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Two remarks on retracts of graph products

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Abstract

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Let H be a bipartite graph and let G_n be the Mycielski graph with $\chi(G)=n$, $n \geq 4$. Then the chromatic number of the strong product of G_n by H is at most $2n-2$. We use this result to show that there exist strong products of graphs in which a projection of a retract onto a factor is not a retract of the factor. We also show that in the Cartesian product of graphs G and H, any triangles of G transfer in H, whenever G and H are connected and G is strongly-triangulated, weakly-triangulated or four-cycle free.

1. Introduction

The work of Sabidussi on graph products is well known. It is not so well known that Sabidussi was apparently the first person to explicitly suggest studying retractions of graphs (see [2, p. 550]). In this paper we continue the investigation of retracts of Cartesian and strong products of graphs in the sense of the papers [3, 5].

All graphs considered in this paper will be undirected, simple graphs, i.e., graphs without loops or multiple edges.

A subgraph R of a graph G is a *retract* of G if there is an edge-preserving map $r: V(G) \rightarrow V(R)$ with $r(x)=x$, for all $x \in V(R)$. The map r is called a *retraction*. If R is a retract of G, then R is an *isometric* subgraph of G, that is $d_G(x, y)=d_R(x, y)$ for all $x, y \in V(R)$, where $d_G(x, y)$ denotes the length of a shortest path in G between x and y.

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An *n-colouring* of a graph G is a function f from $V(G)$ onto $\{1, 2, \ldots, n\}$ such that $[x, y] \in E(G)$ implies $f(x) \neq f(y)$. The smallest number n for which an n-colouring exists is the *chromatic number* $\chi(G)$ of G. It is easy to see that if R is a retract of G then $\chi(R) = \chi(G)$.

The *Cartesian product* $G \square H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ and $[(a, x), (b, y)] \in E(G \square H)$ whenever $[a, b] \in E(G)$ and $x = y$, or $a = b$ and $[x, y] \in E(H)$. The *strong product* $G \boxtimes H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ and $[(a, x), (b, y)] \in E(G \boxtimes H)$ whenever $[a, b] \in E(G)$ and $x = y$, or $a = b$ and $[x, y] \in E(H)$, or $[a, b] \in E(G)$ and $[x, y] \in E(H)$. The notation was suggested by Nešetřil because $\square$ looks like the Cartesian product of an edge with itself, $\boxtimes$ like the strong product.

Let $G \circ H$ be an arbitrary product of graphs G and H. We shall denote the vertices of one factor by $a, b, c, \ldots$ and the vertices of the other factor by x, y, z. Let R be a retract of $G \circ H$. For $x \in V(H)$ set $G_x = G \circ \{x\}$ and $R_x = G_x \cap R$. Analogously we define H_a and R_a for $a \in V(G)$. We call G_x and H_a a *layer* of G and of H, respectively.

In Section 2 we construct examples of strong products of graphs where the projection of a retract on a factor is not a retract of the factor. These examples give some new insight into the internal structure of retracts of strong products. In Section 3 we show that in the Cartesian product of graphs G and H, triangles of G transfer in H, whenever G and H are connected and G is strongly-triangulated, weakly-triangulated or four-cycle free.

2. A remark on the strong product

For the strong product $G \boxtimes H$ of connected graphs G and H, it is shown in [3] that every retract R of $G \boxtimes H$ is of the form $R = G' \boxtimes H'$, where G' is an isometric subgraph of G and H' is an isometric subgraph of H. Furthermore, if both G and H are triangle-free then G' and H' are retracts of G and H, respectively. It is also conjectured in [3] that every retract of strong products of a large class of graphs is a product of retracts of the factors.

The purpose of this section is to show that there exist graphs G and H such that $G' \boxtimes H'$ is a retract of $G \boxtimes H$ yet G' is not a retract G.

Let us denote by G_3 the five-cycle C_5 and construct the graphs G_n, $n \geq 3$, in the following way. Let $V(G_n) = \{a_1, a_2, \ldots, a_p\}$. We construct the graph G_{n+1} from G_n by adding $p + 1$ new vertices $b_1, b_2, \ldots, b_p, b$. The vertex b is joined to each vertex b_i and the vertex b_i is joined to every vertex to which a_i is adjacent. The graphs G_n were constructed by Mycielski [4]. They are triangle-free and, in addition, $\chi(G_n) = n$. The graph G_4 is also called the Grötzsch graph.

Theorem 1. *Let G_n, $n \geq 4$, be the Mycielski graph and let H be a bipartite graph. Then $\chi(G_n \boxtimes H) \leq 2n - 2$.*

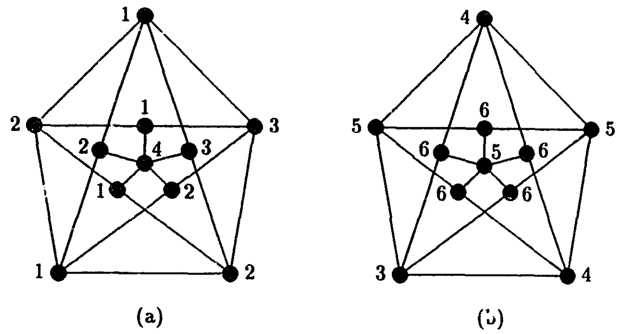

Fig. 1. A 6-colouring of $G_4 \boxtimes K_2$.

