Tiled Partial Cubes

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Abstract: In the quest to better understand the connection between median graphs, triangle-free graphs and partial cubes, a hierarchy of subclasses of partial cubes has been introduced. In this article, we study the role of tiled partial cubes in this scheme. For instance, we prove that

almost-median graphs are tiled and that tiled partial cubes are semi-median. We also describe median graphs as tiled partial cubes without convex Q_{3-} and extend an inequality for median graphs to a larger subclass of partial cubes. © 2002 Wiley Periodicals, Inc. J Graph Theory 40: 91–103, 2002

1.   INTRODUCTION

Median graphs are a class of graphs that capture an essential property of trees: given any three vertices, there exists a unique vertex that lies on shortest paths between any two of the given vertices; a property that is also shared by hypercubes. It turns out that every median graph can be isometrically embedded into hypercubes. Such graphs are called partial cubes, and have been thoroughly investigated; see [16] for a recent survey on median graphs and [1,11,12,20] for partial cubes.

Although median graphs are bipartite, a surprising connection with (bipartite or non-bipartite) triangle-free graphs was recently discovered [15]. It is closely related to the fact that median graphs contain no convex subgraph isomorphic to Q_3^-, that is, to the vertex-deleted cube of dimension three. This fact is the major obstacle for a linear (or almost linear) recognition algorithm for median graphs. Hence, it is natural to investigate classes of graphs that allow convex subgraphs isomorphic to Q_3^- and that lie strictly between median graphs and partial cubes. Two such classes—semi-median graphs and almost-median graphs—were introduced in [13] because of their metric and expansion properties.

Just as median graphs, semi-median graphs can be isometrically embedded into hypercubes in almost linear time [14], once they have been recognized. It would be interesting to find out whether they can also be recognized faster than partial cubes. The solution of this problem may also shed light on the problem of improving the recognition complexity of partial cubes or for finding a nontrivial lower bound for this task, cf. [1].

In Section 2, we introduce basic concepts and a hierarchy of graph classes between median graphs and partial cubes. The results are collected in Section 3. In characterizing median graphs as semi-median graphs that contain no convex Q_3^-, we strengthen a result from [13]. Then, we investigate tiled partial cubes, that is, partial cubes in which cycles can be represented as direct sums of 4-cycles. We prove that graphs that can be obtained by an isometric expansion procedure are tiled and that tiled partial cubes are semi-median. In particular, this implies that the class of median graphs is the class of tiled partial cubes without convex Q_3^-. In the final section, we extend an inequality from [17] for median graphs to graphs that can be obtained by a connected expansion procedure. More precisely, for every such graph on n vertices, m edges, and k equivalence classes with respect to the Djoković-Winkler relation Θ, we show that $2n - m - k \leq 2$. We conclude the article with three open problems.

2.   PRELIMINARIES

The *interval* $I(u, v)$ between two vertices u, v of a connected graph G is the set of vertices of all shortest paths between u and v in G. A graph G is a *median* graph if

$$|I(u, v) \cap I(v, w) \cap I(w, u)| = 1$$

for all triples of vertices u, v, w of G. A graph that admits an isometric embedding into a hypercube is called a *partial cube*. It is well known that median graphs are partial cubes, cf. [19].

Two edges $e = xy$ and $f = uv$ of G are in the Djoković-Winkler [11,21] relation Θ if

$$d_G(x, u) + d_G(y, v) \neq d_G(x, v) + d_G(y, u),$$

where, $d_G(u, v)$ denotes the length of a shortest path in G from u to v. Clearly, Θ is reflexive and symmetric. If G is bipartite, then the edges $e = xy$ and $f = uv$ are in relation Θ precisely, when $d(x, u) = d(y, v)$ and $d(x, v) = d(y, u)$. Winkler [21] proved that a bipartite graph is a partial cube if and only if Θ is transitive.

Partial cubes are a rich class of graphs that encompasses subclasses with widely differing properties. We will pursue two main approaches for the definition of subclasses.

