V(H)|-6$. Consider three vertices $u_j\in V(G_i)\times V(H_j)$, $i\in$

$[m]$, $j \in [3]$. Since S is total mutual-visibility set and $d(u_1, u_2) = 2$, there exists a vertex $w_{12} \in V(G_k) \times V(H_3)$ such that $w_{12} \notin S$. Similarly, there exist vertices $w_{13} \in V(G_k) \times V(H_2)$, $w_{23} \in V(G_k) \times V(H_1)$, which also do not lie in S. Analogously, by considering three vertices $u'_j \in V(G_k) \times V(H_j)$, $k \in [m]$,$k \neq i$, $j \in [3]$ we find three vertices from $V(G) \times V(H_j)$, $j \in [3]$ which do not belong to S. We can conclude that $|S| \leq |V(G)| \cdot |V(H)| - 6$ and hence $\mu_t^1(G \times H) \leq |V(G)| \cdot |V(H)| - 6$.

Let $V(G_i) = \{g_1^i, \ldots, g_{n_i}^i\}$, $i \in [m]$, let $V(H_j) = \{h_1^j, \ldots, h_{r_j}^j\}$, $j \in [3]$, and set

$$A = V(G \times H) \setminus \{(g_1^1, h_1^2), (g_1^1, h_1^3), (g_1^2, h_1^1), (g_1^2, h_1^3), (g_1^3, h_1^1), (g_1^3, h_1^2)\}.$$

We claim that A is a total 1-clique mutual-visibility set for $G \times H$. Let u, v be arbitrary vertices of $G \times H$. Since $\text{diam}(G \times H) = 2$, there are three typical cases to be considered. If $d(u, v) = 1$, there is nothing to be proved. If $d(u, v) = 2$, then we have the following three cases.

Case 1: $u \in V(G_i) \times V(H_j)$, $v \in V(G_{i'}) \times V(H_j)$, where $i \in [m]$, $i \neq i'$, $j \in [3]$.
In this case there exists a vertex $(g_1^l, h_1^{l'})$ in the set

$$\{(g_1^1, h_1^2), (g_1^1, h_1^3), (g_1^2, h_1^1), (g_1^2, h_1^3), (g_1^3, h_1^1), (g_1^3, h_1^2)\},$$

where $l \in [3] \setminus \{i, i'\}$ and $1 \leqslant l' \neq j \leqslant 3$. Now $u \sim (g_1^l, h_1^{l'}) \sim v$ is a shortest u, v-path.

Case 2: $u \in V(G_i) \times V(H_j)$, $v \in V(G_i) \times V(H_{j'})$, where $i \in [m]$, $j, j' \in [3]$, $j \neq j'$.
Now there exists a vertex $(g_1^l, h_1^{l'})$ in the set

$$\{(g_1^1, h_1^2), (g_1^1, h_1^3), (g_1^2, h_1^1), (g_1^2, h_1^3), (g_1^3, h_1^1), (g_1^3, h_1^2)\},$$

where $1 \leqslant l \neq i \leqslant 3$ and $l' \in [3] \setminus \{j, j'\}$. Now $u \sim (g_1^l, h_1^{l'}) \sim v$ is a shortest u, v-path.

Case 3: $u, v \in V(G_i) \times V(H_j)$, where $i \in [m]$, $j \in [3]$.
Then $u \sim (g_1^l, h_1^{l'}) \sim v$ is a shortest u, v-path, where $1 \leqslant l \neq i \leqslant 3$ and $1 \leqslant l' \neq j \leqslant 3$.

We can conclude that $\mu_t^1(G \times H) \geq |V(G)| \cdot |V(H)| - 6$ and henceforth $\mu_t^1(G \times H) = |V(G)| \cdot |V(H)| - 6$. □

Theorem 4.3. *If $G = K_{n_1, \ldots, n_m}$, $m > 3$, and $H = K_{r_1, \ldots, r_{m'}}$, $m' > 3$, then*

$$\mu_t^1(G \times H) = |V(G)| \cdot |V(H)| - 4.$$

Proof. Let G_i, $i \in [m]$, be be the mutipartition part of G of order n_i, and let H_j, $j \in [m']$, be the bipartition part of H of order r_j.

