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Note

Isometric embeddings of subdivided wheels in hypercubes

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Abstract

The k-wheel W_k is the graph obtained as a join of a vertex and the cycle of length k. It is proved that a subdivided wheel G embeds isometrically into a hypercube if and only if G is the subdivision graph $S(K_4)$ of K_4 or G is obtained from the wheel W_k ($k \geqslant 3$) by subdividing any of its outer-edges with an odd number of vertices.

Keywords: Isometric embedding; Hypercube; Wheel

1. Introduction

For a graph G, the *distance* $d_G(u,v)$ (or briefly $d(u,v)$) between vertices u and v is defined as the number of edges on a shortest u,v-path. A subgraph H of G is called *isometric* if $d_G(u,v) = d_H(u,v)$ for all $u,v \in V(H)$. Isometric subgraphs of hypercubes are called *partial cubes*. Partial cubes have first been investigated in the 1970s by Graham and Pollak [9] and Djoković [7]. Avis [1], Winkler [17] and Chepoi [3] followed with different characterizations of these graphs, cf. also [16]. Besides early applications (like in [9]), partial cubes have found several other applications, for instance in chemical graph theory, see [11,13] and references therein. For more information on partial cubes see the books [5,10].

In [12], it is proved that if every edge of a graph G is subdivided by one vertex, then the obtained (bipartite) graph is a partial cube if and only if every block of G

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is complete or a cycle. In addition, subdivisions of wheels are considered in [2] in order to construct bipartite graphs with convex intervals that are not partial cubes, thus answering in negative a question of Chepoi and Tardif whether partial cubes are precisely bipartite graphs with convex intervals. The subdivisions studied in [2] are "uniform", that is, all the edges of the outer cycle of a wheel are subdivided with the same number of vertices, and all the spokes are also subdivided with the same number of vertices. Hence, it is natural to ask when an arbitrary subdivided wheel allows an isometric embedding into a hypercube. A related problem was studied by Deza and Tuma [6]. They have characterized the so-called ℓ_1-graphs among the subdivided wheels in which the spokes are not subdivided. In [15,7], the authors note that ℓ_1-graphs and partial cubes coincide in the case of bipartite graphs. For more information on ℓ_1-graphs we refer to [4,5].

Let W_k be the *k-wheel*, that is, the graph obtained as a join of the one vertex graph K_1 and the k-cycle C_k. In the rest of the paper, we will denote the central vertex of W_k by u and the remaining vertices by $w_1,\dots,w_k$, where adjacencies are defined naturally, cf. Fig. 2. The cycle of W_k induced by the vertices $w_1,\dots,w_k$ will be called the *outer-cycle* of W_k. These notions will also be used for subdivided wheels. In the following the subscript must be read modulo k.

Let $W_k(m_1,\dots,m_k;n_1,\dots,n_k)$ be the graph obtained by subdividing edges of W_k, where m_i is the number of vertices added on the edge w_iw_{i+1}, and n_i the number of vertices added on the inner edge uw_i. In this note we prove the following result:

Theorem 1. *Let $k \geqslant 3$. Then a subdivided wheel W is a partial cube if and only if W is isomorphic to $W_k(m_1,\dots,m_k;n_1,\dots,n_k)$, where m_i is odd for $i=1,\dots,k$ and $n_1=\cdots=n_k=0$, or $W=W_3(1,1,1;1,1,1)$.*

Observe that Theorem 1 implies that $W_k(m_1,\dots,m_k;n_1,\dots,n_k)$ is a median graph if and only if $n_1=\cdots=n_k=0$ and the $m_1=\cdots=m_k=1$ (up to isomorphism).

A particular instance of the graphs from Theorem 1 is $W_3(1,1,1;0,0,0)$ that is known as the bipartite wheel with three spokes BW_3. Note that it is isomorphic to any of the $W_3(1,0,0;1,1,0)$, $W_3(0,1,0;0,1,1)$ and $W_3(0,0,1;1,0,1)$. The graph $W_3(1,1,1;1,1,1)$ is the subdivision graph $S(K_4)$ of K_4. See Fig. 1 for isometric embeddings of these two graphs.

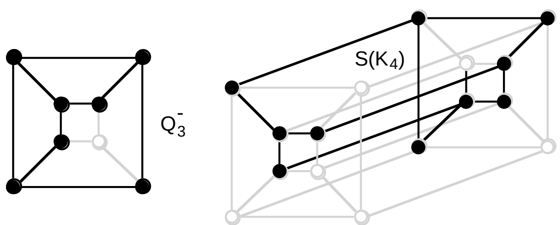

Fig. 1. Isometric embeddings of BW_3 and $S(K_4)$.