We begin with the definition of three types of subsets of the vertex set of a graph and the concept of convexity, cf. [19]. Let $G = (V, E)$ be a connected, bipartite graph and let ab an edge of G. We set

$$W_{ab} = \{w \in V \mid d_G(a, w) < d_G(b, w)\},$$
$$U_{ab} = \{w \in W_{ab} \mid w \text{ has a neighbor in } W_{ba}\},$$
$$F_{ab} = \{e \in E \mid e \text{ is an edge between } W_{ab} \text{ and } W_{ba}\}.$$

Clearly, W_{ab} and W_{ba} are disjoint, and $V = W_{ab} \cup W_{ba}$, since G is bipartite and connected. Moreover, the subgraph $\langle W_{ab} \rangle$ induced by the set W_{ab} is connected, since G is connected. If G is a partial cube, then the sets F_{ab} are precisely the equivalence classes of the relation Θ. This is an immediate consequence of the characterizations of partial cubes by Djoković [11] and Winkler [21].

A subgraph H of a graph G is *convex* if for any two vertices u, v of H all shortest paths between u and v in G are already in H, that is, if $I_G(u, v) \subseteq V(H)$.

In the first approach for the definition of these subclasses, properties of the subgraphs $\langle U_{ab} \rangle$ induced by the sets U_{ab} are used. The starting point is the characterization of median graphs as the class of bipartite graphs with convex $\langle U_{ab} \rangle$'s, see [4]. In this sense, *almost-median graphs* are defined as partial cubes with isometric $\langle U_{ab} \rangle$'s, and *semi-median graphs* as partial cubes with connected $\langle U_{ab} \rangle$'s, see [13].

The second approach characterizes partial cubes by expansion procedures as defined below.

Let G' be a connected graph. A *proper cover* G'_1, G'_2 consists of two isometric subgraphs G'_1, G'_2 of G' such that $G' = G'_1 \cup G'_2$ and $G'_0 = G'_1 \cap G'_2$ is a nonempty subgraph, called the *intersection* of the cover. The *expansion* of G' with respect to G'_1, G'_2 is the graph G constructed as follows. Let G_i be an isomorphic copy of G'_i, for $i = 1, 2$, and, for any vertex u' in G'_0, let u_i be the corresponding vertex in G_i, for $i = 1, 2$. Then, G is obtained from the disjoint union $G_1 \cup G_2$, where for each u' in G'_0 the vertices u_1 and u_2 are joined by an edge. We denote the copy of G'_0 in G_i by G_{0i}, for $i = 1, 2$. If G'_0 is convex, isometric, or connected in G, we speak of a *convex, isometric*, or *connected expansion*, respectively. If G can be obtained from the one-vertex graph K_1 by a sequence of expansions of a given type, then we say that G is obtainable by an *expansion procedure* of that type.

Let G be the expansion of G' with respect to the proper cover G'_1, G'_2. Note that the set of edges between G_{01} and G_{02} is a Θ-class. Moreover, for any edge $v_1 v_2$ with v_1 in G_{01} and v_2 in G_{02}, we have $G_{01} = \langle U_{v_1 v_2} \rangle$ and $G_{02} = \langle U_{v_2 v_1} \rangle$.

These concepts lead to the following hierarchy of classes of partial cubes:

(1) Partial cubes in which the $\langle U_{ab} \rangle$'s are *convex* (median graphs);
(2) Graphs obtainable by a *convex* expansion procedure;
(3) Partial cubes in which the $\langle U_{ab} \rangle$'s are *isometric* (almost-median graphs);
(4) Graphs obtainable by an *isometric* expansion procedure;
(5) Partial cubes in which the $\langle U_{ab} \rangle$'s are *connected* (semi-median graphs);
(6) Graphs obtainable by a *connected* expansion procedure;
(7) Partial cubes in which the $\langle U_{ab} \rangle$'s are *arbitrary*;
(8) Graphs obtainable by an *arbitrary* expansion procedure.

The following implications hold:

$$(1) \Leftrightarrow (2) \begin{smallmatrix} \Rightarrow \\ \not\Leftarrow \end{smallmatrix} (3) \begin{smallmatrix} \Rightarrow \\ \not\Leftarrow \end{smallmatrix} (4) \begin{smallmatrix} \Rightarrow \\ \not\Leftarrow \end{smallmatrix} (5) \begin{smallmatrix} \Rightarrow \\ \not\Leftarrow \end{smallmatrix} (6) \begin{smallmatrix} \Rightarrow \\ \not\Leftarrow \end{smallmatrix} (7) \Leftrightarrow (8).$$

The equivalence between (1) and (2) is due to Mulder [18], while the one between (7) and (8) is due to Chepoi [7]. Clearly (2) implies (3), and the graph $G_1 = Q_3^-$ of Figure 1 shows that the converse is not true.

