Computing graph invariants on rotagraphs using dynamic algorithm approach: the case of $(2,1)$-colorings and independence numbers

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Abstract

Rotagraphs generalize all standard products of graphs in which one factor is a cycle. A computer-based approach for searching graph invariants on rotagraphs is proposed and two of its applications are presented. First, the λ-numbers of the Cartesian product of a cycle and a path are computed, where the λ-number of a graph G is the minimum number of colors needed in a $(2,1)$-coloring of G. The independence numbers of the family of the strong product graphs $C_7 \boxtimes C_7 \boxtimes C_{2k+1}$ are also obtained.
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1. Introduction

A large part of Graph Theory consists of the study of graph invariants, as for instance the chromatic number, the independence number, and the domination number. Unfortunately, the determination of each of these invariants is NP-hard. Therefore many approximation and heuristic algorithms have been developed which obtain near-optimal solutions within reasonable running times. However, such algorithms in general provide only upper and lower bounds.

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One way to approach such invariants is to decompose a graph into smaller and simpler constituents by some basic operations and try to gain a global information from the constituents. For instance, one might try to decompose a graph with respect to one of the standard graphs products, see [16]. Quite often, product graphs having cycles as factors are important—as is the case of the Shannon capacity. A natural generalization of such structures is formed by rotagraphs that are in turn an important class of polygraphs. The later were introduced in chemical graph theory as a model for polymers, cf. [1], and studied in, for instance [20,21,31].

In this paper we propose a dynamic algorithm approach for computing graph properties of rotagraphs. The main idea is to build a function (an invariant) on a rotagraph from corresponding functions on its basic building blocks. Then the problem reduces to a search for a certain subgraph in an associated directed graph. In particular, if the function in question is rather local, one has to look for directed cycles. Although finding a cycle of length n, for an arbitrary n, is NP-complete, we were able to apply our approach for two central invariants. The main reason for the success is that it suffices to deal with small strongly connected components, therefore the computations can be done in a reasonable time.

We first succeeded to compute the λ-numbers of the Cartesian product of a cycle and a path, for which partial results were obtained in [19]. The λ-number of a graph G is the minimum number of colors needed in a $(2,1)$-coloring of G. The concept of $(2,1)$-coloring (or $(2,1)$-labeling) was introduced by Griggs and Yeh in [12], and has since been extensively studied, see for instance [3–5,7,10,15,22,23]. In particular, the λ-numbers (and their generalizations) of Cartesian product graphs (like hypercubes and Hamming graphs) were treated in [6,8,9,12,19,30]. In addition, Jha obtained the λ-numbers of some infinite families of Cartesian products of several cycles [17] as well as of strong products of several cycles [18].

In the second application we determine the independence numbers of the strong products $C_7 \boxtimes C_7 \boxtimes C_{2k+1}$, hence extending the results of [26]. Studies of the independence number of the strong product of odd cycles have been inspired by the Shannon's work [24] on the determination of the zero-capacity of a noisy channel. Shannon formulated the problem in the form of graph theory and supplied some partial results. It turns out that the solution of the problem requires the determination of the independence number of the strong product of graphs containing odd cycles as factors. Note also that the Shannon capacity of the 7-cycle is still open [11,28].

2. Definitions

Let $G = (V(G), E(G))$ be a graph. A *walk* is a sequence of vertices $v_1, v_2, \ldots, v_n$ and edges $v_i v_{i+1}$, $1 \leqslant i \leqslant n-1$. A *path* on n vertices is a walk on n different vertices and denoted P_n. A walk is *closed* if $v_1 = v_n$. A closed walk in which all vertices (except the first and the last) are different, is a *cycle*. The cycle on n vertices is denoted C_n. For $u, v \in V(G)$, $d_G(u,v)$ or $d(u,v)$ denotes the length of a shortest walk (i.e., the number of edges on a shortest walk) in G from u to v. These definitions extend naturally to directed graphs.

The *Cartesian product* of graphs G and H is the graph $G \square H$ with vertex set $G \times H$ and $(x_1, x_2)(y_1, y_2) \in E(G \square H)$ whenever $x_1 y_1 \in E(G)$ and $x_2 = y_2$, or $x_2 y_2 \in E(H)$ and $x_1 = y_1$. The *strong product* of graphs G and H is the graph $G \boxtimes H$ with vertex set $G \times H$ and $(x_1, x_2)(y_1, y_2) \in E(G \boxtimes H)$ whenever $x_1 y_1 \in E(G)$ and $x_2 = y_2$, or $x_2 y_2 \in E(H)$ and $x_1 = y_1$, or $x_1 y_1 \in E(G)$ and $x_2 y_2 \in E(H)$. The Cartesian product and the strong product are commutative and associative, having the trivial graph as a unit, cf. [16].