Let S be an arbitrary total 1-clique mutual-visibility set for $G \times H$. We claim that $|S| \leq |V(G)| \cdot |V(H)| - 4$. Consider two vertices $u \in V(G_i) \times V(H_j)$ and $v \in V(G_{i'}) \times V(H_j)$. Since S is total mutual-visibility set and $d(u, v) = 2$, there exists a vertex $w_1 \in V(G_{i''}) \times V(H_{j'})$, such that $w_1 \notin S$. Now consider two vertices $u \in V(G_i) \times V(H_{j'})$ and $v \in V(G_{i''}) \times V(H_{j'})$, So there exists a vertex $w_2 \in V(G_{i'}) \times V(H_j)$, such that $w_2 \notin S$. By considering two vertices in $V(G_{i'}) \times V(H_j)$ and $V(G_{i'}) \times V(H_{j'})$, we find $w_3 \in V(G_i) \times V(H_{j''})$ which is not in S. Also, by considering two vertices in $V(G_{i'}) \times V(H_{j''})$ and $V(G_{i''}) \times V(H_{j''})$, we find a vertex $w_4 \in V(G_i) \times V(H_j)$ that is not in S. We conclude that $|S| \leq |V(G)| \cdot |V(H)| - 4$ and so $\mu_t^1(G \times H) \leq |V(G)| \cdot |V(H)| - 4$.

Let $V(G_i) = \{g_1^i, \ldots, g_{n_i}^i\}$, $i \in [m]$, and $V(H_j) = \{h_1^j, \ldots, h_{r_j}^j\}$, $j \in [m']$, and set

$$A = V(G \times H) \setminus \{(g_1^1, h_1^1), (g_1^2, h_1^2), (g_1^3, h_1^3), (g_1^4, h_1^4)\}.$$

We claim that A is a total 1-clique mutual-visibility set of $G \times H$. Let u, v be arbitrary vertices of $V(G \times H)$. If $d(u, v) = 1$, there is nothing to be proved. If $d(u, v) = 2$, then we have the following three cases.

Case 1: $u \in V(G_i) \times V(H_j)$, $v \in V(G_{i'}) \times V(H_j)$, where $1 \leqslant i \neq i' \leqslant m$, $j \in [m']$. In this case there exists a vertex (g_1^l, h_1^l) in the set

$$\{(g_1^1, h_1^1), (g_1^2, h_1^2), (g_1^3, h_1^3), (g_1^4, h_1^4)\},$$

where $l \in [4] \setminus \{i, i', j\}$. Now $u \sim (g_1^l, h_1^l) \sim v$ is a shortest u, v-path.

Case 2: $u \in V(G_i) \times V(H_j)$, $v \in V(G_i) \times V(H_{j'})$, where $i \in [m]$, $1 \leqslant j \neq j' \leqslant m'$. Then there exists a vertex (g_1^l, h_1^l) in the set

$$\{(g_1^1, h_1^1), (g_1^2, h_1^2), (g_1^3, h_1^3), (g_1^4, h_1^4)\},$$

where $l \in [4] \setminus \{i, j, j'\}$. Now $u \sim (g_1^l, h_1^l) \sim v$ is a shortest path.

Case 3: $u, v \in V(G_i) \times V(H_j)$, where $i \in [m]$, $j \in [m']$. Now $u \sim (g_1^l, h_1^l) \sim v$ is a shortest path, where $l \in [4] \setminus \{i, j\}$.

Therefore, we conclude that $\mu_t^1(G \times H) \geq |V(G)| \cdot |V(H)| - 4$ and so $\mu_t^1(G \times H) = |V(G)| \cdot |V(H)| - 4$. □

4.2 Total k-clique mutual-visibility in unitary Cayley graphs

By using the above results for the total 1-clique mutual-visibility of direct products of complete multipartite graphs we now determine the 1-clique mutual-visibility number of unitary Cayley graphs of $\mathbb{Z}_n$ for some values of n.

Let R be a finite commutative ring with nonzero identity and $R^\times$ denote the set of all unit elements of R. The *unitary Cayley graph* of R, which is denoted by $G_R = \mathrm{Cay}(R, R^\times)$, is a (simple) graph whose vertex set is R and two distinct vertices x and y are adjacent if and only if $x - y \in R^\times$. We refer to [2, 18, 19, 22, 29] for studies about the unitary Cayley graph of a commutative ring.