That (3) implies (4) was shown in [13]; the graph G_2 of Figure 1 demonstrates that the converse is false. The implications $(4) \Rightarrow (5) \Rightarrow (6)$ were proved by Brešar [6] in a more general setting (for the non-bipartite case). The latter implication is also stated in [13], but it is wrongly asserted that the converse holds as well. This fact was pointed out by Chepoi [9] using the graph G_3 of Figure 1. Indeed, G_3 can be obtained from the semi-median graph Q_3^- by an expansion with respect to its isometric 6-cycle and the subgraph obtained by deletion of a vertex of degree 2. This specific expansion is connected but G_3 is not semi-median: take as ab one of the edges incident with the vertex of degree 2 in G_3, then $\langle U_{ab} \rangle$ is not connected. The graph of Figure 2 is semi-median, but cannot be obtained by a
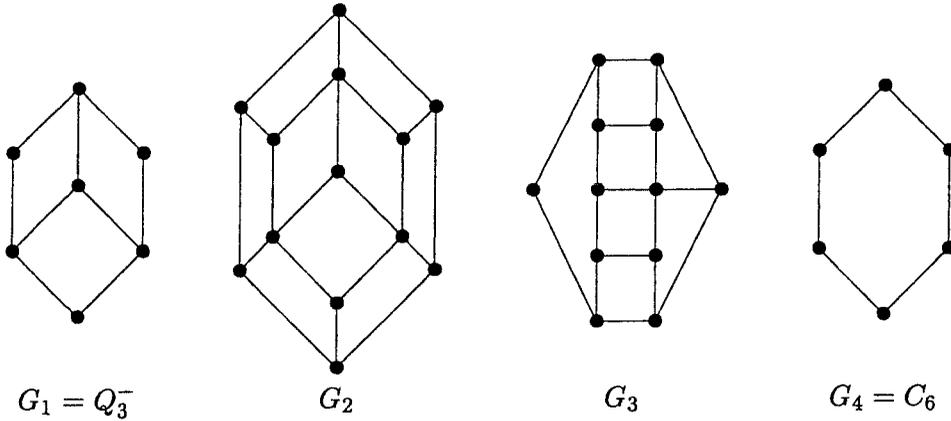
FIGURE 1.   Partial cubes.

sequence of isometric expansions. In particular, this answers the question from [6] whether (4) and (5) are equivalent. Finally, (6) clearly implies (7); C_6 shows that the converse fails in general.

3.   TILED PARTIAL CUBES IN THE HIERARCHY

Let $G_1, G_2, \ldots, G_k$ be subgraphs of a graph G. Then the *symmetric sum* $G_1 \oplus G_2 \oplus \cdots \oplus G_k$ is the subgraph of G induced by those edges of G that appear in
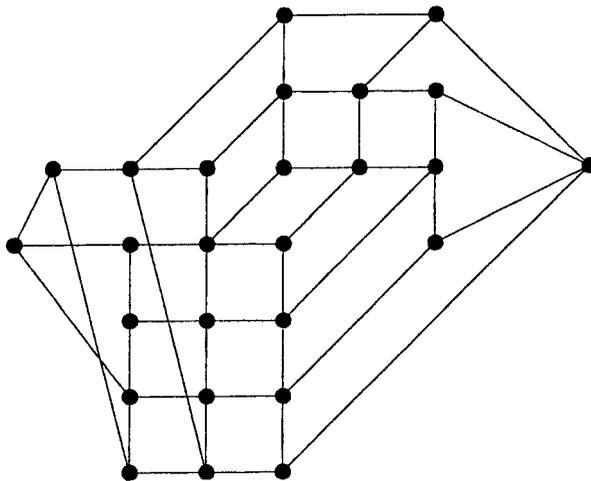

FIGURE 2.   A semi-median graph.

an odd number of the graphs $G_1, G_2, \ldots, G_k$. Let C be a cycle of a graph G. Then a set of 4-cycles $\mathcal{C} = \{C_1, C_2, \ldots, C_p\}$ in G is a *tiling* of C if

$$C = C_1 \oplus C_2 \oplus \cdots \oplus C_p.$$

A graph G is *tiled* if every cycle of G has a tiling.