A $(2, 1)$-coloring of a graph G is a function f from the vertex set $V(G)$ to the set of non-negative integers C (called *colors*) such that

$$|f(u) - f(v)| \geqslant 2 \quad \text{if } d(u, v) = 1$$

and

$$|f(u) - f(v)| \geqslant 1 \quad \text{if } d(u, v) = 2.$$

A $k(2, 1)$-*coloring* is a $(2, 1)$-coloring of G such that $C = \{0, \ldots, k\}$. An optimal $(2, 1)$-coloring of G is a $k(2, 1)$-coloring with k smallest possible. The largest color used by an optimal $(2, 1)$-coloring is called the λ-*number* of G and denoted by $\lambda(G)$. Recall that the problem of determining the λ-number is NP-hard [10].

A set $S \subseteq V(G)$ is *independent* if $xy \notin E(G)$ for any pair of vertices $x, y \in S$. Cardinality of a largest independent set S of G is the *independence number* of G and denoted by $\alpha(G)$.

Let $G_0, G_1, \ldots, G_{n-1}$ be disjoint graphs and let $X_0, X_1, \ldots, X_{n-1}$ be a sequence of sets of edges such that an edge of X_i joins a vertex of G_i with a vertex of G_{i+1} (indices modulo n). A *polygraph*

$$\Omega_n = \Omega_n(G_0, G_1, \ldots, G_{n-1}; X_0, X_1, \ldots, X_{n-1})$$

is defined in the following way:

$$V(\Omega_n) = V(G_0) \cup V(G_1) \cup \cdots \cup V(G_{n-1}),$$

$$E(\Omega_n) = E(G_0) \cup X_0 \cup E(G_1) \cup X_1 \cup \cdots \cup E(G_{n-1}) \cup X_{n-1}.$$

The graphs G_i are called the *fibers* of Ω_n. A *rotagraph* on n fibers is a polygraph $G = \Omega_n(G_0, G_1, \ldots, G_{n-1}; X_0, X_1, \ldots, X_{n-1})$ in which the G_is are pairwise isomorphic and, a vertex of G_i is adjacent to a vertex of G_{i+1} ($i \geqslant 1$) if and only if the corresponding vertices of G_0 and G_1 are adjacent. Note that a rotagraph is uniquely determined by G_0 and X_0, hence we will denote it by $\Omega_n(G_0; X_0)$. A *fasciagraph* $\Psi_n(G_0; X_0)$ is a rotagraph $\Omega_n(G_0; X_0)$ *without* edges between the fibers G_{n-1} and G_0. The subgraph of $\Omega_n(G_0; X_0)$ induced by the vertices of the consecutive fibers $G_i, G_{i+1}, \ldots, G_{i+k}$ (indices modulo n) is a fasciagraph and will be denoted by $\Psi_{k+1}(G_i; X_i)$.

An example of a rotagraph is given in Fig. 1. Its fibers (one being encircled in the figure) are isomorphic to $K_2 \cup K_1$.
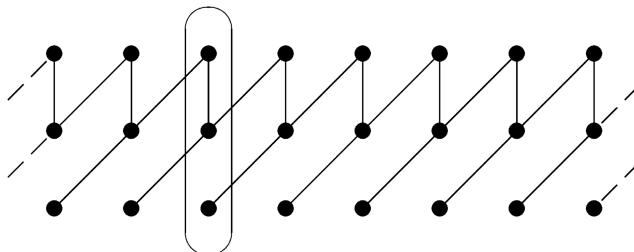
Fig. 1. Rotagraph and its fiber.

3. Graph properties and rotagraphs

For a graph G set

$$\mathscr{F}_q(G) = \{f : V(G) \to \{0, 1, \ldots, q - 1\}\}.$$

A subset of $\mathscr{F}_q(G)$ will be called a *graph q-property*. If q will be clear from the context or not essential, we will shortly say a *graph property*. Let us first list some examples of graph properties. Not all of them are used later in the paper, but we include them in order to demonstrate that the concept is rather general. Moreover, some of them will also be used in a subsequent work.