2. Preliminaries

The set $I(u,v)$ of all vertices of G which lie on a shortest paths between vertices $u,v\in V(G)$ is called *interval*. A graph G is a *median graph* if for any triple of vertices u,v,w we have $|I(u,v)\cap I(u,w)\cap I(v,w)|=1$.

Two edges $e=xy$ and $f=uv$ of a connected graph G are in the Djoković–Winkler [7,17] relation Θ if

$$d_G(x,u)+d_G(y,v)\neq d_G(x,v)+d_G(y,u).$$

If G is bipartite, then the edges $e=xy$ and $f=uv$ are in relation Θ precisely when $d(x,u)=d(y,v)$ and $d(x,v)=d(y,u)$. Winkler [17] proved that a bipartite graph is a partial cube if and only if Θ is transitive. It is easy to see that if e is an edge of an isometric cycle C of a partial cube then the Θ-class of e intersects C in exactly two edges.

Let G' be a connected graph. A *proper cover* G_1', G_2' consists of two isometric subgraphs G_1', G_2' of G' such that $G'=G_1'\cup G_2'$ and $G_0'=G_1'\cap G_2'$ is a nonempty subgraph, called the *intersection* of the cover. The *expansion* of G' with respect to G_1', G_2' is the graph G constructed as follows. Let G_i be an isomorphic copy of G_i', for $i=1,2$, and, for any vertex u' in G_0', let u_i be the corresponding vertex in G_i, for $i=1,2$. Then G is obtained from the disjoint union $G_1\cup G_2$, where for each u' in G_0' the vertices u_1 and u_2 are joined by an edge.

Chepoi [3] proved that a graph is a partial cube if and only if it can be obtained from K_1 by a sequence of expansions. This result is analogous to Mulder's convex expansion theorem for median graphs [14] and was later independently obtained by Fukuda and Handa [8].

For $m_i>0$ we denote by $w_{i,1},\dots,w_{i,m_i}$ the vertices added on the edge w_iw_{i+1} and for $n_i>0$ let $u_{i,1},\dots,u_{i,n_i}$ be the vertices added on the edge uw_i. Let $C_{i,i+1}$ be the cycle of $W_k(m_1,\dots,m_k;n_1,\dots,n_k)$ induced by vertices $u,u_{i,1},\dots,u_{i,n_i}$, $w_i,w_{i,1},\dots,w_{i,m_i}$, $w_{i+1},u_{i+1,n_{i+1}},\dots,u_{i+1,1},u$ (see Fig. 2).

3. Proof of Theorem 1

Lemma 1. *Let $k\geqslant 3$. Then $W_k(m_1,\dots,m_k;0,\dots,0)$ is a partial cube if and only if m_i is odd for $i=1,\dots,k$.*

Proof. Let $W_k(m_1,\dots,m_k;0,\dots,0)$ be a partial cube. Then it is bipartite and hence the m_i's are odd.

Suppose now that the m_i's are odd. We show that $W=W_k(m_1,\dots,m_k;0,\dots,0)$ is a partial cube by induction on $\sum_{i=1}^k m_i$. If $m_1=\cdots=m_k=1$ it is well known that W is a partial cube. (In fact, it is even a median graph.) Without loss of generality we may thus assume $m_1\geqslant 3$. By the induction hypothesis, $W'=W_k(m_1-2,\dots,m_k;0,\dots,0)$ is a partial cube. Then W can be obtained from W' by an expansion over the proper cover $G_1'=\{w_1,w_{1,1},w_{1,2},\dots,w_{1,(m_1+1)/2}\}$ and $G_2'=V(W')\setminus G_1'\cup\{w_1,w_{1,(m_1+1)/2}\}$. (If $m_1=3$ we set $w_{1,2}=w_2$.) $\square$

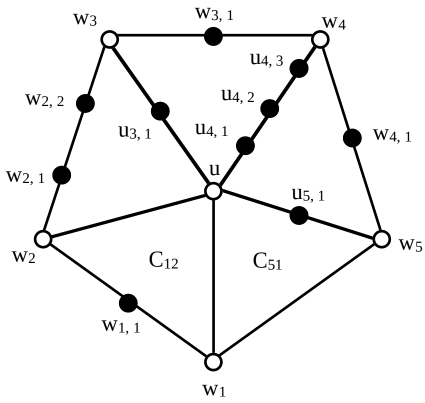

Fig. 2. $W_5(1,2,1,1,0;0,0,1,3,1)$.