Notation 4.4. *Let R be a finite commutative ring. Then, by [15, p. 752], we can write $R \cong R_1 \times \cdots \times R_t$, where R_i is a finite local ring with maximal ideal $\mathfrak{m}_i$ for $i \in [t]$. This decomposition is unique up to permutation of factors. We denote the (finite) residue field $\frac{R_i}{\mathfrak{m}_i}$ by K_i and $f_i = |K_i| = \frac{|R_i|}{|\mathfrak{m}_i|}$. We also assume (after appropriate permutation of factors) that $f_1 \leqslant \cdots \leqslant f_t$.*

The following proposition is a basic consequence of the definition of the unitary Cayley graphs and it was illustrated in [2, Proposition 2.2].

Proposition 4.5. *Let R be a finite commutative ring. We have the following statements.*

(a) The graph G_R is a $|R^\times|$-regular graph.

(b) If R is a local ring with maximal ideal $\mathfrak{m}$, then G_R is a complete multipartite graph whose partite sets are the cosets of $\mathfrak{m}$ in R. In particular, G_R is a complete graph if and only if R is a field.

(c) Let $R \cong R_1 \times \cdots \times R_t$ be a product of local rings, then $G_R \cong \times_{i=1}^{t} G_{R_i}$. Hence, G_R is a direct product of complete multipartite graphs.

We use the below natation in the rest of this section.

Notation 4.6. *Let $\mathbb{Z}_n$ be the ring of integers modulo n. By the prime factorization theorem, we have $n = p_1^{r_1} \dots p_t^{r_t}$, where p_i's are prime numbers with $p_1 < \cdots < p_t$ and this factorization is unique up to the order of the factors. It is easy to see that $\mathbb{Z}_n \cong \mathbb{Z}_{p_1^{r_1}} \times \cdots \times \mathbb{Z}_{p_t^{r_t}}$, which is the direct product of the rings $\mathbb{Z}_{p_i^{r_i}}$, $1 \leqslant i \leqslant t$. Also $\mathbb{Z}_{p_i^{r_i}}$ is a local ring with the maximal ideal $\mathfrak{m}_i = \{rp_i \mid r \in \mathbb{Z}_{p_i^{r_i}}\}$ with $|\mathfrak{m}_i| = p_i^{r_i-1}$ and the number of cosets of $\mathfrak{m}_i$ in $\mathbb{Z}_{p_i^{r_i}}$ is equal to p_i, for each $i \in [t]$.*

Theorem 4.7. *Let $n = p_1^{r_1} \dots p_t^{r_t}$, where p_i's are prime numbers with $p_1 < \cdots < p_t$. Then $G_{\mathbb{Z}_n} \cong \times_{i=1}^{t} K_{\underbrace{p_i^{r_i-1},\dots,p_i^{r_i-1}}_{p_i}}$.*

Proof. By Proposition 4.5 and Notation 4.6, $G_{\mathbb{Z}_{p_i^{r_i}}} \cong K_{p_i^{r_i-1},\dots,p_i^{r_i-1}}$, for each $i \in [t]$. Now since $\mathbb{Z}_n \cong \mathbb{Z}_{p_1^{r_1}} \times \cdots \times \mathbb{Z}_{p_t^{r_t}}$, by the third part of Proposition 4.5, the result holds. □

Using Proposition 2.2 and Theorems 4.1, 4.2 and 4.3, we can compute $\mu_t^1(G_{\mathbb{Z}_n})$ when $n = p_1^{r_1} p_2^{r_2}$.

Theorem 4.8. *If $n = p_1^{r_1} p_2^{r_2}$, where $p_1 < p_2$ and $r_1, r_2 \geqslant 0$, then*

$$\mu_t^1(G_{\mathbb{Z}_n}) = \begin{cases} p_1; & r_1 = 1, r_2 = 0, \\ p_1^{r_1} - 2; & r_1 > 1, r_2 = 0, \\ p_1^{r_1} p_2^{r_2} - 6; & p_1 \in \{2,3\}, r_1, r_2 \geqslant 1, \\ p_1^{r_1} p_2^{r_2} - 4; & p_1 \geqslant 4, r_1, r_2 \geqslant 1. \end{cases}$$

5 Corona product and computational complexity

This section first addresses the computation of the total k-clique mutual-visibility number in corona product graphs. These results are then used to explore the complexity of the associated computational problem involving μ_t^k.