1. Let $\mathscr{C}_q(G) \subseteq \mathscr{F}_q(G)$ be the set of functions f with the following characterizing property: If $uv \in E(G)$ then $f(u) \neq f(v)$. It is clear that $\mathscr{C}_q(G)$ represents the set of all proper q-colorings of G. In particular $\mathscr{C}_q(G) \neq \emptyset$ if and only if $\chi(G) \leqslant q$.
2. Let $\mathscr{I}_2(G) \subseteq \mathscr{F}_2(G)$ be the set of functions f with the following property: Let $f \in \mathscr{I}_2(G)$, then there is an independent set I of G such that $I = f^{-1}(1)$. Then $\mathscr{I}_2(G)$ is the set of characteristic functions of the independent sets of G.
3. Let the functions f of $\mathscr{D}_2(G) \subseteq \mathscr{F}_2(G)$ be defined with the following property: For any $f \in \mathscr{D}_2(G)$ and any vertex $u \in V(G)$ with $f(u) = 0$ there exists a vertex $v \in N(u)$ such that $f(v) = 1$. Observe that $\mathscr{D}_2(G)$ represents the dominating sets of G.
4. Let $\mathscr{L}_q(G) \subseteq \mathscr{F}_q(G)$ be the set of functions f with the following property: Let $f \in \mathscr{L}_q(G)$, then if $uv \in E(G)$ we have $|f(u) - f(v)| \geqslant 2$, and if $d(u, v) = 2$ we have $|f(u) - f(v)| \geqslant 1$. Clearly, $\mathscr{L}_q(G)$ describes the admissible $(2, 1)$-colorings of G.
5. Let $\mathscr{B}_2(G) \subseteq \mathscr{F}_2(G)$ be the set of functions f with the following property: Let $f \in \mathscr{B}_2(G)$, then $f^{-1}(0)$ and $f^{-1}(1)$ are independent sets of G. Then $\mathscr{B}_2(G)$ describes the bipartitions of G. In particular $\mathscr{B}_2(G) \neq \emptyset$ if and only if G is bipartite.

We now turn our attention to rotagraphs. Let $\Omega_n(G_0; X_0)$ be a rotagraph with consecutive fibers $G_0, G_1, \ldots, G_{n-1}$. Then the restriction of $f \in \mathscr{F}_q(\Omega_n(G_0; X_0))$ to the subgraph induced by the vertices of the consecutive fibers $G_i, G_{i+1}, \ldots, G_{i+k}$ (indices modulo n) will be denoted f_i^{i+k}. We say that a graph property $\mathscr{P}_q$ is *hereditary* (for rotagraphs),

if for any rotagraph $\Omega_n(G_0; X_0)$,

$$f \in \mathcal{P}_q(\Omega_n(G_0; X_0)) \Rightarrow f_i^{i+k} \in \mathcal{P}_q(\Psi_{k+1}(G_i; X_i)); \quad i, k = 0, 1, \ldots, n-1.$$

Note that $\mathcal{C}_q$, $\mathcal{I}_2$, $\mathcal{L}_q$, and $\mathcal{B}_2$ are hereditary properties, while $\mathcal{D}_2$ is not. To see that $\mathcal{D}_2$ is not hereditary, consider the rotagraph of Fig. 1 and its dominating set consisting of the middle vertices of each fiber.

A graph property $\mathcal{P}_q$ is called *d-local* (for rotagraphs), $d \geqslant 1$, if for any rotagraph $\Omega_n(G_0; X_0)$, $n \geqslant 2d + 1$, and any $f \in \mathcal{F}_q(\Omega_n(G_0; X_0))$,

$$f_i^{i+d} \in \mathcal{P}_q(\Psi_{d+1}(G_i; X_i)), \quad 0 \leqslant i \leqslant n-1 \Rightarrow f \in \mathcal{P}_q(\Omega_n(G_0; X_0)).$$

Note that $\mathcal{C}_q$, $\mathcal{I}_2$, and $\mathcal{D}_2$ are 1-local properties (for rotagraphs), $\mathcal{L}_q$ is a 2-local property, while $\mathcal{B}_2$ is not d-local for any $d \geqslant 1$.

The basic building stone of a rotagraph $\Omega_n(G_0; X_0)$ is the fasciagraph $\Psi_2(G_0; X_0)$. Therefore, our main idea is to build functions (alias graph properties) on $\Omega_n(G_0; X_0)$ from corresponding functions on $\Psi_2(G_0; X_0)$. We proceed as follows.

