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On a Vizing-like conjecture for direct product graphs[1]

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Abstract

Let $\gamma(G)$ be the domination number of a graph G, and let $G \times H$ be the direct product of graphs G and H. It is shown that for any $k \geqslant 0$ there exists a graph G such that $\gamma(G \times G) \leqslant \gamma(G)^2 - k$. This in particular disproves a conjecture from [5].

1. Introduction

A set D of vertices of a simple graph G is called *dominating* if every vertex $w \in V(G) - D$ is adjacent to some vertex $v \in D$. The *domination number* of a graph G, $\gamma(G)$, is the order of a smallest dominating set of G. A dominating set D with $|D| = \gamma(G)$ is called a *minimum dominating set.*

The *direct product* $G \times H$ of graphs G and H is a graph with $V(G \times H) = V(G) \times V(H)$ and $E(G \times H) = \{\{(a,x),(b,y)\} \mid \{a,b\} \in E(G) \text{ and } \{x,y\} \in E(H)\}$. This product is also known as *Kronecker product*, *tensor product*, *categorical product* and *graph conjunction*. The *Cartesian product* $G \Box H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ and $(a,x)(b,y) \in E(G \Box H)$ whenever $x = y$ and $\{a,b\} \in E(G)$, or $a = b$ and $\{x,y\} \in E(H)$.

Most of the interest for domination in graph products is due to Vizing's conjecture [11] from 1963. Vizing conjectured that

$$\gamma(G \Box H) \geqslant \gamma(G)\gamma(H)$$

hold for any graphs G and H. Despite considerable efforts (cf. [1–4,6–9]) it seems that presently there is no 'winning way' to the conjecture.

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Another graph product which offers interesting and non-trivial problems on domination is the direct product. Gravier and Khelladi [5] posed the following Vizing-like conjecture for the direct product:

$$\gamma(G \times H) \geqslant \gamma(G)\gamma(H).$$

Here we show that for any $k \geqslant 0$ there exists a graph G such that $\gamma(G \times G) \leqslant \gamma(G)^2 - k$. This result in particular disproves the above-mentioned conjecture. Moreover, it also supports the following statement: although the direct product of graphs is the most natural graph product, it is also the most difficult and unpredictable among standard graph products.

In fact, as far as we know Nowakowski and Rall were the first who observed that the above-mentioned conjecture does not hold. In their manuscript [10] they report a graph with $\gamma(G) = 2$ yet $\gamma(G \times G) = 3$. We wish to add that the paper of Nowakowski and Rall is a nice and relevant paper which considers several graph parameters (related to independence, domination and irredundance) of all main associative graph products.

2. The construction

Let G_1 be the graph depicted in Fig. 1 and let H be the graph $G_1 \setminus \{u, w\}$ (see Fig. 1 again). Then we have:

Lemma 2.1. (i) $\gamma(G_1) = \gamma(H) = 3$.
(ii) $\gamma(G_1 \times G_1) \leqslant 7$.

Proof. (i) Partition $V(G_1)$ into the sets $V_1 = \{x, y, z, \bar{x}, \bar{y}, \bar{z}\}$, $V_2 = \{u, v, w\}$ and $V_3 = \{\bar{u}, \bar{v}, \bar{w}\}$ and note that the domination number of the subgraph of G_1 induced by the set V_1 is equal 2.

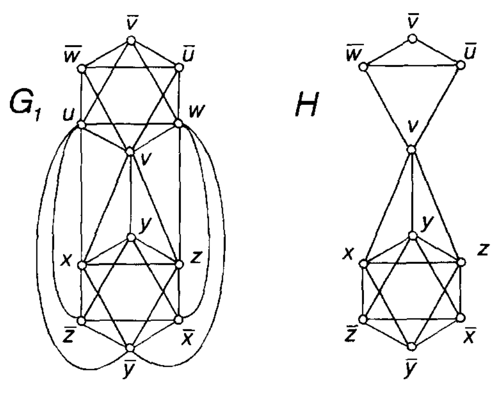

Fig. 1. Graphs G_1 and H.