Let G be a graph with $V(G) = \{v_1, \ldots, v_n\}$ and let H be a graph. The *corona product* $G \circ H$ was defined in [16] as the graph obtained from G and H by taking one copy of G and n copies of H and the edge set of $G \circ H$ is the union of the edge set of G, the edges of H_i (the i-th copy of H) and the edges which joining each vertex H_i to v_i, for all $i \in [n]$, cf. [21].

Theorem 5.1. *If G is a connected graph of order at least two, and $\omega(H) \geq k \geq 1$, then*

$$\mu_t^k(G \circ H) = |V(G)| \cdot |V(H)|.$$

Proof. Let $V(G) = \{v_1, \ldots, v_n\}$ and let H_i, $i \in [n]$, be the i-th copy of H in $G \circ H$. Let $A = V(G \circ H) \setminus V(G)$. We claim that A is a total k-clique mutual-visibility set for $G \circ H$. To prove it, consider arbitrary distinct k-cliques X and Y in $G \circ H$.

Case 1: $V(X) \cup V(Y) \subseteq V(H_i)$ for some $i \in [n]$.
In this case we have $d(X, Y) \leqslant 2$. There is nothing to prove if $d(X, Y) \leq 1$. If $d(X, Y) = 2$, then there exists a shortest X, Y-path of the form $x \sim v_i \sim y$, where $x \in V(X)$ and $y \in V(Y)$. Since $v_i \notin A$, the cliques X and Y are A-visible.

Case 2: $V(X) \subseteq V(H_i)$ and $V(Y) \subseteq V(H_j)$, where $i \neq j$.
Since any shortest X, Y-path is of the form $x \sim v_i \sim g_1 \sim \cdots \sim g_r \sim v_j \sim y$, where $x \in V(X)$, $y \in V(Y)$, and $g_i \in V(G)$, $i \in [r]$, the cliques X and X are A-visible.

Case 3: $V(X) \subseteq V(H_i)$ for some $i \in [n]$, and $V(Y) \subseteq V(G)$.
In this case the inner vertices of any any shortest X, Y-path lie in $V(G)$, hence the reuqired conclusion.

Case 4: $V(X) \cup V(Y) \subseteq V(G)$.
The argument is the same as in Case 3.

Case 5: $V(X) \subseteq V(H_i) \cup \{v_i\}$, $V(Y) \subseteq V(H_j) \cup \{v_j\}$, $v_i \in V(X)$, $v_j \in V(Y)$.
If $i = j$, then $d(X, Y) = 0$ and we are done. If $i \neq j$, then any shortest X, Y-path is of the form $v_i \sim g_1 \sim \cdots \sim g_r \sim v_j$, where $g_i \in G$. Hence the conclusion.

Case 6: $V(X) \subseteq V(G)$, $V(Y) \subseteq V(H_j) \cup \{v_j\}$ and $v_j \in V(Y)$.
The argument is is parallel to the one of Case 5.

By the above, $\mu_t^k(G \circ H) \geq |V(G)| \cdot |V(H)|$. To prove the reverse inequality it suffices to prove that no total k-clique mutual-visibility set of $G \circ H$ contains a vertex of G. To do this, let A' be an arbitrary total k-clique mutual-visibility set of $G \circ H$ and suppose that $v_i \in A'$ for some $i \in [n]$. Now consider a k-clique X with $V(X) \subseteq V(H_i)$ and a k-clique Y with $V(Y) \subseteq V(H_j)$, where $i \neq j$, to reach a contradiction. □

Theorem 5.2. *If $k \geq 3$, G is a connected graph of order at least two, and H is a graph, then the following hold.*

(1) If $\omega(H) = k - 1$, then $\mu_t^k(G \circ H) = |V(G)| \cdot |V(H)| + \mu_t^1(G)$.