Let $\mathcal{P}_q$ be a d-local property, and let $\Omega_n(G_0; X_0)$ be a rotagraph with $n \geqslant 2d + 1$ fibers. We now introduce a directed graph $D_d(G_0; X_0)$ as follows. (We do not know whether this concept was introduced elsewhere.) Its vertices are the functions from $\mathcal{P}_q(\Psi_2(G_0; X_0))$, while its arcs are of two types: the first type arcs will be simply called *arcs*, and the second type arcs will be *d-arcs*. Now, in $D_d(G_0; X_0)$ make an arc from f to g if and only if f restricted to the second fiber of $\Psi_2(G_0; X_0)$ equals to g restricted to the first fiber of $\Psi_2(G_0; X_0)$. In addition, if $d \geqslant 2$, then for any directed path (consisting of arcs) of length $d - 1$, say $f_1 \to f_2 \to \cdots \to f_d$, we make a d-arc from f_1 to f_d whenever the composition of $f_1, f_2, \ldots, f_d$ belongs to $\mathcal{P}_q(\Psi_{d+1}(G_0; X_0))$. Here by the composition we mean that f_i is applied to the subgraph induced by the fibers G_{i-1} and G_i. In the particular case when $d = 2$ we interpret this as follows: If the composition of f_1 and f_2 belongs to $\mathcal{P}_q(\Psi_3(G_0; X_0))$ then we leave the arc from f_1 to f_2, otherwise we remove it. Now we can state:

Theorem 1. *Let $\mathcal{P}_q$ be a hereditary, d-local property, and let $\Omega_n(G_0; X_0)$ be a rotagraph with $n \geqslant 2d + 1$ fibers. Then $\mathcal{P}_q(\Omega_n(G_0; X_0)) \neq \emptyset$ if and only if $D_d(G_0; X_0)$ contains* (*not necessarily different*) *vertices $f_0, f_1, \ldots, f_{n-1}$ connected with arcs (f_i, f_{i+1}) and d-arcs (f_i, f_{i+d-1}) for $i = 0, 1, \ldots, n-1$* (*indices modulo n*).

Proof. Assume $\mathcal{P}_q(\Omega_n(G_0; X_0)) \neq \emptyset$ and let $f \in \mathcal{P}_q(\Omega_n(G_0; X_0))$. Since $\mathcal{P}_q$ is hereditary, f_i^{i+1} is a vertex of $D_d(G_0; X_0)$ for $i = 0, 1, \ldots, n-1$. Set $f_i = f_i^{i+1}$, $i = 0, 1, \ldots, n-1$. Clearly, there is an arc from f_i to f_{i+1} for all i. If $d = 1$ we are done, so let $d \geqslant 2$. Then, again as $\mathcal{P}_q$ is hereditary, f_i^{i+d} belongs to $\mathcal{P}_q(\Psi_{d+1}(G_i; X_i))$ and we have a d-arc from f_i to f_{i+d-1}.

Suppose now that $D_d(G_0; X_0)$ contains vertices $f_0, f_1, \ldots, f_{n-1}$, arcs (f_i, f_{i+1}) and d-arcs (f_i, f_{i+d-1}). Define a function $f \in \mathcal{F}_q(\Omega_n(G_0; X_0))$ with the property that f_i^i is equal to f_i restricted to the first fiber of $\Psi_2(G_i; X_i)$. Since we have a d-arc from f_i to f_{i+d-1}, f_i^{i+d} belongs to $\mathcal{P}_q(\Psi_{d+1}(G_i; X_i))$. Moreover, as $\mathcal{P}_q$ is a d-local property we conclude that $f \in \mathcal{P}_q(\Omega_n(G_0; X_0))$. $\quad\square$

The most practically important special case of the theorem is:

Corollary 2. *Let $\mathscr{P}_q$ be a hereditary, d-local property, $1 \leqslant d \leqslant 2$, and let $\Omega_n(G_0; X_0)$ be a rotagraph with $n \geqslant 5$ fibers. Then $\mathscr{P}_q(\Omega_n(G_0; X_0)) \neq \emptyset$ if and only if $D_d(G_0; X_0)$ contains a directed closed walk of length n.*