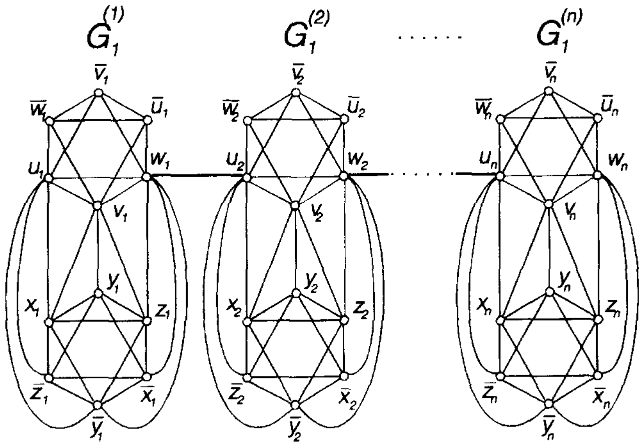

Fig. 2. Graph G_n.

Suppose that $\gamma(G_1) = 2$ and let D be a minimum dominating set. Since any pair of vertices from the set $\{\bar{x}, y, \bar{z}\}$ have no common neighbour in V_2, at least one vertex of V_1 must belong to D. Therefore, the other vertex of D must lie in V_2. But this means that at least one vertex of V_3 is not dominated by D, a contradiction. Clearly, $\gamma(G_1) \leqslant 3$.

Analogous argument (with $V_2 = \{v\}$) also gives $\gamma(H) = 3$.

(ii) It is straightforward to check that the set

$$\{(u,u),(v,v),(w,w),(v,y),(y,v),(u,z),(z,u)\}$$

constitutes a dominating set of $G_1 \times G_1$. □

For any $i \geqslant 1$ let $G_1^{(i)}$ be an isomorphic copy of the graph G_1 (where $G_1^{(i)} = G_1$). Label the vertices of the graphs $G_1^{(i)}$ as it is shown in Fig. 2. Let G_n be the graph which we obtain from the disjoin union of the graphs $G_1^{(1)}, G_1^{(2)}, \ldots, G_1^{(n)}$ with the addition of edges $\{w_i, u_{i+1}\}$, $1 \leqslant i < n$ (see Fig. 2).

Theorem 2.2. (i) $\gamma(G_n) = 3n$.

(ii) $\gamma(G_n \times G_n) \leqslant 7n^2$.

Proof. (i) By Lemma 2.1(i) the domination number of any graph $G_1^{(i)}$ does not depend on the vertices u_i and w_i. Therefore, each $G_1^{(i)}$ must contain at least 3 vertices of a minimum dominating set of G_n.

(ii) Let $G'_n = \bigcup_{i=1}^{n} G_1^{(i)}$ be the disjoint union of the graphs $G_1^{(i)}$. Clearly, $G'_n \times G'_n$ is a (proper) subgraph of the product $G_n \times G_n$ and hence

$$\gamma(G_n \times G_n) \leqslant \gamma(G'_n \times G'_n).$$

The graph $G_n' \times G_n'$ consists of n^2 connected components which are all isomorphic to the product $G_1 \times G_1$. Using Lemma 2.1(ii) we thus infer

$$\gamma(G_n' \times G_n') = n^2\gamma(G_1 \times G_1) \leqslant 7n^2,$$

which completes the proof. □

We can now state the main result of this note.

Corollary 2.3. *For any $k \geqslant 0$ there exists a graph G such that*

$$\gamma(G \times G) \leqslant \gamma(G)^2 - k.$$

Proof. Set $n = \lceil \sqrt{k/2} \rceil$. Theorem 2.2 immediately gives

$$\gamma(G_n \times G_n) \leqslant \gamma(G_n)^2 - 2n^2$$

which in turn implies that

$$\gamma(G_n \times G_n) \leqslant \gamma(G_n)^2 - k. \qquad \square$$

To conclude we wish to add that the above result indicates that domination problems are not only interesting on the Cartesian product graphs but also on the direct product graphs.

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