We now extend the hierarchy of partial cubes by the following subclass:

(9) Partial cubes in which every cycle can be tiled.

We call these graphs *tiled partial cubes* and will eventually show that

$$(4) \Rightarrow (9) \; \genfrac{}{}{0pt}{}{\Rightarrow}{\nLeftarrow} \; (5).$$

This result enables us to characterize median graphs as the class of tiled partial cubes without convex Q_3^-. In addition, a partial converse of the implication $(9) \Rightarrow (5)$ is also proved. Note that tiled partial cubes are precisely the partial cubes in which the 4-cycles constitute a basis of the cycle space.

We show first that $(9) \Rightarrow (5)$.

Theorem 3.1. *Tiled partial cubes are semi-median graphs.*

Proof. Assume the contrary, and let G be a tiled partial cube that is not a semi-median graph. Since G is not a semi-median graph, there exists an edge ab in $E(G)$ such that $\langle U_{ab} \rangle$ (and $\langle U_{ba} \rangle$) is not connected. Let $\langle U_{ab}^1 \rangle$ and $\langle U_{ab}^2 \rangle$ be two different connected components of $\langle U_{ab} \rangle$, and let $\langle U_{ba}^1 \rangle$ and $\langle U_{ba}^2 \rangle$ be the corresponding connected components of $\langle U_{ba} \rangle$. Let $u_1 v_1$ and $u_2 v_2$ be two edges from F_{ab} with $u_i \in \langle U_{ab}^i \rangle$ and $v_i \in \langle U_{ba}^i \rangle$ for $i = 1, 2$. Then, there is an induced cycle $C = u_1 P u_2 v_2 Q v_1 u_1$ of G such that the path P lies in W_{ab} and Q lies in W_{ba}. Since G is tiled, there exists a nonempty set of 4-cycles $\mathcal{C} = \{C_1, \ldots, C_p\}$ of G such that $C = C_1 \oplus \cdots \oplus C_p$. Note that, if the 4-cycle C_i has an edge in F_{ab}, then it has exactly two edges in F_{ab}. So, denote by $\mathcal{C}_r = \{C_{i_1}, \ldots, C_{i_r}\}$ the subset of 4-cycles of $\mathcal{C}$ that have two edges in common with F_{ab}.

By the choice of the paths P and Q, it follows that $u_1 v_1$ and $u_2 v_2$ are the only two edges of C which are in F_{ab}. Hence, $u_1 v_1$ and $u_2 v_2$ are the only edges of F_{ab} that are contained in an odd number of cycles from $\mathcal{C}_r$.

Now, consider the graph $H = C_{i_1} \oplus \cdots \oplus C_{i_r}$. Clearly $u_1 v_1$ and $u_2 v_2$ are edges in H, and any other edge of H is contained in $\langle U_{ab} \rangle$ or $\langle U_{ba} \rangle$. Furthermore, H is an *even* graph, that is, a graph every vertex of which has even degree, or, equivalently, a graph in that all components are Eulerian. Because of the parity condition on the number of occurrences of the edges of F_{ab} in $\mathcal{C}_r$, we infer that $u_1 v_1$ and $u_2 v_2$ are in the same component of H. In particular, u_1 and u_2 are connected by a path in $\langle U_{ab} \rangle$. This contradicts the assumption that the vertices u_1, u_2 are in different components of $\langle U_{ab} \rangle$. ∎

To see that the converse of Theorem 3.1 is not true, i.e., that (5) does not imply (9), we may again use the graph from Figure 2. It is semi-median, but certain 6-cycles of this graph have no tiling: take for instance the 6-cycle involving the two leftmost vertices and the horizontal edges incident with these. In the following example, we show how a tiling can be destroyed by an expansion. Consider the tiling of the indicated 6-cycle in Figure 3, consisting of the two indicated 4-cycles and the outer 4-cycle. This tiling is destroyed by the expansion. Of course, in this example there exists another tiling that is retained in the expansion. But, we have no procedure to find such tilings in all cases or to distinguish between the two types of tilings in the smaller graph.