4. (2, 1)-colorings of the Cartesian product of a cycle and a path

In this section, we apply the proposed general approach from Section 3 to obtain the λ-numbers of the Cartesian product of a path and a cycle. Whittlesey et al. [30] obtained the λ-numbers of the Cartesian product of two paths. They proved that if $m \geqslant 4$, then $\lambda(P_m \square P_2) = 5$ and if $m, n \geqslant 4$, or $m \geqslant 5$ and $n \geqslant 3$, then $\lambda(P_m \square P_n) = 6$. For the Cartesian product of paths and cycles, the following partial results from [19] are known:

Theorem 3. *Let $m \geqslant 3$. Then*:

(i) *If $m \equiv 0 \,(mod \, 3)$, then $\lambda(C_m \square P_2) = 5$.*
(ii) *If $m \not\equiv 0 \,(mod \, 3)$, then $\lambda(C_m \square P_2) \leqslant 6$.*
(iii) *If $m \equiv 0 \,(mod \, 7)$ and $n \geqslant 3$, then $\lambda(C_m \square P_n) = 6$.*
(iv) *If $m, n \geqslant 3$, then $\lambda(C_m \square P_n) \leqslant 7$.*

In this section we show:

Theorem 4. *Let $m \geqslant 3$. Then:*

(i) $\lambda(C_m \square P_2) = \begin{cases} 5, & m \equiv 0 \,(mod \, 3), \\ 6, & \textit{otherwise.} \end{cases}$

(ii) $\lambda(C_m \square P_3) = \begin{cases} 7, & m = 4, 5, \\ 6, & \textit{otherwise.} \end{cases}$

(iii) *For $n \geqslant 4$,* $\lambda(C_m \square P_n) = \begin{cases} 6, & m \equiv 0 \,(mod \, 7), \\ 7, & \textit{otherwise.} \end{cases}$

In the rest of this section we prove Theorem 4. For this sake we need the following lower bound due to Griggs and Yeh [12]:

Lemma 5. *Let G be graph with maximum degree $\Delta \geqslant 2$. If G contains three vertices of degree Δ such that one of them is adjacent to the other two, than $\lambda(G) \geqslant \Delta + 2$.*

Case (i): $\lambda(C_m \square P_2)$.
The Cartesian product $C_m \square P_2$ is isomorphic to the rotagraph $\Omega_m(P_2; X_0)$, where X_0 consists of two edges inducing an identity between the corresponding fibers, that is, between the P_2s. Consider the graph property $\mathscr{L}_6$ and recall that it is hereditary,
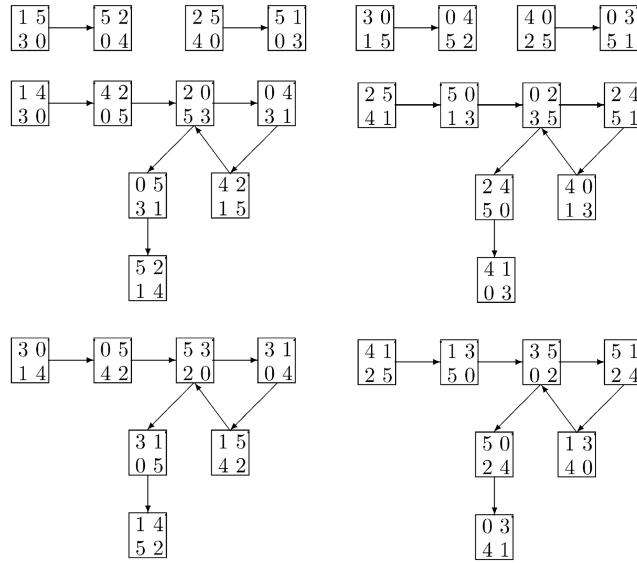
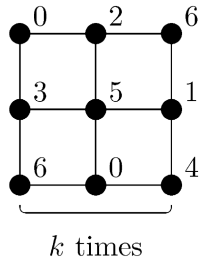Fig. 2. Graph $D_2(P_2; X_0)$ (without isolated vertices).

k times

Fig. 3. 6(2, 1)-coloring of $C_{3k} \square P_3$, $k \geqslant 1$.

2-local property. From Corollary 2 we know that $\mathscr{L}_6(\Omega_m(P_2; X_0)) \neq \emptyset$ if and only if $D_2(P_2; X_0)$ contains a directed closed walk of length m. The graph $D_2(P_2; X_0)$ (without isolated vertices) is depicted in Fig. 2. The two columns of a box represent colors of the first and the second fiber, respectively.

Fig. 2 immediately implies that all directed closed walks in $D_2(P_2; X_0)$ are of length $3k$, $k \geqslant 1$. By Lemma 5, $\lambda(C_m \square P_2) \geqslant 5$, and by Theorem 3, $\lambda(C_m \square P_2) \leqslant 6$, hence the conclusion.