To make this paper self-contained, we have included the above proof, although Lemma 1 follows from a result of Deza and Tuma [6]. They proved that $W_k(m_1,\ldots,m_k;0,\ldots,0)$ is an ℓ_1-graph if and only if it is not one of the graphs $W_3(0,2m+1,2k+1;0,0,0)$, $m,k \geqslant 0$, $W_3(0,2m+1,2k;0,0,0)$, $m,k \geqslant 0$, and $W_4(0,2m+1,0,2k+1;0,0,0,0)$, $m,k \geqslant 0$. Since in the bipartite case ℓ_1-graphs and partial cubes coincide, see [15], it follows that $W_k(m_1,\ldots,m_k;0,\ldots,0)$ is a partial cube if and only if it is bipartite.

Lemma 2. *Let G be a graph and let K be an isometric subgraph of G which is isomorphic to a subdivision of $K_{2,3}$. Then G is not a partial cube.*

Proof. It is enough to show that K is not a partial cube since isometric subgraphs of partial cubes are partial cubes. K can be described with two vertices a and b plus three vertex-disjoint paths P_1, P_2, and P_3 from a to b each of length at least 2. For $i=1,2,3$ let $a=x_{i,0},x_{i,1},\ldots,x_{i,n_i}=b$ be the vertices of the path P_i. We may assume that P_1 is the shortest among these three paths. Note that n_1+n_i is even for $i=2,3$. Consider the edge e between a and $x_{1,1}$, the edge f between $x_{2,(n_1+n_2)/2-1}$ and $x_{2,(n_1+n_2)/2}$, and the edge g between $x_{3,(n_1+n_3)/2-1}$ and $x_{3,(n_1+n_3)/2}$. It is straightforward to see that $f\Theta e$, $e\Theta g$ but f is not in relation Θ to g, hence Θ is not transitive. □

For $k \geqslant 3$, the *k-fan* F_k is the graph obtained as the join of a vertex u and a path on k vertices $w_1,\ldots,w_k$.

Lemma 3. *Let G be a graph and let K be an isometric subgraph of G which is isomorphic to a subdivision of F_k $(k \geqslant 3)$ such that at least one of the edges $uw_2,\ldots,uw_{k-1}$ is subdivided. Then G is not a partial cube.*

Proof. We proceed by induction on k. If $k=3$ then we are done by Lemma 2. Let $k \geqslant 4$ and let uw_i with $2 \leqslant i \leqslant k-1$ be a subdivided edge. Suppose that there is an $j \neq i$ such that the path in K corresponding to the edge uw_j in F_k is not a geodesic.

If $j=1$ (or $j=k$) then we remove the inner vertices of the path from u to w_2 that goes through w_1. Else we remove the inner vertices of the u,w_j-path. In both cases, we obtain an isometric subdivision of F_{k-1}. By induction hypothesis it is not a partial cube. In the other case, all the corresponding subdivided u,w_j-paths are geodesics. Then the subgraph of K induced by the vertices of the corresponding subdivided u,w_{i-1}-path, u,w_i-path and u,w_{i+1}-path in F_k are isometric and we can apply Lemma 2 again. □

We are now ready for the proof of Theorem 1. Let $W=W_k(m_1,\dots,m_k;n_1,\dots,n_k)$ be a bipartite wheel. Lemma 1 takes care for the case when all the n_i's are zero. Hence we may, without loss of generality, assume that $n_1 \geqslant 1$.

Case 1: $k \geqslant 4$. Assume, first that $n_2=\cdots=n_k=0$. Then $C_{12}\cup C_{k1}$ induces an isometric subdivided $K_{2,3}$, and by Lemma 2 W is not a partial cube.

Assume, next that there exists an $i \neq 1$ such that $n_i \geqslant 1$, $i \neq 2$ and $i \neq k$. Let H be the subgraph of W induced by u,w_1,w_2 and w_k and the corresponding paths between them. Let, in addition, H' be the subgraph induced by $u,w_2,w_3,\dots,w_k$ and the corresponding paths. We claim that H or H' is an isometric subgraph of W. Indeed, if the path $w_k,u_{k,n_k},\dots,u,u_{2,1},\dots,w_2$ is a geodesic then H and H' are isometric subgraphs. Otherwise w_1 or w_i is on geodesic between w_2 and w_k. In the first case H is isometric, and in the second case H' is such. In any case W is not a partial cube by Lemma 3.

In the final subcase, we may assume that $i=2$ and $n_k=n_3=0$. Note that w_k,u,w_3 is a geodesic (if $k=4$, observe that the edge w_kw_3 must be subdivided because W is bipartite). Now, the subgraph induced by w_3,u,w_k,w_1,w_2 and the corresponding paths between them is an isometric subdivided F_4. Hence, again using Lemma 3, W is not a partial cube.