Proof. We first prove the theorem for the case when H is a single edge K_2. The proof is by induction on n.

Colour one layer of G_4 in the product $G_4 \boxtimes K_2$ as shown in Fig. 1(a) and the other layer of G_4 as in Fig. 1(b). One can easily see that this is a 6-colouring of the product, hence $\chi(G_4 \boxtimes K_2) \leqslant 6$.

Suppose now that $\chi(G_n \boxtimes K_2) \leqslant 2n-2$, $n \geqslant 4$. Let $V(G_{n+1}) = \{a_1, \ldots, a_p, b_1, \ldots, b_p, b\}$ and let $V(K_2) = \{x, y\}$. Let g be a $(2n-2)$-colouring of the subgraph in $G_{n+1} \boxtimes K_2$ induced by the vertices $\{a_1, \ldots, a_p\} \times \{x, y\}$. Extend g to a colouring f of $G_{n+1} \boxtimes K_2$ in the following way. For $i = 1, 2, \ldots, p$ set $f(b_i, x) = g(a_i, x)$ and $f(b_i, y) = g(a_i, y)$. Finally set $f(b, x) = 2n-1$ and $f(b, y) = 2n$.

We claim that f is a $(2n)$-colouring of $G_{n+1} \boxtimes K_2$. Note first that $f(b_i, x) \neq f(b_i, y)$, $i = 1, 2, \ldots, p$. If $[(a_i, x), (b_j, x)] \in E(G_{n+1} \boxtimes K_2)$, $i \neq j$, then $[(a_i, x), (a_j, x)] \in E(G_{n+1} \boxtimes K_2)$ and hence $f(a_i, x) = g(a_i, x) \neq g(a_j, x) = f(b_j, x)$. It follows that the layer $(G_{n+1})_x$ (and by symmetry, the layer $(G_{n+1})_y$) is properly coloured. Assume next that $[(a_i, x), (b_j, y)] \in E(G_{n+1} \boxtimes K_2)$, $i \neq j$. But then $[(a_i, x), (a_j, y)] \in E(G_{n+1} \boxtimes K_2)$, and with the same argument as above, $f(a_i, x) \neq f(b_j, y)$. This proves the claim and the proof for the case K_2 is complete.

Let H be a bipartite graph with a vertex partition $V(H) = V_1 + V_2$. Let f be a $(2n-2)$-colouring of $G_n \boxtimes K_2$, let f_1 be the induced colouring of one layer of G_n and let f_2 be the induced colouring of the second layer. Colour the graph $G_n \boxtimes H$ in the following way: for $x \in V_1$ colour the layer $(G_n)_x$ with the colouring f_1 and for $y \in V_2$ colour the layer $(G_n)_y$ with f_2. It is easy to see that we have obtained a $(2n-2)$-colouring of $G_n \boxtimes H$, and the proof is complete. □

Vesztergombi [6] showed that if both G and H have at least one edge then $\chi(G \boxtimes H) \geqslant \max\{\chi(G), \chi(H)\} + 2$. It follows from this fact and Theorem 1 that $\chi(G_4 \boxtimes H) = 6$, whenever H is a bipartite graph with at least one edge.

We now are ready for the following construction. Denote by H_n, $n \geqslant 4$, a graph which we get from a copy of the Mycielski graph G_n and the complete graph K_{n-1} by joining an arbitrary vertex of G_n with a vertex of K_{n-1}. As $\chi(K_{n-1} \boxtimes K_2) =$

$2n-2$ it is easy to see that there is a retraction from $V(H_n \boxtimes K_2)$ onto a subgraph $K_{n-1} \boxtimes K_2$. (Take any colour preserving map.) But as $\chi(G_n)=n$ there is no retraction $V(G_n) \to V(K_{n-1})$. In fact, it is easy to see that graphs H_n contain no proper retracts at all.

3. A remark on the Cartesian product

In [5], decomposition theorems for retracts of Cartesian products of graphs are derived for strongly-triangulated and weakly-triangulated graphs as well as for graphs without four-cycles.

The proofs make repeated use of the statement that 'triangles transfer'. The example in Fig. 2 (where the filled vertices induce a retract and a corresponding retraction is indicated with arrows) shows a counterexample to this statement. We prove however, that triangles transfer for strongly-triangulated graphs, weakly-triangulated graphs and graphs without four-cycles. This fills a gap in the proofs of Theorems 1 to 5 of [5].

A subgraph S of a graph G *transfers* in H if, for every retract R of $G \square H$, $S \square \{x\} \subseteq R$ and $(a, y) \in V(R)$, where $a \in V(G)$ and $x, y \in V(H)$, we have $S \square \{y\} \subseteq R$.

A connected graph G with $|V(G)| \geq 3$ is *weakly-triangulated* if each edge of G is in a triangle. Call G *strongly-triangulated* if every pair of vertices are joined by a sequence of triangles with consecutive ones sharing an edge.