Case (ii): $\lambda(C_m \square P_3)$.

For $m = 4, 5$ the result follows from a simple computer program based on backtracking. The same program gives the 6(2, 1)-colorings for $m = 3k$ (Fig. 3), for $m = 3k + 1$ (Fig. 4), and for $m = 3k + 2$ (Fig. 5).

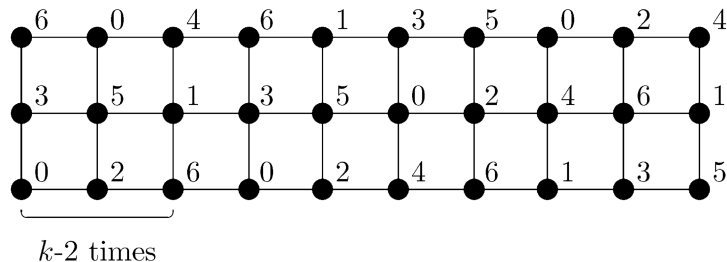Since by Lemma 5, $\lambda(C_m \square P_3) \geqslant 6$, Case (ii) is settled.

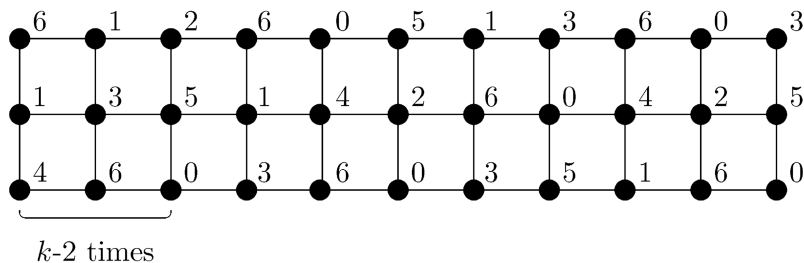Fig. 4. $6(2, 1)$-coloring of $C_{3k+1} \square P_3$, $k \geqslant 2$.

Fig. 5. $6(2, 1)$-coloring of $C_{3k+2} \square P_3$, $k \geqslant 2$.

Case (iii): $\lambda(C_m \square P_n)$, $n \geqslant 4$.

For $m \equiv 0 \,(\text{mod } 7)$ the conclusion follows from Theorem 3.

Let $n = 4$ and consider the rotagraph $\Omega_m(P_4; X_0)$, where X_0 consists of four edges inducing an identity between the corresponding P_4s. Consider the graph property $\mathscr{L}_7$. Corollary 2 implies that $\Omega_m(P_4; X_0)$ has a $6(2, 1)$-coloring if and only if $D_2(P_4; X_0)$ contains a directed closed walk of length m. The graph $D_2(P_4; X_0)$ consists of 1248 vertices and 912 edges (determined by a computer program). In order to search for cycles in $D_2(P_4; X_0)$, the strongly connected components were detected first. Linear-time algorithms that obtain strongly connected components are well known (see e.g. [29]). Eight strongly connected components of $D_2(P_4; X_0)$ were computed, each of them consisting of seven vertices and exactly one directed cycle. Therefore, all directed closed walks in H are of length $7k$, $k \geqslant 1$, thus a $6(2, 1)$-coloring of $\Omega_m(P_4; X_0)$ for $m \not\equiv 0 \,(\text{mod } 7)$ does not exist. Since $\lambda(C_m \square P_4) \leqslant 7$ by Theorem 3, and $\lambda(C_m \square P_4) \geqslant 6$ by Lemma 5, the argument for the subcase $n = 4$ of is complete.

For $n \geqslant 5$ just note that $C_m \square P_4$ is a subgraph of $C_m \square P_n$, $n \geqslant 4$, and apply Lemma 5 and Theorem 3.