We can avoid the above problems by considering a slightly smaller class of graphs. In the expansion procedure for a median graph (a partial cube), it turns out that the graphs in proper covers induce median graphs (partial cubes). Thus, it makes sense to call a semi-median graph a *proper semi-median graph*, if it can be obtained by a connected expansion procedure in which the proper cover consists of proper semi-median graphs in each step:

(10) Proper semi-median graphs.

Theorem 3.2 fits this class into the hierarchy, as displayed by the following diagram:

$$(2) \quad \begin{matrix} \Rightarrow (3) \Rightarrow (4) \Rightarrow \\ \Rightarrow (10) \Rightarrow \end{matrix} \quad (9) \Rightarrow (5).$$

Recall that $\triangle$ denotes the *symmetric difference* of sets.

Theorem 3.2. *A proper semi-median graph is tiled.*

Proof. The proof is by induction on the number of Θ-classes of G. If this number is zero, then G is the one vertex graph, which is trivially tiled. Now, let G be a proper semi-median graph with at least one Θ-class. Then G can be obtained by a connected expansion from a semi-median graph G' with respect to the proper cover consisting of proper semi-median subgraphs, which are tiled by the
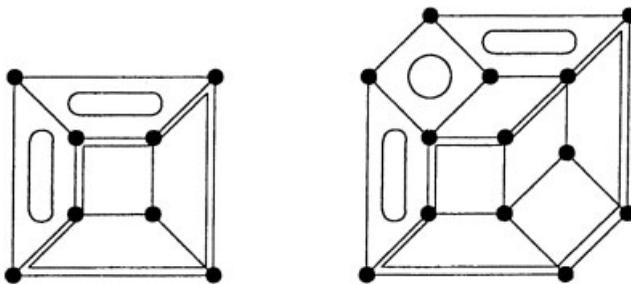

FIGURE 3. A connected expansion.

induction hypothesis. Let F_{ab} be the Θ-class obtained in the last expansion step. It is well-known and also easy to see that the isometric cycles of a graph constitute a basis of the cycle space of that graph. So, it suffices to prove that an arbitrary isometric cycle C of G has a tiling.

Since G is a partial cube it follows that $\langle W_{ab} \rangle$ and $\langle W_{ba} \rangle$ are convex subgraphs. In particular, this implies that every isometric cycle of G has at most 2 edges in F_{ab}. Thus, $|E(C) \cap F_{ab}| = 0$ or 2.

Suppose first that $|E(C) \cap F_{ab}| = 0$. Then, we may assume that C is a cycle of $\langle W_{ab} \rangle$. Clearly, $\langle W_{ab} \rangle$ is a proper semi-median graph with fewer Θ-classes than G. Hence, by induction, C is tiled in $\langle W_{ab} \rangle$ and so it is in G.

Assume next that $E(C) \cap F_{ab} = \{u_1 v_1, u_2 v_2\}$, where $u_1, u_2 \in U_{ab}$ and $v_1, v_2 \in U_{ba}$. Let P_u be a path in $\langle U_{ab} \rangle$ between u_1 and u_2. Denote the isomorphic copy of P_u in $\langle U_{ba} \rangle$ by P_v. Note that P_u and P_v always exist, since $\langle U_{ab} \rangle$ and $\langle U_{ba} \rangle$ are connected and isomorphic.

Let $H = P_u \cup P_v \cup \{u_1 v_1, u_2 v_2\}$. Then, H is a cycle, and H has the obvious tiling $\mathcal{C}_H$ such that each 4-cycle of $\mathcal{C}_H$ has two edges in F_{ab}, one on P_u, and the corresponding one on P_v. Clearly, we can consider the even graph $C \oplus H$ as a set (possibly empty) of pairwise edge-disjoint cycles $C_1, C_2, \ldots, C_k$ such that each one is either in $\langle W_{ab} \rangle$ or in $\langle W_{ba} \rangle$. Thus, by the previous case, for each such cycle C_i in $C \oplus H$ there exists a tiling $\mathcal{C}_i$. Let

$$\mathcal{C} = \mathcal{C}_1 \triangle \mathcal{C}_2 \triangle \cdots \triangle \mathcal{C}_k.$$

Now, it is easy to see that $\mathcal{C}$ is a tiling of $C \oplus H$. Finally, $\mathcal{C} \triangle \mathcal{C}_H$ is a tiling of C. ∎

We do not know how to fit proper semi-median graphs into the hierarchy of Section 2. This is easier for the tiled partial cubes. The following theorem generalizes the corresponding result for median graphs from [5], it shows that $(4) \Rightarrow (9)$.