Proposition 1. *Let G be a connected, weakly-triangulated graph and let H be a connected graph. Then every triangle of G transfers in H.*

Proof. Let R be a retract of $G \square H$ and let $r: V(G \square H) \to V(R)$ be a retraction. Let $T=\{a, b, c\}$ be a triangle in G and suppose that $T \square \{x\}$ is contained in R for some $x \in V(H)$. Assume that for some $y \in V(H)$, R_y is non-empty, but $T \square \{y\}$ is not contained in R. As H is connected we may suppose that $[x, y] \in E(H)$.

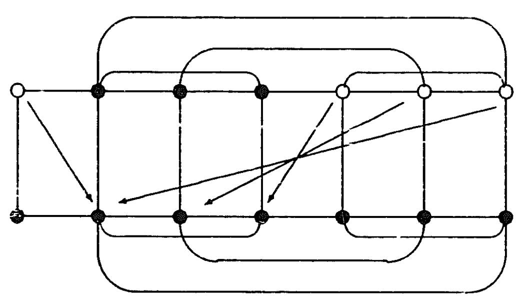

Fig. 2. A triangle which does not transfer.

Case 1: *At least one of the vertices* (a, y), (b, y), (c, y) *belongs to* $V(R)$.

We may assume that $(a, y) \in V(R)$. If $(b, y) \notin V(R)$ and $(c, y) \notin V(R)$ then $r(b, y) = (a, x)$ and $r(c, y) = (a, x)$, a contradiction. But if $(b, y) \in V(R)$ and $(c, y) \notin V(R)$ then (c, y) must be mapped to both (a, x) and (b, x), which is again impossible.

Case 2: *None of the vertices* (a, y), (b, y), (c, y) *belong to* $V(R)$.

The layer R_y is non-empty, hence let $(d, y) \in V(R)$ be a vertex such that the distance in G between d and T is small as possible. Let P be a shortest path in R connecting a vertex of $T \square \{x\}$, say (a, x), with (d, y). The projection of P onto G is also a shortest a–d path. Because R is an isometric subgraph of $G \square H$, we have $P = (a_0, x)(a_1, x) \cdots (a_{k-1}, x)(a_k, x)(a_k, y)$ with $a_0 = a$, $a_k = d$ and $k \geqslant 1$. Note that $(a_i, y) \notin V(R)$ for $i = 0, 1, \ldots, k-1$. Now $r(a_{k-1}, y) = (a_k, x)$. Since G is weakly-triangulated, the edge $[a_{k-1}, a_k] \in E(G)$ is contained in a triangle T'. Let e be the third vertex of T'. Since $r(T' \square \{y\})$ is a triangle and two of the vertices of $r(T' \square \{y\})$ lie in the layer H_d, it follows that $r(e, y) = (d, z)$ for some $z \in V(H)$. Clearly, $z \neq x, y$. Hence $T'' = \{x, y, z\}$ is a triangle in H. Consider now $\{d\} \square T''$ and $\{a_{k-1}\} \square T''$. As $\{d\} \square T''$ belongs to R and $(a_{k-1}, x) \in V(R)$ we can use the argument of Case 1 to show that $(a_{k-1}, y) \in V(R)$, which contradicts the choice of d. □

As not every strongly-triangulated graph is weakly-triangulated, we must also prove the following proposition.

Proposition 2. *Let G be a strongly-triangulated graph and let H be a connected graph. Then every triangle of G transfers in H.*

Proof. Let $R, r: V(G \square H) \to V(R)$, $T = \{a, b, c\}$, $x, y \in V(H)$ be defined as in Proposition 1. Suppose that $(T \square \{y\}) \cap R = \emptyset$ and let $(d, y) \in V(R)$. As G is strongly-triangulated, there is a sequence of triangles in G_y, $T_1, T_2, \ldots, T_n$, $n \geqslant 1$, such that $(d, y) \in T_1$, $(a, y) \in T_n$, consecutive ones sharing an edge. Since $(d, y) \in V(R)$, $r(T_1)$ lies completely in R_y or in R_d. In the former case, every triangle $r(T_i)$, $i = 1, 2, \ldots, n$ lies in R_y, hence $r(a, y) \in R_y$. Since (a, y) is not fixed by r, $r(a, y)$ is not adjacent to (a, x), a contradiction. Hence $r(T_i)$ is contained in R_d, for every $i = 1, 2, \ldots, n$. Since $r(a, y)$ is adjacent to (a, x), it follows $r(a, y) = (d, x)$. Consider now a sequence of triangles between (d, y) and (b, y). If r maps these triangles on R_d, then since $r(b, y)$ is adjacent to (b, x), $r(b, y) = (d, x)$, a contradiction. But if r maps these triangles on G_y, then $r(b, y) = (c, y)$, $c \neq b$, which is also impossible. □

To prove that triangles transfer also for four-cycle free graphs, one can use the arguments in the last paragraph of the proof of Lemma 7 in [5]. Omitting details we thus state the following.

Proposition 3. *Let G be a connected graph without any four-cycle and let H be a connected graph. Then every triangle of G transfers in H.*

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