5. Independent sets in strong products of cycles

The independence numbers of the strong product of cycles where at most one cycle is odd are well-known. The problem is incomparably more difficult for the strong product

of cycles with at least two odd factors. Hales [14], as well as Sonnemann and Krafft [25], proved that for $j, k \in \mathbb{N}$, $j \geqslant k$, $\alpha(C_{2j+1} \boxtimes C_{2k+1}) = jk + \lfloor k/2 \rfloor$. For the strong product of three (or more) odd cycles only partial results are known. The independence numbers of the strong products of three cycles of (moderate) sizes $C_5 \boxtimes C_7 \boxtimes C_{2k+1}$, $C_5 \boxtimes C_9 \boxtimes C_{2k+1}$, $C_7 \boxtimes C_7 \boxtimes C_{2k+1}$, and $C_7 \boxtimes C_9 \boxtimes C_{2k+1}$ were considered in [2,26]. The obtained results are:

$$\alpha(C_5 \boxtimes C_7 \boxtimes C_k) = 7k + 2, \; 3 \leqslant k \leqslant 5,$$
$$\alpha(C_5 \boxtimes C_7 \boxtimes C_k) = 7k + 3, \; k \geqslant 6,$$
$$\alpha(C_5 \boxtimes C_9 \boxtimes C_k) = 9k + 4, \; k \geqslant 4,$$
$$\alpha(C_7 \boxtimes C_7 \boxtimes C_7) = 33,$$
$$\alpha(C_7 \boxtimes C_7 \boxtimes C_9) = 44,$$
$$\alpha(C_7 \boxtimes C_7 \boxtimes C_k) = \begin{cases} 10k + 4, \\ 10k + 5, \end{cases} k \geqslant 5,$$
$$\alpha(C_7 \boxtimes C_9 \boxtimes C_k) = \begin{cases} 13k + 5, \\ 13k + 6, \end{cases} k \geqslant 4.$$

The above results were obtained mostly using backtracking algorithms. For instance, the computations for $\alpha(C_7 \boxtimes C_7 \boxtimes C_9)$ took about one week. In [26] it was also conjectured that the independence numbers of $C_7 \boxtimes C_7 \boxtimes C_{2k+1}$ and $C_7 \boxtimes C_9 \boxtimes C_{2k+1}$ reach the upper bound from the above if k is large enough. This is indeed true for the first case, as follows from the main result of this section:

Theorem 6. *Let* $k \geqslant 3$. *Then*:

$$\alpha(C_7 \boxtimes C_7 \boxtimes C_{2k+1}) = \begin{cases} 33, & k = 3, \\ 10k + 4, & 4 \leqslant k \leqslant 9, \\ 10k + 5, & k \geqslant 10. \end{cases}$$

In the rest of the section we prove Theorem 6. It is well-known that for any graph G and $n \in \mathbb{N}$, $\alpha(G \boxtimes K_n) = \alpha(G)$. Hence:
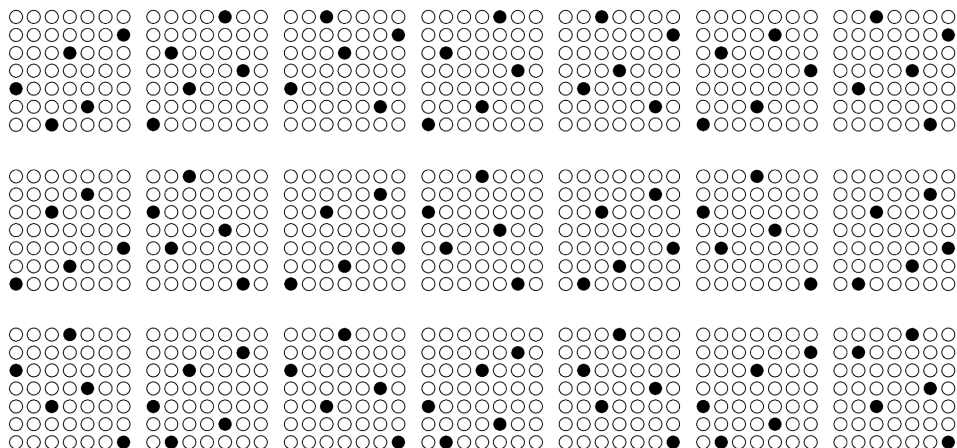
Corollary 7. $\alpha(C_7 \boxtimes C_7 \boxtimes K_2) = \alpha(C_7 \boxtimes C_7) = 10$.