Theorem 3.3. *Graphs obtainable by an isometric expansion procedure are tiled.*

Proof. Assume the contrary and let G be the smallest non-tiled graph obtainable by an isometric expansion from a tiled graph G', which in itself is obtainable by an isometric expansion procedure. Let F_{ab} be the Θ-class obtained in the last expansion step. Note that $\langle U_{ab} \rangle$ and $\langle U_{ba} \rangle$ are both isometric subgraphs of G.

Claim. *Every cycle C of G with $|E(C) \cap F_{ab}| = 0$ is tiled.*

We may assume that C is a cycle of $\langle W_{ab} \rangle$. Let C', U', W'_{ab}, and W'_{ba} in G' correspond to C, U_{ab}, W_{ab}, and W_{ba}, respectively. Note that U' also corresponds to U_{ba}, and that $U' = W'_{ab} \cap W'_{ba}$. Then C' is a cycle contained in $\langle W'_{ab} \rangle$, but it may

have vertices in $U' \subseteq W'_{ba}$ as well. By the induction hypothesis, there is a tiling $\mathcal{C}' = \{C'_1, \ldots, C'_p\}$ of C' in G'. Since G' is a partial cube it does not contain $K_{2,3}$ as a subgraph. Now, since $\langle U' \rangle$ is an isometric subgraph of G', we conclude that every 4-cycle of $\mathcal{C}'$ is contained either in $\langle W'_{ab} \rangle$ or in $\langle W'_{ba} \rangle$ (or in both). Denote the set of 4-cycles of $\mathcal{C}'$ that have a vertex in $W'_{ba} \backslash W'_{ab}$ by $\mathcal{C}'_b$ and write $\mathcal{C}'_a = \mathcal{C}' \backslash \mathcal{C}'_b$. Finally, let $\mathcal{C}_b$ be the set of 4-cycles in $\langle W_{ba} \rangle$ that naturally correspond to the 4-cycles of $\mathcal{C}'_b$ and let $\mathcal{C}_a$ be the set of 4-cycles in $\langle W_{ab} \rangle$ that correspond to the 4-cycles of $\mathcal{C}'_a$, say $\mathcal{C}_a = \{C_1, \ldots, C_q\}$ and $\mathcal{C}_b = \{C_{q+1}, \ldots, C_p\}$.

Consider the two subgraphs

$$H_a = C \oplus C_1 \oplus \cdots \oplus C_q \quad \text{and} \quad H_b = C_{q+1} \oplus \cdots \oplus C_p.$$

Note that H_a and H_b are even graphs. Since C is contained in $\langle W_{ab} \rangle$, the cycle C' is contained in $\langle W'_{ab} \rangle$. Therefore, each edge in a 4-cycle of $\mathcal{C}'_b$ with an end in $W'_{ba} \backslash W'_{ab}$ occurs in an even number of 4-cycles in $\mathcal{C}'_b$. Hence, each edge in a 4-cycle of $\mathcal{C}_b$ with an end in $W_{ba} \backslash U_{ba}$ occurs in an even number of 4-cycles of $\mathcal{C}_b$. This implies that H_b is contained in $\langle U_{ba} \rangle$. Note that C is involved in the definition of H_a. Now, by a similar argument, we deduce that H_a is contained in $\langle U_{ab} \rangle$. Moreover, since $C' = C'_1 \oplus \cdots \oplus C'_p$, the graph H_a must be an isomorphic copy of H_b in U_{ab}.