(2) If $\omega(H) \leq k - 2$, then $\mu_t^k(G \circ H) = |V(G)| \cdot |V(H)| + \mu_t^k(G)$.

Proof. (1) Let $V(G) = \{v_1, \ldots, v_n\}$, and let H_i, $i \in [n]$, be the i-th copy of H in $G \circ H$. Let A_G be a μ_t^1-set for G and set $A = \bigcup_{i=1}^n V(H_i) \cup A_G$. We claim that A is a total k-clique mutual-visibility set for $G \circ H$. Consider arbitrary distinct k-cliques X and Y in $G \circ H$ and distinguish the following cases.

Case 1: $V(X) \subseteq V(H_i) \cup \{v_i\}$ and $V(Y) \subseteq V(H_j) \cup \{v_j\}$, where $i, j \in [n]$.
If $i = j$, then $d(X, Y) = 0$ and there is nothing to show. So let $i \neq j$. Since $\omega(H) = k - 1$, we have $v_i \in X$ and $v_j \in Y$. Since v_i and v_j are A_G-visible in G, it follows that X and Y are A-visible in $G \circ H$.

Case 2: $V(X) \subseteq V(H_i) \cup \{v_i\}$ for some $i \in [n]$, and $V(Y) \subseteq V(G)$.
Let y be a vertex of Y closest to v_i. Then, since v_i and y are A_G-visible in G, the cliques X and Y are A-visible in $G \circ H$.

Case 3: $V(X) \cup V(Y) \subseteq V(G)$.
Let $d(X, Y) = d(v_i, v_j)$, where $v_i \in X$ and $v_j \in Y$. Since v_i and v_j are A_G-visible, we again can conclude that X and Y are A-visible.

We have thus proved that $\mu_t^k(G \circ H) \geq |V(G)| \cdot |V(H)| + \mu_t^1(G)$. To prove the reverse inequality, suppose for a contradiction that there exists a k-clique mutual-visibility set S of $G \circ H$ with $|S| > |V(G)| \cdot |V(H)| + \mu_t^1(G)$. It follows that $|S_G| > \mu_t^1(G)$, where $S_G = S \cap V(G)$. Hence S_G cannot be a k-clique mutual-visibility set of G. Let v_i and v_j be two vertices of G which are not S_G-visible. But then a k-clique from $V(H_i) \cup \{v_i\}$ which contains v_i and a k-clique from $V(H_j) \cup \{v_j\}$ which contains v_j, are not S-visible, a contradiction.

(2) By assumption, every k-clique of $G \circ H$ lies completely in G. Hence if A_G is a μ_t^k-set for G, then we infer that $\bigcup_{i=1}^n V(H_i) \cup A_G$ is a μ_t^k-set for $G \circ H$. □

In the rest of the section we apply the results obtained for the corona product to study the complexity of the μ_t^k problem.

The problem of enumerating all k-cliques in a general graph is known to be NP-hard [13], but it can be solved in polynomial time for many well-structured graphs such as complete and bipartite graphs. Anyhow, for our purposes we will assume that the set of all k-cliques of a given graph is part of the input.

NP-hardness of problem of computing μ_t^1 was proven in [10]. Hence it remains to address the cases when $k > 1$, for which we have the following decision μ_t^k PROBLEM.

- INSTANCE: A positive integer $k \geq 2$, a connected graph G, the set of all k-cliques of G, and a positive integer $r \leq |V(G)|$.
- QUESTION: Is it satisfied that $\mu_t^k(G) \geq r$?

Theorem 5.3. *For a given $k \geq 2$, the μ_t^k PROBLEM is NP-complete.*

Proof. First we observe the μ_t^k *problem* is in NP. To show NP-hardness of this problem, consider an arbitrary connected graph G and set $G' = G \circ K_{k-1}$. Then by Theorem 5.2, $\mu_t^k(G') = (k-1)|V(G)| + \mu_t^1(G)$. Clearly, constructing G' from G can be done in polynomial time. Therefore, if there would exist a polynomial-time algorithm for computing $\mu_t^k(G')$, then there would exist a polynomial-time algorithm for finding $\mu_t^1(G)$, but the latter problem was proved to be NP-complete in [10]. □

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Declaration of interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Our manuscript has no associated data.

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