We will also need the following lemma from [25]:

Lemma 8. *For any graph G and any $k \in \mathbb{N}$,*

$$\alpha(C_{2k+3} \boxtimes G) \geqslant \left\lceil \frac{2k+3}{2k+1} \alpha(C_{2k+1} \boxtimes G) \right\rceil.$$

Since $G = C_7 \boxtimes C_7 \boxtimes C_{2k+1} = (C_7 \boxtimes C_7) \boxtimes C_{2k+1}$, we can consider G as the rotagraph $\Omega_{2k+1}(C_7 \boxtimes C_7; X_0)$ where X_0 is defined in the obvious way. We now consider those functions f of $\mathscr{I}_2(\Omega_{2k+1}(C_7 \boxtimes C_7; X_0))$ for which $|f^{-1}(1)| = 10k + 5$. By Corollary 7,

Fig. 6. 105 independent vertices of $C_7 \boxtimes C_7 \boxtimes C_{21}$.

any two adjacent $(C_7 \boxtimes C_7)$-fibers contain at most 10 independent vertices. Thus, in order to obtain $10k + 5$ independent vertices altogether, having in mind that the number of fibers is odd, every $(C_7 \boxtimes C_7)$-fiber must contain 5 independent vertices. In other words, for any i we have $|(f_i^i)^{-1}(1)| = 5$. By Corollary 2, $\Omega_{2k+1}(C_7 \boxtimes C_7; X_0)$ has an independent set with $10k + 5$ independent vertices if and only if $D = D_1(C_7 \boxtimes C_7; X_0)$ contains a directed closed walk of length $2k + 1$.

We have first computed the directed graph D, giving us $980\binom{10}{5} = 246\,960$ vertices of D. (Recall from [26] that $C_7 \boxtimes C_7$ has 980 independent sets with 10 vertices if its vertices have fixed labels.) Using depth-first search we then discovered that D contains an odd cycle. In order to establish the smallest k, such that $\alpha(C_7 \boxtimes C_7 \boxtimes C_{2k+1}) = 10k+5$, a shortest odd cycle was to be found. We used a construction described by Grötschel et al. in [13]. An auxiliary bipartite graph $\Gamma = ((V^1, V^2), E)$ was constructed as follows. Each vertex $u \in D$ was split into two vertices u^1 and u^2, with u^i included in V^i $(i = 1, 2)$. For each arc uv of D we have added the arcs u^1v^2 and u^2v^1 to E. Note that a shortest path from u^1 to u^2 in Γ corresponds to a shortest cycle of odd length in D containing u. Thus we have constructed Γ and determined the length of a shortest path from u^1 to u^2 for every $u \in D$. The program found out that a shortest cycle in D is of length 21. The corresponding set with 105 independent vertices is depicted in Fig. 6.

In this way we have established the theorem for $k = 10$. The result for $k > 10$ follows from Lemma 8.

6. Concluding remarks

In this paper we have proposed a dynamic algorithm approach for computing graph properties of rotagraphs. Essentially the same approach can be used for computing graph properties of fasciagraphs. For instance, in the case of 1- or 2-local properties,

the only difference being that instead of searching for directed closed walks of length n in $D_d(G_0; X_0)$ we need to look for directed walks of length n.

Suppose we consider a d-local, hereditary property $\mathscr{P}_q$ with $d > 2$. Then one might select the fasciagraphs $\Psi_{d+1}(G_0; X_0)$ as the basic building stone of the rotagraphs $\Omega_n(G_0; X_0)$ and combine functions restricted to these fasciagraphs to search for functions on rotagraphs. In this way a corresponding directed graph D can be introduced such that $\mathscr{P}_q(\Omega_n(G_0; X_0))$ is nonempty if and only if D contains a directed closed walk of length n, a result analogous to Corollary 2. Of course, if depends on the functions considered which approach would be more efficient. In particular, if the approach via fasciagraphs $\Psi_d(G_0; X_0)$ would be used, the number of the corresponding functions on $\Psi_d(G_0; X_0)$ must be of a moderate size.

Not all interesting graph functions are d-local and hereditary. For instance, we have observed that the functions from $\mathscr{D}_2$ describing dominating sets are not hereditary. Nevertheless, a slightly altered approach can still be used to obtain new exact domination numbers of the Cartesian product of two cycles. It will be described in a subsequent paper [27].

Finally, we wish to add that in the case of rotagraphs, the concept of a d-local property seems to be most naturally defined in terms of the rotagraph's fibers. However, in a more general situation where we have no fiber structure, it might have been natural to define a graph property $\mathscr{P}_q$ to be d-local, if for any graph (from a certain class of graphs) G, a function $f \in \mathscr{F}_q(G)$ belongs to $\mathscr{P}_q(G)$ if and only if for any vertex u of G the function f, restricted to disk $N_d[u]$ of radius d around u, belongs to $\mathscr{P}_q(\langle N_d[u] \rangle)$.

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