Now, let $\mathcal{C}_{ab}$ be the set of 4-cycles $D = cdefc$ such that $cd \in H_a$, $ef \in H_b$, and $de, cf \in F_{ab}$. Finally, since H_a and H_b are even graphs, we infer that $\mathcal{C}_a \cup \mathcal{C}_b \cup \mathcal{C}_{ab}$ is a tiling of C in G. This proves the claim.

Let C be a cycle in G that is not tiled. As in the proof of Theorem 3.2, we may assume that C is isometric, and therefore $|E(C) \cap F_{ab}| \leq 2$. By the claim, we infer that $|E(C) \cap F_{ab}| = 2$. Let uv and xy be the two edges of C in F_{ab} with u, x in W_{ab}. Let P be the path on C between u and x in $\langle W_{ab} \rangle$ and, similarly, let Q be the path on C between v and y in $\langle W_{ba} \rangle$. Then, $C = uvQyxPu$.

Let R_u be a shortest path from u to x in $\langle U_{ab} \rangle$. Denote the corresponding path between v and y in $\langle U_{ba} \rangle$ by R_v. Note that $R_u \oplus P$ is a (possible empty) set of cycles in $\langle W_{ab} \rangle$. Thus, by the claim, it follows that there exists a set of 4-cycles $\mathcal{C}_u$ in G such that $R_u \oplus P = \oplus_{D \in \mathcal{C}_u} D$. Similarly, let $\mathcal{C}_v$ be a set of 4-cycles such that $R_v \oplus Q = \oplus_{D \in \mathcal{C}_v} D$. Now, denote by $\mathcal{C}_{uv}$ the set of 4-cycles, of which two edges are in F_{ab} and one edge in each of R_u and R_v. Note that $R_u \cup R_v \cup \{uv, xy\} = \oplus_{D \in \mathcal{C}_{uv}} D$. Now, clearly, $\mathcal{C}_u \triangle \mathcal{C}_v \triangle \mathcal{C}_{uv}$ is a tiling of C. This contradiction completes the proof. ∎

In [13], it was proved that a graph is a median graph if and only if it is an almost-median graph that contains no convex Q_3^- as a subgraph. The following theorem strengthens this result. For its proof we recall the following lemma from [13].

Convexity Lemma. *Let H be an induced subgraph of a bipartite graph G and let ∂H be the set of all edges of G with precisely one end-vertex in H. Then, H is a convex subgraph of G if and only if no edge of ∂H is in relation Θ to an edge in H.*

The equivalence between (i) and (ii) of the following theorem is due to Chepoi [10]. More precisely, it is an implicit consequence of [10, Theorem 6.1]. An induced connected subgraph H of a graph G is 2-*convex* if for any two vertices u and v of H with $d_G(u, v) = 2$, every common neighbor of u and v belongs to H. In [8] (see also [3]), it is proved that a subgraph in a median graph is convex if and only it is 2-convex.

Theorem 3.4. *For a graph G, the following conditions are equivalent:*

(i) *G is a median graph;*
(ii) *G is a tiled partial cube that contains no convex Q_3^-;*
(iii) *G is a semi-median graph that contains no convex Q_3^-;*
(iv) *G is obtainable by connected expansions from K_1 and contains no convex Q_3^-.*

Proof. Median graphs are partial cubes that contain no convex Q_3^- and are tiled by Theorem 3.3. Thus, (i) implies (ii). By Theorem 3.1 (ii) implies (iii), while (iv) follows from (iii) by the hierarchy of the classes of graphs introduced in Section 2. Thus it remains to prove that (iv) implies (i).

Suppose that G can be constructed from the one-vertex graph by a series of connected expansions. If G is not median, then there must be a first expansion step which leads to a non-median graph. Let uv be an edge of the Θ-class produced by this expansion and let us assume for the time being that this is the last expansion step in constructing G. Let this expansion be with respect to the cover G_1', G_2', where $G_0' = G_1' \cap G_2'$ is not convex. Therefore, G_0' is not 2-convex and hence, there is a 4-cycle consisting of the edges ab, bc, cd, da, where a, b, c are in G_0', and d is in $G_1' \backslash G_2'$. Let aa', bb', and cc' be edges of the new Θ-class. Then a, b, c, d, a', b', c' form a convex Q_3^- in G, which readily follows from the Convexity Lemma.