Case 2: $k=3$. If $n_1>0$ and $n_2=n_3=0$ then, since W is bipartite, there is at least one vertex on the outer-cycle between w_2 and w_3. Hence, $C_{12}\cup C_{31}$ is isometric and by Lemma 2 we get that W is not a partial cube.

Assume now that $n_1>0$ and $n_2>0$. If $n_3=0$ then consider the 3-wheel centered in w_3. The case when none of the edges w_3u, w_3w_1 and w_3w_2 is subdivided was treated in Lemma 1; moreover, the case when exactly one of these edges is subdivided has been considered above. Hence $m_3>0$ and $m_2>0$. Without loss of generality, assume that $m_3 \leqslant m_2$. If $m_3 \geqslant n_1+1$ then $C_{12}\cup C_{23}$ is isometric and no partial cube is possible. So $m_3 \leqslant n_1$. Clearly, $m_3<n_1$, for otherwise we have an odd cycle. Removing from W the inner vertices of the u,w_1-path we get an isometric subdivided $K_{2,3}$. Conclude as above.

Now we have that $n_i>0$ for all $i=1,2,3$. Then considering a vertex w_i (for any i) as the central vertex of the wheel, we also obtain that $m_i>0$ for all $i=1,2,3$. We claim that the u,w_i-path is the unique geodesic between u and w_i. Indeed, in the opposite case, remove from W the inner vertices of the u,w_i-path to get an isometric subdivided $K_{2,3}$. By the same argument we also infer that the corresponding w_i,w_{i+1}-paths are isometric. In particular, this implies that the cycles C_{12},C_{23},C_{31} and the outer-cycle are all isometric. Let $v=u_{1,1}$ and $e=uv$. Then e is in relation Θ with exactly one edge $f=v'u'$ of C_{12} and one edge $g=v''u''$ of C_{13}. Note that $v,u,u_{2,1},\dots,w_2$ is a geodesic because W is bipartite. Hence f lies on the w_1,w_2-geodesic. Analogously, we have that g is an edge of the w_1,w_3-geodesic (see Fig. 3).

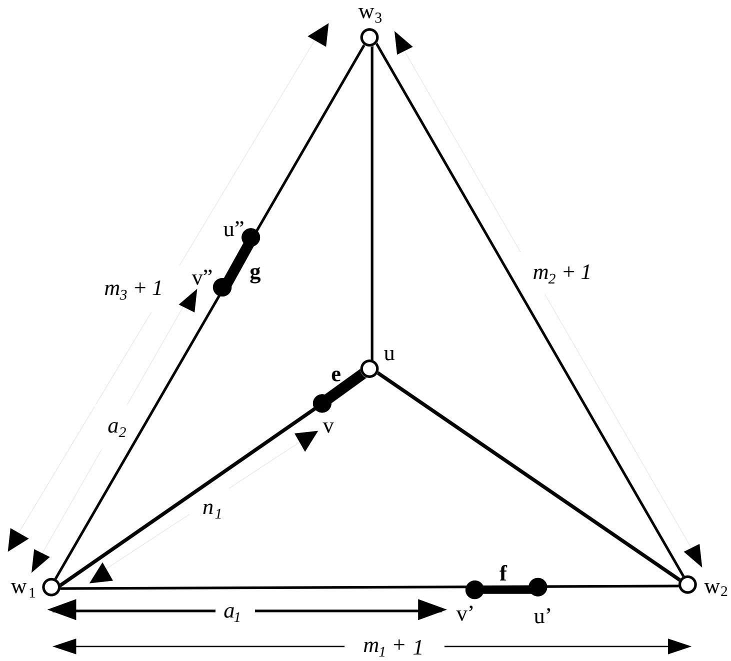

Fig. 3. Situation from the proof.

Set $a_1 = d(w_1, v')$ and $a_2 = d(w_1, v'')$. By the definition of Θ and since the corresponding cycles are isometric, we get the following equalities:

$$a_1 + n_1 = m_1 - a_1 + n_2 + 1,$$

$$a_2 + n_1 = m_3 - a_2 + n_3 + 1,$$

$$a_1 + a_2 = m_1 - a_1 + m_2 + 1 + m_3 - a_2.$$

From these equalities we obtain

$$m_2 + 2n_1 = n_2 + n_3 + 1.$$

Considering the edges $uu_{2,1}$ and $uu_{3,1}$ we also obtain

$$m_3 + 2n_2 = n_3 + n_1 + 1,$$

$$m_1 + 2n_3 = n_1 + n_2 + 1.$$

Hence $m_1 + m_2 + m_3 = 3$, and so $m_1 = m_2 = m_3 = 1$. Replacing the role of vertex u with vertex w_1 (w_2, w_3), we conclude that $n_1 = n_2 = n_3 = 1$ holds if W is a partial cube. □

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