If this is not the last expansion step, then all further connected expansions add edges, which belong to Θ-classes different from all previous ones, so this Q_3^- remains convex by the Convexity Lemma. ∎

4. AN EULER-TYPE FORMULA

Let G be a median graph with n vertices, m edges, and k equivalence classes of G with respect to relation Θ. It was proved in [17] that $2n - m - k \leq 2$, and that equality holds if and only if G is a cube-free median graph.

As observed above, in the case of bipartite graphs, the relation Θ may be defined by the property $uv\Theta xy$ if and only if $d(x, u) = d(y, v)$ and $d(x, v) = d(y, u)$. Because of this property, one may consider an equivalence class with respect to Θ as a *class of parallel edges*. This approach is prominent in [5]. In the network case, where one assigns weights (lengths) to the edges, such an idea of parallelism is an essential aspect of the results, see [2]. Metaphorically, we might

consider vertices as 0-dimensional objects, edges as 1-dimensional objects, and classes of parallel edges as 2-dimensional objects. Thus, in the formula $2n - m - k \leq 2$, the successive numbers n, m, k count the "0-dimensional objects," the "1-dimensional objects," and the "2-dimensional objects." In this way, the formula reflects an essential property of the classical Euler formula for planar graphs. Moreover, replacing each 4-cycle of a plane semi-median graph G with a solid square and taking all the cells sharing edges from the same Θ-class, their union can be viewed as a kind of a strip that is in fact the Cartesian product of a (solid) edge from this class with a graph H of smaller dimension than the strip. In the case of cube-free semi-median graphs H is a tree, hence the strip can be considered as a 2-dimensional object. On the other hand, our formula is not an alternating sum. So it is *not* an Euler formula in the proper sense.

In what follows, we show that this inequality holds for graphs obtainable by connected expansion procedures as well. Thus, we 'push' the inequality 'down' along the hierarchy. We do not know yet whether this is as far as possible, see Problem 1 below.

Theorem 4.1. *Let G be a graph with n vertices, m edges, and k equivalence classes of G with respect to Θ, that is obtainable by a connected expansion procedure. Then $2n - m - k \leq 2$. Moreover $2n - m - k = 2$ if and only if G is obtainable by an expansion procedure with proper covers having trees as intersections in every step.*

Proof. We prove the inequality by induction on the number of vertices. The inequality reduces to $2 \leq 2$ if $G = K_1$. So assume that G is the connected expansion of G' with respect to isometric subgraphs G'_1, G'_2 with $G'_0 = G'_1 \cap G'_2$. By induction, we have $2n' - m' - k' \leq 2$ for G', where k', n', m' are the corresponding parameters of G'. Let t be the number of vertices in G'_0, so that G'_0, being connected, has at least $t - 1$ edges. Then, we have $n = n' + t$ and $m \geq m' + 2t - 1$. Moreover, the expansion step yields one more Θ-equivalence class, and so we have $k = k' + 1$. Thus,

$$\begin{aligned} 2n - m - k &\leq 2(n' + t) - (m' + 2t - 1) - (k' + 1) \\ &= 2n' - m' - k' \\ &\leq 2. \end{aligned}$$

Note that the first inequality is an equality if and only if G'_0 has precisely $t - 1$ edges, i.e., if G'_0 is a tree. Hence, $2n - m - k = 2$ if and only if the intersection of the cover induces a tree in every expansion step. ∎

Combining Euler's classical formula $n - m + f = 2$ for planar graphs with Theorem 4.1, we obtain the following corollary.

Corollary 4.2. *Let G be a planar semi-median graph with n vertices, k equivalence classes with respect to the relation Θ, and f faces in its planar*

embedding. Then $f \geq n - k$. Moreover $f = n - k$ if and only if G is obtainable by an expansion procedure with proper covers having trees as intersections in each step.

We conclude the paper with three open problems.

1. Is there an Euler-type relation for partial cubes? In particular, is it true that $2n - m - 2k \leq 0$ for partial cubes with more than 2 vertices?
2. Can one obtain every tiled partial cube by a sequence of isometric expansions? In other words, does (9) imply (4)?
3. Where exactly are the proper semi-median graphs located in the hierarchy? In other words, where is (10) located with respect to (3) and (4)?

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